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Trigonometric approximation of the conjugate series of a function of generalized Lipchitz class by product summability

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Abstract Trigonometric Fourier approximation and Lipchitz class of function had been introduced by Zygmund and McFadden respectively. Dealing with degree of approximation of conjugate series of a Fourier series of a function of Lipchitz class Misra et al. have established certain theorems. Extending their results, in this paper a theorem on trigonometric approximation of conjugate series of Fourier series of a function $f \in Lip(\xi(t), r)$ by product summability $(E, S)(N, p_n, q_n)$

Keywords and phrases Fourier Approximation, $Lip(\xi(t), r)$ class function, (E, q) mean,(N, p_n, q_n) mean,$(E, S)(N, p_n, q_n)$ product mean, Conjugate

Fourier Series and Lebesgue Integral.

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1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$.Let$\{p_n\}$ and $\{q_n\}$ be the sequences of positive real numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \text{ and } Q_n = \sum_{\nu=0}^{n} q_\nu \tag{1}$$

Let

$$t_n = \frac{1}{r_n} \sum_{\nu=0}^{n} p_{n-\nu} q_\nu s_\nu \tag{2}$$

where $r_n = p_0 q_n + p_1 q_{n-1} + ... + p_n q_0 (\neq 0), p_{-1} = q_{-1} = r_{-1} = 0$. Then $\{t_n\}$ is called the sequence of (N, p_n, q_n) mean of the sequence $\{s_n\}$. If

$$t_n \to s \text{ as } n \to \infty \tag{3}$$

then the series $\sum a_n$ is said to be (N, p_n, q_n) summable to s. The necessary and sufficient conditions for regularity of (N, p_n, q_n) method are [1]:

$$\frac{p_{n-\nu} q_\nu}{r_n} \to 0, \text{ for each integer } \nu \geq 0 \text{ as } n \to \infty \tag{4}$$

and

$$\sum_{\nu=0}^{n} |p_{n-\nu} q_\nu| < H|r_n| \text{ where H is a positive integer independent of} n. \tag{5}$$

The sequence-to-sequence transformation[2]

$$T_n = \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} C(n, \nu) q_{n-\nu} s_\nu \tag{6}$$

defines the (E, q) mean of the sequence $\{s_n\}$.If

$$T_n \to s \text{ as } n \to \infty \tag{7}$$

then the series $\sum a_n$ is said to be (E, q) summable to s.Clearly, (E, q) method is regular.

Further, the (E,q) transform of (N,p_n,q_n) transform of$\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n}\sum_{k=0}^{n} C(n,k)q^{n-k}t_k \ = \frac{1}{(1+q)^n}\sum_{k=0}^{n} C(n,k)q^{n-k}\{\frac{1}{r_k}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu s_\nu\} \quad (8)$$

If

$$\tau_n \to s \text{ as } n \to \infty \quad (9)$$

then the series $\sum a_n$ is said to be $(E,q)(N,p_n,q_n)$ summable to s. Let $f(t)$ be a periodic function with period 2π and L-integrable over $(-\pi,\pi)$.The Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2}\sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) \equiv \sum_{n=0}^{\infty} A_n(x) \quad (10)$$

and the conjugate Fourier series of (10) is

$$\sum_{n=1}^{\infty}(b_n cosnx - a_n sinnx) \equiv \sum_{n=0}^{\infty} B_n(x) \quad (11)$$

The L_∞ norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\} \quad (12)$$

and L_ν norm is defined by

$$\|f\|_\nu = \Big\{\int_0^{2\pi} |f(x)|^\nu dx\Big\}^{\frac{1}{\nu}}, \nu \geq 1. \quad (13)$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under the norm $\|.\|_\infty$ is defined by [7]

$$\|P_n - f\|_\infty = \sup\{|P_n(x) - f(x)| : x \in R\} \quad (14)$$

and the degree of approximation $E_n(f)$ of a function $f \in L_\nu$ is given by [6]

$$E_n(f) = \min_{p_n} \|P_n - f\|_\nu \quad (15)$$

This method is called Trigonometric Fourier approximation. A function $f \in Lip\alpha$,if [3]

$$|f(x+t) - f(x)| = O(|t|^\alpha), 0 < \alpha \leq 1 \quad (16)$$

and $f \in Lip(\alpha, r)$, $0 < \alpha \leq 2\pi$, if[3]

$$\Big(\int_0^{2\pi} |f(x+t) - f(x)|dx\Big)^{\frac{1}{r}} = O(|t|^{\alpha}), 0 < \alpha \leq 1, r \geq 1, t > 0 \tag{17}$$

For a positive increasing function $\xi(t)$ and an integer $p > 1$, we define[13], $f \in Lip(\xi(t), r)$ if

$$\Big(\int_0^{2\pi} |f(x+t) - f(x)|dx\Big)^{\frac{1}{r}} = O(\xi(t)), \tag{18}$$

We use the following notation through out this paper:

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-ct)\} \tag{19}$$

$$\overline{s_n}(f;x) = n\text{-th partial sum of the conjugate Fourier series} \tag{20}$$

and

$$\overline{K_n}(t) = \frac{1}{\pi(1+s)^n} \sum_{k=0}^{n} C(n,k) s^{n-k} \Big\{\frac{1}{r_k} \sum_{\nu=0}^{k} p_{k-\nu} q_\nu \frac{cos\frac{t}{2} - cos\Big(\nu + \frac{1}{2}\Big)t}{sin\frac{t}{2}}\Big\} \tag{21}$$

Further the method $(E,q)(N,p_n,q_n)$ is regular and this case is supposed through out this paper.

2. KNOWN THEOREMS

Dealing with the degree of approximation by product summability, Misra et al[4] proved the following theorem using $(E,q)(\overline{N},p_n)$ mean of the conjugate series of a Fourier series.

2.1. **Theorem:** If f is a 2π periodic function of class $Lip\alpha$, then the degree of approximation by the product mean $(E,q)(\overline{N},p_n)$ summability means of the conjugate series (11) of the Fourier series (10) is given by $\|\tau_n - f\|_\infty = O\Big(\frac{1}{(n+1)^\alpha}\Big), 0 < \alpha < 1$ where τ_n is defined in(8).

Subsequently, Misra et al[5] established another theorem on degree of approximation by the product mean $(E, q)(\overline{N}, p_n)$ of the conjugate series of the fourier series of a function of class$Lip(\alpha, r)$. They prove:

2.2. **Theorem:** If f is a 2π periodic function of class $Lip(\alpha, r)$, then the degree of approximation by the product mean $(E, q)(\overline{N}, p_n)$ summability means of the conjugate series (11) of the Fourier series (10) is given by $\|\tau_n - f\|_\infty = O\Big(\frac{1}{(n+1)^{(\alpha+\frac{1}{r})}}\Big), 0 < \alpha < 1, r \geq 1$ where τ_n is defined in(8).

3. MAIN THEOREM

In this paper, we have studied a theorem on the degree of approximation by the product mean $(E, S)(N, p_n, q_n)$ of the conjugate series of the Fourier series of a function of class $Lip(\xi(t), r)$.We prove:

3.1. **Theorem:** If f is a 2π periodic function of class $Lip(\xi(t), l)$, then the degree of approximation by the product mean $(E, s)(N, p_n, q_n)$ summability means of the conjugate series (11) of the Fourier series (10) is given by $\|\tau_n - f\|_\infty = O\Big((n+1)^{(\alpha+\frac{1}{r})}\xi\Big(\frac{1}{n+1}\Big)\Big), l \geq 1$ where τ_n is defined in (8).

4. REQUIRED LEMMAS

We require the following lemmas for the proof of the theorem.

4.1. **Lemma:**

$$|\overline{K_n}(t)| = O(n), 0 \leq t \leq \frac{1}{(n+1)}$$

Proof of Lemma 4.1: For $0 \leq t \leq \frac{1}{(n+1)}$, we have $\sin nt \leq nsint$

$$\begin{aligned}
|\overline{K_n}(t)| &= \frac{1}{\pi(1+s)^n}\Big|\sum_{k=0}^{n} C(n,k)s^{n-k}\Big\{\frac{1}{r_k}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu \frac{\cos\frac{t}{2}-\cos(\nu+\frac{1}{2})t}{\sin\frac{t}{2}}\Big\}\Big| \\
&\leq \frac{1}{\pi(1+s)^n}\Big|\sum_{k=0}^{n} C(n,k)s^{n-k}\Big\{\frac{1}{r_k}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu \Big(O\Big(2\sin\nu\frac{t}{2}\cos\nu\frac{t}{2}+\nu\sin t\Big)\Big\}\Big| \\
&\leq \frac{1}{\pi(1+s)^n}\Big|\sum_{k=0}^{n} C(n,k)s^{n-k}\Big\{\frac{1}{r_k}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu \Big(O(\nu)+O(\nu)\Big)\Big\}\Big| \\
&\leq \frac{1}{\pi(1+s)^n}\Big|\sum_{k=0}^{n} C(n,k)s^{n-k}\Big\{\frac{O(K)}{R_k}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu\Big)\Big\}\Big| \\
&= O(n)
\end{aligned}$$

This proves the Lemma.

4.2. **Lemma:**

$$|\overline{K_n}(t)| = O\Big(\frac{1}{t}\Big), \frac{1}{(n+1)} \leq t \leq \pi$$

Proof of Lemma 4.2:
By Jordan's lemma, for $\frac{1}{(n+1)} \leq t \leq \pi$, we have $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$. Then

$$\begin{aligned}
|\overline{K_n}(t)| &= \frac{1}{\pi(1+s)^n}\Big|\sum_{k=0}^{n} C(n,k)s^{n-k}\Big\{\frac{1}{r_k}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu \frac{\cos\frac{t}{2}-\cos(\nu+\frac{1}{2})t}{\sin\frac{t}{2}}\Big\}\Big| \\
&\leq \frac{\pi}{2\pi(1+s)^n t}\Big|\sum_{k=0}^{n} C(n,k)s^{n-k}\Big\{\frac{1}{r_k}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu\Big\}\Big| \\
&= \frac{1}{2(1+s)^n t}\Big|\sum_{k=0}^{n} C(n,k)s^{n-k}\Big\{\frac{1}{r_k}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu\Big\}\Big| \\
&= \frac{1}{2(1+s)^n t}\Big|\sum_{k=0}^{n} C(n,k)s^{n-k}\Big| \\
&= O\Big(\frac{1}{t}\Big).
\end{aligned}$$

This proves the lemma.

Proof of Theorem 3.1:

Using Riemann-Lebesgue theorem, for the nth partial sum $\overline{s_n}(f;x)$ of the conjugate Fourier series (11) of $f(x)$ and following Titchmarch[6], we have

$$\overline{s_n}(f;x) - f(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\frac{\cos\frac{t}{2} - \sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt$$

using (2), the (N, p_n, q_n) transform of $\overline{s_n}(f;x)$ is given by

$$t_n - f(x) = \frac{2}{\pi r^n}\int_0^\pi \psi(t)\sum_{k=0}^{n} p_{n-k}q_k\frac{\cos\frac{t}{2} - \sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt$$

Denoting the $(E,q)(N,p,q)$ transform of $\overline{s_n}(f;x)$ by τ_n, we have

$$\begin{aligned}
\|\tau_n - f\| &= \frac{2}{\pi(1+s)^n}\int_0^\pi \psi(t)\sum_{k=0}^{n} C(n,k)s^{n-k}\Big\{\frac{1}{r_k}\sum_{k=0}^{n} p_{n-k}q_k\frac{\cos\frac{t}{2} - \sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\Big\}dt \\
&= \int_0^\pi \psi(t)\overline{K_n}(t)dt \\
&= \Big\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\Big\}\psi(t)\overline{K_n}(t)dt \\
&= I_1 + I_2, \text{say}
\end{aligned}$$

Now

$$
\begin{aligned}
|I_1| &= \frac{2}{\pi(1+s)^n}\int_0^{\frac{1}{n+1}}\psi(t)\sum_{k=0}^{n}C(n,k)s^{n-k}\Big\{\frac{1}{r_k}\sum_{k=0}^{n}p_{n-k}q_k\frac{\cos\frac{t}{2}-\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\Big\}dt \\
&= \Big|\int_0^{\frac{1}{n+1}}\psi(t)\overline{K_n}(t)dt\Big| \\
&= \Big(\int_0^{\frac{1}{n+1}}\Big|\frac{\psi(t)}{\xi(t)}\Big|^l\Big)^{\frac{1}{l}}\Big(\int_0^{\frac{1}{n+1}}\Big|\xi(t)K_n(t)\Big|^m\Big)^{\frac{1}{m}}dt \\
&\text{where}\frac{1}{l}+\frac{1}{m}=1,\text{using Holder's inequality} \\
&= O(1)\Big(\int_0^{\frac{1}{n+1}}\xi(t)n^m dt\Big)^{\frac{1}{m}} \\
&= O\Big(\xi\Big(\frac{1}{n+1}\Big)\Big)\Big(\frac{n^m}{n+1}\Big)^{\frac{1}{m}} \\
&= O\Big(\xi\Big(\frac{1}{n+1}\Big)\Big)\Big(\frac{1}{n+1}\Big)^{\frac{1}{m}-1}\Big) \\
&= O\Big(\xi\Big(\frac{1}{n+1}\Big)\Big)\Big(\frac{1}{n+1}\Big)^{\frac{-1}{l}}\Big) \\
&= O\Big(\xi\Big(\frac{1}{n+1}\Big)\Big)(n+1)^{\frac{1}{l}}\Big) \qquad (3.1.1)
\end{aligned}
$$

Next

$$
\begin{aligned}
|I_2| &\leq \left(\int_{\frac{1}{(n+1)}}^{\pi}\Big|\frac{\psi(t)}{\xi(t)}\Big|^l dt\right)^{\frac{1}{l}}\left(\int_{\frac{1}{n+1}}^{\pi}\Big|\xi(t)K_n(t)\Big|^m dt\right)^{\frac{1}{m}}\text{By the Holder's inequality} \\
&= O(1)\left(\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{\xi(t)}{t}\Big)^m dt\right)^{\frac{1}{m}}\text{using lemma 4.2} \\
&= O(1)\left(\int_{\frac{1}{\pi}}^{n+1}\left|\frac{\xi\left(\frac{1}{y}\right)}{\left(\frac{1}{y}\right)}\right|^m dy\right)^{\frac{1}{m}} \qquad (3.1.2)
\end{aligned}
$$

Since,$\xi(t)$ is a positive increasing function, so is $\left\{\dfrac{\xi\left(\frac{1}{y}\right)}{\left(\frac{1}{y}\right)}\right\}$. Using the second mean value theorem we get

$$= O\left((n+1)\xi\Big(\frac{1}{n+1}\Big)\right)\left(\int_{\delta}^{n+1}\frac{1}{y^2}dy\right)^{\frac{1}{m}}, \text{for some} \frac{1}{\pi} \leq \delta \leq n+1$$
$$= O\left((n+1)^{\frac{1}{l}}\xi\Big(\frac{1}{n+1}\Big)\right)$$

Then from (3.1.1) and (3.1.2),we have

$$|\tau_n - f(x)| = O\left((n+1)^{\frac{1}{l}}\xi\Big(\frac{1}{n+1}\Big)\right), \text{for } l \geq 1$$

Hence,

$$\|\tau_n - f(x)\|_\infty = \sup_{-\pi<x<\pi}|\tau_n - f(x)| = O\left((n+1)^{\frac{1}{l}}\xi\Big(\frac{1}{n+1}\Big)\right), \text{for } l \geq 1$$

This completes the proof of the theorem.

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