ON ITERATES OF q-ANALOUGE BERNSTEIN OPERATOR VIA FIXED POINTS

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Abstract. In this article, we derive the convergence of successive iterations for certain nonlinear operators on complete normed linear space. As a consequence of our result we investigate the convergence of iterates for the Lupaş q-analouge Bernstein operators on $C[0,1]$. Our theorem indeed generalize the Kelisky-Rivlin result about the convergence of iterates for the Bernstein operator. The convergence of successive iterations for certain uniformly local nonlinear operator is also derived.

1. Introduction

The well-known Bernstein operator B_n (for $n \geq 1$) on the space $C[0,1]$ (the collection of all continuous real-valued mappings on $[0,1]$) is defined by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \text{ for } f \in C[0,1] \text{ and } x \in [0,1].$$

The operator has important contribution in approximation theory as well as it has a wide range of applications in numerical analysis, differential equation and probability theory. Kelisky and Rivlin [6] in 1967 first scrutinized the convergence of iterates of this operator through linear algebra. In fact, they established that for any $f \in C[0,1]$ and a fixed $n \geq 1$, $\lim_{j\to\infty}(B_n^j f)(x) = (1-x)f(0) + xf(1)$ where x lies in $[0,1]$. Later Rus [12] proved this result in a very simplified manner, in the perspective of fixed point theory.

The quick improvement of q-calculus has brought to the new extension of the Bernstein operator incorporating with q-integers. The notion of q-analouge Bernstein operator $L_{n,q}$ ($n \in \mathbb{N}$ and $q > 0$) on the domain $C[0,1]$ was first initiated by Lupaş [8] in the year 1987. For an

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arbitrary f in the space,

$$L_{n,q}(f(x)) = \sum_{i=0}^{n} f\left(\frac{[i]_q}{[n]_q}\right) b_{n,i}(q,x) \quad \text{for } x \in [0,1], \tag{1.1}$$

in which for each i,

$$b_{n,i}(q,x) = \frac{\begin{bmatrix} n \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}} x^i (1-x)^{n-i}}{\prod_{j=0}^{n-1}\{1-x+q^j x\}}.$$

These are linear operators and the operators $L_{n,q}$ is nothing but the Bernstein operators [6] for $q = 1$. It is worth to note that these operators give rational functions instead of polynomials for $q \neq 1$. In 2008, Agratini [1] first discussed the convergence of iterates of q-analouge Bernstein operator over $C[0,1]$. For more papers in this direction, one can refer to [1, 7, 9, 11].

In 2008, Jachymski [5] derived the below stated result for ensuring the convergence of successive approximation for certain linear operator on a complete normed linear space through the language of fixed point theory.

Theorem 1.1. [5] *Assume that P_0 is a closed subspace of a complete normed linear space P and $T : P \to P$ is a linear operator in order that $\|T|_{P_0}\| < 1$. Then for $p \in P$, $\{\lim_{n\to\infty} T^n p\} = (p + P_0) \cap \{p^* \in P : Tp^* = p^*\}$ if $(I - T)(P) \subseteq P_0$.*

As a consequence of this result, the author derived the Kelisky-Rivlin theorem for Bernstein operator. In 2014, Sultana and Vetrivel [13, Theorem 6] generalized aforementioned Jachymski's result for some nonlinear operator T. They also studied the convergence of successive iterations of nonlinear Bernstein type operator.

In this manuscript, we study the convergence of iterates of certain class of nonlinear operators over a complete normed linear space through the concept of fixed point theory. In particular, our result is an extension of the preceding Theorem 1.1 due to Jachymski [5]. As an implementation of our theorem, we investigate the convergence of iterates for Lupaş q-analouge Bernstein operator over the space $C[0,1]$. Moreover, this article deals with the convergence of iterates for uniformly local nonlinear operator on a normed linear space.

2. Notations and Definitions

In connection with q-calculus, we now recall the subsequent definitions and symbols from [11] which will be used in Section 4. For any

$q > 0$ and $n \in \{0\} \cup \mathbb{N}$, the q-fractional [11] identified by $[n]_q!$ is the following

$$[n]_q! = \prod_{i=1}^{n} [i]_q \quad (\text{for} \quad n \geq 1), \qquad [0]_q! = 1,$$

in which $[i]_q = \sum_{k=0}^{i-1} q^k$ for $i \geq 1$ and $[0]_q = 0$. Moreover, for $0 \leq i \leq n$, we define the Gaussian coefficients as below

$$\left[\begin{array}{c} n \\ i \end{array} \right]_q = \frac{[n]_q!}{[n-i]_q![i]_q!}.$$

It can be noted that the Gaussian coefficients are the ordinary binomial coefficients for $q = 1$.

To the end of this article, we indicate by Ψ the set consisting of all upper semi-continuous map $\psi : [0, \infty) \to [0, \infty)$ satisfying $\psi(s) < s$ for each positive s. We recall the subsequent result due to Jachymsky [4, Lemma 1], that will be used in the sequel.

Lemma 2.1. [4] *For any $\psi \in \Psi$, we have a continuous non-decreasing map $\phi : [0, \infty) \to [0, \infty)$ meeting $\psi(s) \leq \phi(s) < s$ for $s > 0$.*

3. Convergence of iterates for nonlinear operators

We commence this section with the upcoming result, which ensures the convergence of successive approximation for certain class of non-linear operators on Banach space. From this result, the convergence of iterates for Lupaş q-analouge of the Bernstein operator due to O. Agratini [1] has been established.

Theorem 3.1. *Assume P_0 is a closed subspace of a complete normed space $(P, ||.||)$. Let $T : P \to P$ satisfy $(I - T)(P) \subseteq P_0$ and for $p, q \in P$ with $p - q \in P_0$,*

$$||Tp - Tq|| \leq \psi(||p - q||) \quad \text{whenever } \psi \in \Psi. \tag{3.1}$$

Then for each p in P, $\{\lim_{n\to\infty} T^n p\} = (p+P_0) \cap \{p^* \in P : T(p^*) = p^*\}$.

Proof. Choose an element $p \in P$. As $(I - T)(P) \subseteq P_0$, it occurs for every $n \in \mathbb{N}$ that $T^n p - T^{n+1} p \in P_0$. Now, for all $n \geq 1$ we get

$$\|T^n p - T^{n+1} p\| \leq \psi(\|T^{n-1} p - T^n p\|) < \|T^{n-1} p - T^n p\|. \tag{3.2}$$

Hence it appears that $\{d_n\}_n = \{\|T^n p - T^{n+1} p\|\}_n$ is monotone decreasing. Thus $\lim_{n\to\infty} d_n = a$ exists where $a \geq 0$. If possible, let us suppose that $a > 0$. We see from (3.2) with the help of upper semi-continuity of ψ that

$$\begin{aligned} a \leq \limsup_{d_{n-1}\to a} \psi(d_{n-1}) &\leq \psi(a) \\ &< a, \text{ [Since } \psi(s) < s, \ \forall\, s > 0] \end{aligned}$$

which is not true. Thus, $\lim_{n\to\infty} d_n = 0$.

Since $\psi \in \Psi$, according to Lemma 2.1 we have a continuous strictly increasing map ϕ on $[0, \infty)$ satisfying

$$\psi(s) \leq \phi(s) < s \quad \text{for any } s \in (0, \infty). \tag{3.3}$$

On the account of $\|T^n p - T^{n+1}p\| \to 0$ whenever n approaches to ∞, thus by choosing $\delta = \varepsilon - \phi(\varepsilon) > 0$, we see that

$$\|T^n p - T^{n+1}p\| < \varepsilon - \phi(\varepsilon) \quad \forall\, n \geq M$$

for some $M \in \mathbb{N}$. Now, for all $n \geq M$, it yields

$$\begin{aligned} \|T^n p - T^{n+2}p\| &\leq \|T^n p - T^{n+1}p\| + \|T^{n+1}p - T^{n+2}p\| \\ &\leq \varepsilon - \phi(\varepsilon) + \psi(\|T^n p - T^{n+1}p\|) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(\|T^n p - T^{n+1}p\|) \quad \text{[By (3.3)]} \\ &< \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon. \qquad [\because\ \phi \text{ is increasing}] \end{aligned}$$

Hence it can be shown in a similar way that $\|T^n p - T^{n+k}p\| < \varepsilon$ for every $k \in \mathbb{N}$ and $n \geq M$. Thus $\{T^n p\}_n$ is Cauchy and hence $\lim_{n\to\infty} T^n p$ exists. Suppose $\lim_{n\to\infty} T^n p = p^*$ for some $p^* \in P$.

As P_0 is closed and $T^n p - T^{n+1}p \in P_0$ where $n \geq 1$, it appears that $T^n p - p^* \in P_0$ for each n. Now for any $n \geq 1$,

$$\begin{aligned} \|Tp^* - p^*\| &\leq \|Tp^* - T^{n+1}p\| + \|T^{n+1}p - p^*\| \\ &\leq \psi(\|p^* - T^n p\|) + \|T^{n+1}p - p^*\| \\ &\leq \|p^* - T^n p\| + \|T^{n+1}p - p^*\|. \end{aligned}$$

Taking limit $n \to \infty$, it appears $Tp^* = p^*$. Hence $\lim_{n\to\infty} T^n p \in \{p^* \in P : Tp^* = p^*\}$.

Since $(I - T)P \subseteq P_0$ and P_0 is closed, it appears that $\lim_{n\to\infty} T^n p \in (p + P_0)$. Hence, $\lim_{n\to\infty} T^n p \in (p + P_0) \cap \{p^* \in P : T(p^*) = p^*\}$. Let $z_1, z_2 \in (p + P_0) \cap \{p^* \in P : T(p^*) = p^*\}$. Then $z_1 - z_2 \in P_0$. Hence and by equation (3.1), we have

$$||Tz_1 - Tz_2|| \leq \psi(||z_1 - z_2||) < ||z_1 - z_2|| \quad \text{[As } \psi(s) < s, \ \forall\, s > 0].$$

Therefore we get $z_1 = z_2$. Hence proved.

$\square$

The renowned Banach contraction principle was extended by Edelstein [3] in 1961 for uniformly local contraction [3] over a complete metric space. Later Nadler [10] investigated the fixed points for the set-valued uniformly local contraction. Few more results in this direction can be found in [5, 13]. In the succeeding theorem, we investigate the convergence of successive approximation for uniformly local nonlinear operators over a complete normed linear space.

Theorem 3.2. *Let $(P, \|.\|)$ be a complete normed linear space and $T : P \to P$ in order that for each $p, q \in P$ having $\|p - q\| < r\,(r > 0)$,*

$$\|Tp - Tq\| \leq \psi(\|p - q\|) \text{ where } \psi \in \Psi. \tag{3.4}$$

Then for each p lies in P, $\{\lim_{n\to\infty} T^n p\} = \{p^* \in P : Tp^* = p^*\}$.

Proof. Let $p \in P$ be arbitrary. For given $r > 0$, we can find a finite sequence $(y_0^i)_{i=0}^N$ in order that $y_0^0 = p$, $y_0^N = Tp$ and $\|y_0^{i-1} - y_0^i\| < r$ for each $i = 1, 2, \cdots, N$.

For every i, we now construct a sequence $\{y_n^i\}_{n\in\mathbb{N}}$ where $y_n^i = T(y_{n-1}^i)$ for $n \geq 1$. As $\|y_0^{i-1} - y_0^i\| < r$ and T satisfies (3.4), it appears that $\|y_1^{i-1} - y_1^i\| < r$ for any $1 \leq i \leq N$. It can be shown in a similar pattern that $\|y_n^{i-1} - y_n^i\| < r$ for each $n \geq 1$. Clearly, $y_n^0 = T^n p$ and $y_n^N = T^{n+1} p$. Now for each $1 \leq i \leq N$ and $n \geq 1$,

$$\|y_n^{i-1} - y_n^i\| \leq \psi(\|y_{n-1}^{i-1} - y_{n-1}^i\| < \|y_{n-1}^{i-1} - y_{n-1}^i\|.$$

Hence it occurs that for each i, $\{d_n^i\}_n = \{\|y_n^{i-1} - y_n^i\|\}_n$ is monotone decreasing. By following the same proof line of the Theorem 3.1, it happens that $\lim\limits_{n\to\infty} d_n^i = 0$, for every $i = 1, 2, \cdots, N$. Now for any $n \geq 1$,

$$\begin{aligned}
\|T^n p - T^{n+1} p\| &= \|y_n^0 - y_n^N\| \\
&\leq \|y_n^0 - y_n^1\| + \|y_n^1 - y_n^2\| + \cdots + \|y_n^{N-1} - y_n^N\| \\
&= d_n^1 + d_n^2 + \cdots + d_n^N.
\end{aligned}$$

Hence we get that $\lim_{n\to\infty} \|T^n p - T^{n+1} p\| = 0$.

Since $\psi \in \Psi$, according to Lemma 2.1 we have a continuous strictly increasing map ϕ on $[0, \infty)$ satisfying

$$\psi(s) \leq \phi(s) < s \quad \text{for any } s \in (0, \infty).$$

Let $\varepsilon > 0$ be arbitrary. On the account of $\|T^n p - T^{n+1} p\| \to 0$, then for $0 < \delta < \min\{r, \varepsilon\}$, we obtain $M \in \mathbb{N}$ in order that,

$$\|T^n p - T^{n+1} p\| < \delta - \phi(\delta), \text{ for every } n \geq M.$$

Clearly, $\|T^np - T^{n+1}p\| < \delta < r$. Thus for every $n \geq M$, it yields

$$\begin{aligned}\|T^np - T^{n+2}p\| &\leq \|T^np - T^{n+1}p\| + \|T^{n+1}p - T^{n+2}p\| \\ &< \delta - \phi(\delta) + \phi(\|T^np - T^{n+1}p\|) \\ &< \delta - \phi(\delta) + \phi(\delta) = \delta < \varepsilon.\end{aligned}$$

By following the same methodology, for each $k \geq 1$ and $n \geq M$, we conclude that $\|T^np - T^{n+k}p\| < \varepsilon$. Thus $\{T^np\}_n$ becomes a Cauchy sequence in P, hence $T^np \to p^*$ for some $p^* \in P$.

As $T^np \to p^*$, then we obtain $N_1 \in \mathbb{N}$ satisfying $\|T^np - p^*\| < r$ for every $n \geq N_1$. Now for $n \geq N_1$,

$$\begin{aligned}\|Tp^* - p^*\| &\leq \|Tp^* - T^{n+1}p\| + \|T^{n+1}p - p^*\| \\ &\leq \psi(\|p^* - T^np\|) + \|T^{n+1}p - p^*\| \\ &\leq \|p^* - T^np\| + \|T^{n+1}p - p^*\|.\end{aligned}$$

Taking limit $n \to \infty$, we have $Tp^* = p^*$ and hence $\lim_{n\to\infty} T^np \in \{p^* \in P : Tp^* = p^*\}$. Now we shall demonstrate that the collection of fixed points of the operator T is nothing but a singleton set.

Suppose $T(a) = a$ and $T(b) = b$ for some $a, b \in P$. Consequently, we have $(x_i)_{i=0}^L$ in P where $x_0 = a$, $x_L = b$ and $\|x_{i-1} - x_i\| < r$ for $1 \leq i \leq L$. Consequently (3.4) leads to $\|T^nx_{i-1} - T^nx_i\| < r$ for each i and $n \geq 1$. Thus for every $i = 1, 2, \cdots, L$ and $n \geq 1$ we obtain,

$$\|T^nx_{i-1} - T^nx_i\| \leq \psi\big(\|T^{n-1}x_{i-1} - T^{n-1}x_i\|\big) < \|T^{n-1}x_{i-1} - T^{n-1}x_i\|.$$

Then as in the first part of the proof, we can show that for every i, the sequence $s_i^n = \|T^nx_{i-1} - T^nx_i\|$ is convergent and $\lim_{n\to\infty} s_i^n = s_i = 0$.

$$\|a - b\| = \|T^na - T^nb\| \leq \sum_{i=1}^{L} \|T^nx_{i-1} - T^nx_i\| = \sum_{i=1}^{L} s_i^n.$$

Letting $n \to \infty$, we conclude that $a = b$. Hence proved. □

In 1969, Boyd and Wong [2] generalized the Banach contraction principle by using nonlinear contraction map. The authors [2] established that for a complete metric space (P, d), there is a unique fixed point for the map $T : P \to P$ if for all $p, q \in P$, $d(T(p), T(q)) \leq \psi(d(p, q))$ where $\psi \in \Psi$. An extension of this result for uniformly local nonlinear map on r-chainable metric space can be established by following the same proof line of the Theorem 3.2. The result is described in the following manner.

Theorem 3.3. *Assume (P, d) is r-chainable (where $r > 0$), that is, for any $p, q \in P$, there is sequence $(y^i)_{i=0}^N \subseteq P$ satisfying $y^0 = p$, $y^N = q$*

and $d(y^i, y^{i+1}) < r$ *where* $i = 0, \cdots, N-1$. *Suppose* $T : P \to P$ *is satisfying*

$$d(T(p), T(q)) \leq \psi(d(p,q)), \quad \forall\, p, q \in P \text{ with } d(p,q) < r, \tag{3.5}$$

where $\psi \in \Psi$. *Then for each* p *in* P, $\{\lim_{n\to\infty} T^n p\} = \{p^* \in P : Tp^* = p^*\}$ *if* P *is complete.*

It is worth to note that any mapping which satisfies the nonlinear contractive condition due to Boyd-Wong [2] fulfils (3.5), whereas the converse statement may not be true and the underneath example illustrates that.

Example 3.4. Suppose $P = (-\infty, -1] \cup [1, \infty)$ having standard metric and a map T on P is as follows

$$\begin{aligned} T(p) &= \frac{1}{2}(p+1) \qquad \text{for } p \geq 1; \\ &= \frac{1}{2}(p-1) \qquad \text{otherwise.} \end{aligned}$$

Consider $r = 1$, then we observe that for every $p, q \in P$ with $d(p,q) < 1$, it appears $d(Tp, Tq) \leq \psi(d(p,q))$, where ψ defined on $[0, \infty)$ by

$$\begin{aligned} \psi(s) &= s/2 \qquad \text{for } 0 \leq s < 2; \\ &= (s+1)/2 \qquad \text{otherwise.} \end{aligned}$$

Obviously, we can see that $\psi \in \Psi$. Therefore T is a uniformly local nonlinear operator. However, for $p = 1$ and $q = -1$,

$$d(T(1), T(-1)) = 2 = d(1, -1) > \varphi(d(p,q)),$$

for any map $\varphi \in \Psi$. This indicates that the map T is not a nonlinear contraction however it is a uniformly local nonlinear operator.

4. Convergence of Iterates for Lupaş q-analouge of the Bernstein Operator

As an application of the above mention Theorem 3.1, we derive the convergence of iterates of Lupaş q-analouge of the Bernstein operator $L_{n,q}$ on $C[0,1]$.

Corollary 4.1. *Let* $L_{n,q}$ $(q > 0)$ *be an operator as defined in* (1.1). *Then, we have for any* $f \in C[0,1]$ *and* $n \geq 1$,

$$\lim_{j\to\infty} (L^j_{n,q} f)(x) = f(0) + [f(1) - f(0)]x \qquad \text{for } x \in [0,1].$$

Proof. Let us take a Banach space P as $C[0,1]$ having the supremum norm. Let $P_0 \subseteq P$ consist of all $f \in P$ which takes value 0 at $x = 0, 1$. Take two distinct elements α and β in P where $\alpha - \beta \in P_0$. Since $(\alpha-\beta)(0) = 0$ and $(\alpha-\beta)(1) = 0$, it occurs for any $x \in [0,1]$ that

$$\begin{aligned} \left|L_{n,q}\alpha(x) - L_{n,q}\beta(x)\right| &\leq \sum_{i=1}^{n-1} b_{n,i}(q,x) \left|(\alpha-\beta)\left(\frac{[i]_q}{[n]_q}\right)\right| \\ &\leq [1 - b_{n,n}(q,x) - b_{n,0}(q,x)]\,||(\alpha-\beta)||. \end{aligned} \tag{4.1}$$

Let us represent $g_n(x) = q^{n(n-1)/2}x^n + (1-x)^n$ and $h_n(x) = \prod_{s=0}^{n-1}(1 - x + q^s x)$. For $n \geq 2$, the minimum value of $g_n(x)$ over $[0,1]$ occurs at $x = (1+q^{n/2})^{-1}$ and its value is $\left(q^{n/2}/(1+q^{n/2})\right)^{n-1}$. For $n = 1$, the minimum value is 1. On the other hand, it is simple to visualize that $\max_{x\in[0,1]} h_n(x) \leq \max\{1, q^{(n-1)^2}\}$, for any $q > 0$. Hence we can write that

$$b_{n,n}(q,x) + b_{n,0}(q,x) = \frac{g_n(x)}{h_n(x)} \geq \left(\frac{q^{n/2}}{1+q^{n/2}}\right)^{n-1} \frac{1}{\max\{q^{(n-1)^2}, 1\}}. \tag{4.2}$$

Let us set $a_{n,q} = \left(\frac{q^{n/2}}{1+q^{n/2}}\right)^{n-1} \frac{1}{\max\{1, q^{(n-1)^2}\}}$ and we see that $0 < a_{n,q} \leq 1$. It follows from (4.1) and (4.2) that $||L_{n,q}\alpha - L_{n,q}\beta|| \leq \psi(||\alpha-\beta||)$, where $\psi(s) = (1 - a_{n,q})s$, for each $s \geq 0$. Evidently, $\psi \in \Psi$ and hence $L_{n,q}$ meets the condition (3.1) of Theorem 3.1.

Let us take f in P. Now, $f(0) - L_{n,q}(f(0)) = [f(0) - f(0)]b_{n,0}(q,0) = 0$ and $f(1) - L_{n,q}(f(1)) = [f(1) - f(1)]b_{n,0}(q,1) = 0$. Hence $(I - L_{n,q})(P) \subseteq P_0$. According to Theorem 3.1, $\left\{\lim_{j\to\infty} L_{n,q}^j f\right\} = (f + P_0) \cap Fix\, L_{n,q}$ for any $f \in C[0,1]$. We observe $e_0(x) = x$ and $e_1(x) = 1 - x$ are fixed points of $L_{n,q}$. This implies that $e_2(x) = f(0)(1-x) + f(1)x \in Fix\, L_{n,q}$. Moreover, it appears $e_2 \in f + P_0$ and thus $(f+P_0) \cap Fix\, L_{n,q} = \{e_2\}$. Hence proved. □

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