Journal of the Orissa Mathematical Society ISSN:0975-2323
Volume 33, No.2(2014), p.71-108

# DOMINATED ASSIGNMENT SIMULATION TECHNIQUE

*Prasanta Kumar Das[1] and Susanta Kumar Das[2]
[1]School of Applied Sciences (Mathematics)
KIIT University, Bhubaneswar
Orissa, India, 751024.
Email: pkdasfma@kiit.ac.in
[2]School of Applied Sciences (Physics)
KIIT University, Bhubaneswar
Orissa, India, 751024.
Email: skdasfpy@kiit.ac.in

**Abstract:** The aim of this paper is to introduce a new methodology, called the dominated assignment simulation (or selection) technique (**DAS**-technique) to solve the linear assignment problems. We introduce the new concepts such as, $(2\times 2)$-decision matrix, Minimin simulation criterion and Maximax simulation criterion, dominated column. Here the concept of $(2 \times 2)$-decision matrix is introduced to find the final two assignments from the cost matrix which are important to this method. The minimized linear assignment problems (loss matrix) and the maximized linear assignment problems (profit matrix) are solved using Minimin simulation criterion and Maximax simulation criterion respectively in the presence of $(2 \times 2)$-decision matrix. The dominated (dominating) column concept is used to break the tie cases arises in minimized (maximized) assignment problem. In this technique, the $n \times n$ linear assignment problem is solved in $n$ steps. We reduce the order of assignment problem by removing the corresponding rows and columns of the obvious assignments from the cost matrix.

**Keywords and phrases:** Minimized assignment problem, maximized assignment problem, off diagonal, trace, offtrace, dominated column, dominating column, decision matrix, Minimin simulation criterion, Maximax simulation criterion, simulated assignment problems, dominated assignment simulation technique, simulated solution.

**2000 Mathematics Subject Classification No:** 05C50, 90B12, 90C60, 65K15.

## 1. Introduction

The most widely used and most written about method for solving the assignment problem is the "Hungarian Method". Originally suggested by Kuhn [7] in 1955 and it has appeared in many variants (e.g., Flood [5], Ford and Fulkerson [6], Kuhn [7], Kuhn [8]). The original version of the assignment problem is discussed in almost every textbook for an introductory course in either management science/operations research or production and operations management. As usually described, the problem is to find a one-to-one matching between $n$ tasks and $n$ agents, the objective being to minimize the total cost of the assignments. Classic examples involve such situations as assigning jobs to machines, jobs to workers or workers to machines.

"The best person for the job" is an apt description of the assignment problem. The situation can be illustrated by the assignment of workers with varying degrees of skill to jobs. A job that happens to match a worker's skill cost less than that in which the operator is not a skillful. The objective of the model is to determine the optimum (least cost) assignment of workers to jobs. The general assignment model with $n$ workers and $n$ jobs is represented in the Table 1.1.

| Worker↓ \ Job→ | $B_1$ | $B_2$ | $\ldots$ | $B_n$ | $\sum_{j=1}^{n} a_{ij} \downarrow$ |
|---|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | $\ldots$ | $a_{1n}$ | 1 |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\ldots$ | $a_{2n}$ | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| $A_n$ | $a_{n1}$ | $a_{n2}$ | $\ldots$ | $a_{nn}$ | 1 |
| $\sum_{i=1}^{n} a_{ij} \rightarrow$ | 1 | 1 | $\ldots$ | 1 | |

Cost Matrix: Table 1.1

The element $a_{ij}$ represents the cost of assigning worker $i$ to job $j$ $(i, j = 1, 2, \cdots, n)$. Classical (linear) assignment problem $(AP)$ is the following: For a given $n \times n$ real matrix $A$; find $n$ entries of $A$, no two belonging to the same row or column, so that their sum is optimal. This problem has numerous applications and belongs to basic combinatorial optimization problems. It has been studied by many authors since the 1950's. A thorough overview of the achievements can be found in the papers studied by Burkard and Cęla [2], Bukard [3].

However, although the basic version of the $AP$ can be solved very efficiently (say by the Hungarian method in $O(n^3)$ steps [9]), there are variants of this problem which are much harder, some being $NP$-complete or with undecided computational complexity. One of them is the parity $AP$: Obviously, $n$ entries of an $n \times n$ matrix, no two belonging to the same row or column, correspond to a permutation of the set $N = \{1, 2, ..., n\}$. In the classical $AP$, no additional conditions are set on the optimal permutation. In the parity $AP$, this permutation has to be of a prescribed parity.

In this paper, we solve the linear minimized assignment problem (or linear maximized assignment problem) with a new concept **DAS**-technique associated with a decision matrix and the corresponding dominated column (or dominating column) for the undecided computational complexity.

Firstly we discuss about the minimized assignment problem.

1.1. **MINIMIZED ASSIGNMENT PROBLEM.** The assignment problem in which $n$ workers are assigned to $n$ jobs can be represented as an $LP$ model in the following manner: Let $a_{ij}$ be the cost of assigning worker $i$ to job $j$ $(i, j = 1, 2, \cdots, n)$ which are given in the Table 1.2.

| Worker↓ \ Job→ | $B_1$ | $B_2$ | $\ldots$ | $B_n$ |
|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | $\ldots$ | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\ldots$ | $a_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $A_n$ | $a_{n1}$ | $a_{n2}$ | $\ldots$ | $a_{nn}$ |

Table 1.2

and define

$$x_{ij} = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ job is assigned to } j^{\text{th}} \text{ person;} \\ 0 & \text{otherwise.} \end{cases}$$

Then, the $LP$ model for assignment problem is defined by

$$\text{minimize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = 1,$$

$$\sum_{i=1}^{n} x_{ij} = 1,$$

$$i, j = 1, 2, \cdots, n \quad \text{where} \quad x_{ij} = 0 \,\text{or}\, 1.$$

It is to be noted that if the additional requirement is that the permutation is cyclic, then the arising task is the well-known ($NP$-complete) traveling salesman problem. Again a diagonally dominant matrix can be transformed to a normal form by adding constants to the rows and/or columns and no permutations of the rows or columns are needed. These constants can be found in a straightforward way, without using the Hungarian method or other method for solving the $AP$ [4]. To analyze it, we recall the definition the Monge matrix which is defined as follows.

**Definition 1.1.1.** [1] A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ the set of square matrix of order $n$, is said to be Monge if $a_{ij} + a_{kl} \geq a_{il} + a_{kj}$ for every $i, j, k$ and $l$ such that $1 \leq i \leq k \leq n$ and $1 \leq j \leq l \leq n$.

According to Burkard, Klinz and Rudolf [1], every Monge matrix is diagonally dominant. It is easy to see that adding constants to the rows and columns does not change the Monge property. For our need, we recall the Hungarian method.

1.2. **HUNGARIAN METHOD.** Earlier, the Hungarian method was used as the most useful tool to solve the assignment problems as classical method that was designed primarily for hand computation. The optimal solution of the preceding $LP$ model remains unchanged if a constant is added to or subtracted from any row or any column of the cost matrix $(a_{ij})$. To prove this point, let $p_i$ and $q_j$ be constants subtracted from row $i$ and column $j$. Thus the cost elements $a_{ij}$ is changed to

$$a'_{ij} = a_{ij} - p_i - q_j.$$

Now we have

$$\begin{aligned} \sum_i \sum_j a'_{ij}\, x_{ij} &= \sum_i \sum_j a_{ij}\, x_{ij} - \sum_i p_i - \sum_j p_j \\ &= \sum_i \sum_j a_{ij}\, x_{ij} - \text{constant}. \end{aligned}$$

Thus the new objective function differs from the original by a constant.

The steps of the Hungarian method [10] to solve the minimized assignment problem is given as follows.

Step 1. For the original cost matrix, identify each row's minimum, and subtract all the entries from all the entries of the row.

Step 2. For the matrix resulting from step 1, identify each column's minimum, and subtract it from all the entries of the column.

Step 3. Identify the optimal solution as the feasible assignment associated with the zero elements of the matrix obtained in step 2.

Step 4. If no feasible assignment (with all zero entries) can be secured from step 1 and 2, then

(*i*) Draw the minimum number of horizontal and vertical lines in the last reduced matrix that will cover all the zero entries.

(*ii*) Select the smallest uncovered element, subtract it from every uncovered element, and then add it to every element at the intersection of two lines.

(*iii*) If no feasible assignment can be found among the resulting zero entries, repeat step 4. Otherwise, go to step 3 to determine the optimal assignment.

Let the cost matrix is to be obtained by step 4(*ii*) of the Hungarian method. The dilemma is that which zero is to be encounted as an assignment of any particular row if there is more than one zeros.

To avoid the ambiguity of the Hungarian method, we introduce a concept as *dominated column* or *dominating column* to choose the assignments by simulation technique. Using this concept, we introduce **DAS**-technique (*Dominated Assignment Simulation Technique*) to solve the higher order assignment problems. In this technique, we use the concept *Minimin simulation criterion* to solve the minimized linear assignment problem. The *Maximax simulation criterion* is used for solving the maximized linear assignment problem. The Minimin simulation criterion means minimum of the minimum assignment sums and the Maximax simulation criterion means maximum of the maximum assignment sums. This technique provides the easy steps to choose the original assignments to solve the problem with hand computation.

## 2. SIMULATED ASSIGNMENT PROBLEM ($SAP$) AND ITS SIMULATED SUM

We observed that the optimal solution of the preceding $LP$ model remains unchanged if a constant is added to or subtracted from any row or any column of the cost matrix $(a_{ij})$. Then, the new objective function differs from the original by a constant. So, in the assignment problem, the problem is invariant under row interchange in the cost matrix. Selection of the first assignment is the key point of the assignment problem. To find the first assignment of the problem, if we shift the rows one by one and write the rows in a cyclic form then for each case, we shall get a new assignment problem whose objective function will be greater than or equal to (or less than or equal to) the original objective function of the minimized assignment problem (or maximized assignment problem). Thus, minimum (or maximum) of all the optimal solutions of the reformulated assignment problems is same as the optimal solution of the original assignment problem.

Suppose the linear assignment problem is to find the optimal solution (minimal or maximal solution). For the selection of first assignment from any row cyclicwise, we have $n$ number of ways. Suppose we want to select assignment from the first row, then shifting of any row to the first position out of $n$ rows of the $AP$ and the other rows are positioned in that in a cyclic form gives a new problem which is called simulated assignment problem ($SAP$) and the sum of assignments of the simulated assignment problems is denoted by simulated sum $(S_i)$. Here, we reformed the original $n \times n$ assignment problem is into $n$ different simulated assignment problems from which we can find the optimal simulated solution $(S)$ that equals to optimal solution of the assignment problems. The mathematical modeling of this concept can be represented in the following manner.

Let $r_i$, $i = 1, 2, \cdots, n$ be the $i^{\text{th}}$ row of the problem $AP$ placed on a circle ordered wise. Then, the formation of simulated assignment problems are defined as follows: The first simulated assignment problem is

$$SAP_1 = AP = [r_1, r_2, \cdots, r_{i-1}, r_i, r_{i+1}, \cdots, r_n]^T$$

and the $i^{\text{th}}$ simulated assignment problem is

$$\begin{aligned} SAP_i = AP \quad &= \quad [r_i, r_{i+1}, \cdots, r_n, r_1, r_2, \cdots, r_{i-1}]^T, \\ & \qquad i = 2, 3, \cdots, n, \end{aligned}$$

where $i^{\text{th}}$ row of the $AP$ is the first row of the $SAP_i$ and the other $(n-1)$ rows are placed in a cyclic form. Thus, the $n$ simulated assignment problems are listed as follows.

$$\begin{aligned} SAP_1 &= [r_1, r_2, \cdots, r_{i-1}, r_i, r_{i+1}, \cdots, r_n]^T \\ SAP_2 &= [r_2, r_3 \cdots, r_{i-1}, r_i, r_{i+1}, \cdots, r_n, r_1]^T \\ SAP_3 &= [r_3, r_4 \cdots, r_{i-1}, r_i, r_{i+1}, \cdots, r_n, r_1, r_2]^T \\ &\vdots \\ SAP_n &= [r_n, r_1, r_2, \cdots, r_{i-1}, r_i, r_{i+1}, \cdots, r_{n-1}]^T \end{aligned}$$

The optimal sum of the $SAP_i$ is called the $i^{\text{th}}$ simulated sum and is denoted by $S_i$ which is the sum of all the assignments of the problem $SAP_i$. Since for each $SAP_i$, there is a new simulated sum $S_i$, so the optimal solution of the given assignment problem $(AP)$ is the minimum of all the simulated sums for the minimized assignment problem. This technique gives the upper (or lower) limit of the minimized assignment problem (maximized assignment problem).

2.1. **Offtrace of Matrix, Decision Matrix and its application.** Let $\mathbb{R}^{n\times n}$ be the set of square matrix of order $n$. For our need we make the following definitions.

**Definition 2.1.1.** Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. Then the other diagonals are so-called the *off-diagonals*, So the element $a_{ij}$ is the *off-diagonal* element of $A$ if

$$i + j = n + 1$$

and *offtrace* of $A$ is the sum of off-diagonals of $A$, denote by offtr$(A)$.

**Definition 2.1.2.** The *decision cost matrix* is a matrix of order 2 given by

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

if it is obtained from any of the simulated assignment problems of the given assignment problem of **DAS**-technique. In this matrix the diagonals are $d_{11}$, $d_{22}$ and off-diagonals are $d_{21}$, $d_{12}$

2.2. **Rule of finding the assignments from the Decision Matrix:** The diagonals and off-diagonals of the decision matrix are playing the most important role to find the last two simulated assignments.

Let $\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$ be a given decision cost matrix. Then trace and offtrace of $D$ are

$$\text{tr}(D) = \text{trace of } D = d_{11} + d_{22}$$

and

$$\text{offtr}(D) = \text{offtrace of } D = d_{12} + d_{21}$$

respectively. Here we can obtain the last two assignments from the decision matrix for the cost matrix of minimized (or maximized) assignment problem.

The rules to get the assignments of the decision matrix are given as follows.

$(a)$ For the minimized assignment problem, if

$$\text{tr}(D) \leq \text{offtr}(D)$$

then assignments of $D$ are the diagonal elements, otherwise off-diagonal elements, that is, if

$$d_{11} + d_{22} \leq d_{12} + d_{21},$$

then $d_{11}$ and $d_{22}$ are the assignments of $D$, otherwise $d_{12}$ and $d_{21}$.

$(b)$ For the maximized assignment problem, if

$$\mathrm{tr}(D) \geq \mathrm{offtr}(D)$$

then assignments of $D$ are the diagonal elements, otherwise off-diagonal elements, that is, if

$$d_{11} + d_{22} \geq d_{12} + d_{21},$$

then $d_{11}$ and $d_{22}$ are the assignments of $D$, otherwise $d_{12}$ and $d_{21}$.

**Example 2.2.1.** Let us take the decision cost matrix as $\begin{bmatrix} 10 & 8 \\ 18 & 12 \end{bmatrix}$. Here we have

$$d_{11} + d_{22} = 22 \quad \text{and} \quad d_{21} + d_{12} = 26.$$

Since $d_{11} + d_{22} < d_{12} + d_{21}$, the assignments for the minimized assignment problem obtained by $(a)$ are 10 and 12, and the optimal solution is 22. But the assignments for the maximized assignment problem obtained by $(b)$ are 8 and 18, and the optimal solution is 26.

**Remark 2.2.2.** Let the decision cost matrix of a minimized assignment problem is

$$D = \begin{bmatrix} 10 & 8 \\ 18 & 12 \end{bmatrix}$$

given in Example 2.2.1. Using step 1 and step 2 of the Hungarian method, the changing matrix obtained is

$$\widetilde{D} = \begin{bmatrix} \tilde{d}_{11} & \tilde{d}_{12} \\ \tilde{d}_{21} & \tilde{d}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & 0 \\ 4 & \mathbf{0} \end{bmatrix}$$

which gives the allocation for the assignments are the diagonals. Therefore the optimal solution is $10 + 12 = 22$.

2.3. **Steps to find decision cost matrix.** Let the first simulated assignment problem ($SAP_1$), that is, the original $AP$ be given. We choose the minimum (maximum) element of the row to select as an assignment of that row if the $AP$ is minimized (maximized). To find the decision matrix from the simulated assignment problem, we follow the steps accordingly:

(1) Select the first assignment from the first row and cover its corresponding row and column.

(2) Select the second assignment from the second row and cover its corresponding row and column, continue this process upto $(n-2)^{\text{th}}$ row.

(3) If $(n-2)$ numbers of assignments are selected, then we have the decision matrix whose entities are the uncovered elements of the last two rows (that is, $(n-1)^{\text{th}}$ row and $n^{th}$ row) from which we find the last two assignments of $SAP_1$. Use this process to find the decision matrix of the problems $SAP_2, SAP_2, \cdots, SAP_n$.

## 3. Main Procedures of DAS-Technique

The main procedure of **DAS**-technique is based on three important steps which are given below.

(1) Depending on the classification of the objective function (i.e. minimization or maximization) of the $AP$, select $(n-2)$ numbers of assignments from the first $(n-2)$ rows of the simulated assignment problems and then find the last two assignments from their corresponding decision matrices.

(2) Find $n$ numbers of simulated sums of the $n$ numbers of simulated assignment problems.

(3) Find of the optimal solution as follows.

$(a)$ For minimized assignment problem, the optimal solution is the minimum of all simulated sums.

($b$) For maximized assignment problem, the optimal solution is the maximum of all simulated sums.

3.1. **Steps for the minimized *SAP*.** We have given the procedures to find the first simulated sum ($S_1$) of the $SAP_1$ from the cost matrix of original minimized assignments problem (i.e., loss matrix) are given below. The cost matrix of $SAP_1$ is given by

| | $B_1$ | $B_2$ | $\dots$ | $B_n$ |
|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | $\dots$ | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\dots$ | $a_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $A_n$ | $a_{n1}$ | $a_{n2}$ | $\dots$ | $a_{nn}$ |

Table 3.1

The steps are given as follows:

(1) Select the minimum element of the first row as $a_{1k}$, $1 \leq k \leq n$ to get the first assignment. Draw the horizontal line for first row and vertical line for $k^{\text{th}}$ column to cover $a_{1k}$.

(2) Select the minimum element of the second row as $a_{2j}$, $1 \leq j \leq n, j \neq k$ to get the second assignment. Draw the horizontal line and vertical line to cover $a_{2j}$.

(3) If there is tie case to select the assignment from a particular row, then find the dominated column. To find the dominated column, follow the steps.

($i$) Let the tie case arises in $i^{\text{th}}$ row, then find the maximum element from the corresponding column elements of $(i+1)^{\text{th}}$ row. The column which contains the unique maximum element from $(i+1)^{\text{th}}$ row that corresponds to tie elements of the $i^{\text{th}}$ row is *dominated column*, otherwise go to step ($ii$). If the dominated column is $l^{th}$ column, then the assignment for $i^{th}$ row is $a_{il}$. Cover the corresponding row and column of $a_{il}$.

($ii$) The column which contains the unique maximum element from $(i+2)^{\text{th}}$ row that corresponds to tie elements of the $(i+1)^{\text{th}}$ row is dominated column, and so on. Continue this process to get the dominated column, then cover the row and columns of the assignment for $i^{th}$ row.

(4) Cover the rows and columns of the selected assignments from top to bottom.

(5) Using step 1 to step 4, find $(n-2)$ numbers of assignments from the first $(n-2)$ rows (top to bottom) of the simulated assignment problem. Finally find the last two assignments from its corresponding decision matrix whose elements are uncovered elements of last two rows.

(6) Find the first simulated sum $S_1$ where $S_1$ equals to summation of all selected assignments of the $SAP_1$ obtained using the step 1 to step 5.

(7) Using the process from steps 1 to 6, find all the simulated sums $S_1, S_2, S_3, \cdots, S_n$ of the simulated assignment problems $SAP_1$, $SAP_2$, $SAP_3$, $\cdots, SAP_n$ respectively.

(8) The approximate optimal solution of $AP$ is $S$ equals to minimum of all the simulated sums $S_1, S_2, S_3, \cdots, S_n$, that is,

$$S = \min_{1 \le i \le n} S_i = S_k \quad \text{(say)}.$$

(9) Allocate the feasible assignments corresponds to $S_k$.

(10) For tie case of simulated sums, i.e.,

$$S = \min_{1 \le i \le n} S_i = \{S_{k_1}, S_{k_2}, \cdots, S_{k_m}\}$$

where $k_1 < k_2 < \cdots < k_m$, take the first indexed simulated sum $S_{k_1}$ by index priority concept and allocate the assignments corresponds to $S_{k_1}$.

**Note 3.1.1.** If the approximate optimal solution $S$ is equal to original optimal solution, then the corresponding assignments of the $SAP$ are exact.

3.2. **Examples of Minimized Assignment Problem.** We have studied some concrete examples to establish the concept of **DAS**-technique for minimized assignment problems.

**Example 3.2.1.** *Let the cost matrix of the minimized assignment problem be given by:*

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | *13* | *8* | *16* | *18* | *19* |
| $A_2$ | *9* | *15* | *24* | *9* | *12* |
| $A_3$ | *12* | *9* | *4* | *4* | *4* |
| $A_4$ | *6* | *12* | *10* | *8* | *13* |
| $A_5$ | *15* | *17* | *18* | *12* | *20* |

*Table 3.2*

Since the problem is minimized assignment problem, so we solve the problem using Minimin criterion of the **DAS**-technique which means minimum of the simulated sums. The first simulated assignment problem is the assignment problem given in Table 3.2. In the first row, $a_{12} = 8$ is the unique minimum element. So it is selected as first assignment for first row. Shading the first row and second column, we get, the next cost matrix is

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | 9 | 15 | 24 | 9 | 12 |
| $A_3$ | 12 | 9 | 4 | 4 | 4 |
| $A_4$ | 6 | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | 12 | 20 |

Table 3.3

In the second row, the minimum element is 9 and $a_{21} = a_{24} = 9$, i.e., we have a tie case in first and fourth column. To find the dominated column, we search the unique maximum in third row. Since in the third row, $a_{31} > a_{34}$, therefore first column ($B_1$) is dominated column. Thus the assignment for the second row is $a_{21} = 9$. Covering the second row and first column in Table 3.3, we get, the next cost matrix as

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | **9** | 15 | 24 | 9 | 12 |
| $A_3$ | 12 | 9 | 4 | 4 | 4 |
| $A_4$ | 6 | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | 12 | 20 |

Table 3.4

In the third row, there is a tie case in third, fourth and fifth columns for selecting the minimum element, i.e., $a_{33} = a_{34} = a_{35} = 4$. To find the dominated column, we see the next row, that is, fourth row. Since in the fourth row, $a_{45} = 13$ is the unique maximum element among the uncovered elements, so fifth column is the dominated column. Therefore, $a_{35} = 4$ is selected as third assignment for third row. Covering the third row and fifth column, we get, the next cost matrix as

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | **9** | 15 | 24 | 9 | 12 |
| $A_3$ | 12 | 9 | 4 | 4 | **4** |
| $A_4$ | 6 | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | 12 | 20 |

Table 3.5

Since the cost matrix of the $AP$ is of order 5 and we have selected 3 numbers of assignments from the first three rows of the matrix, so the last two assignments will be obtained from the decision matrix $D_1$ whose elements are the uncovered elements fourth row and fifth row. The decision cost matrix $D_1$ for

$SAP_1$ is

$$D_1 = \begin{bmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 18 & 12 \end{bmatrix}$$

which gives the assignments $a_{43} = 10$ and $a_{54} = 12$, since

$$\mathrm{tr}(D_1) < \mathrm{offtr}(D_1).$$

Hence the first simulated sum is

$$S_1 = a_{12} + a_{21} + a_{35} + (a_{43} + a_{54}) = 8 + 9 + 4 + (10 + 12) = 43$$

where the bracket is given because of decision matrix assignments.

In $SAP_2$, for first three rows the assignments selected in Table 3.6

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_2$ | **9** | 15 | 24 | 9 | 12 |
| $A_3$ | 12 | 9 | 4 | 4 | **4** |
| $A_4$ | 6 | 12 | 10 | **8** | 13 |
| $A_5$ | 15 | 17 | 18 | 12 | 20 |
| $A_1$ | 13 | 8 | 16 | 18 | 19 |

Table 3.6

are $a_{21} = 9$, $a_{35} = 4$ and $a_{42} = 8$. Last two assignments obtained from the decision matrix

$$D_2 = \begin{bmatrix} a_{52} & a_{53} \\ a_{12} & a_{13} \end{bmatrix} = \begin{bmatrix} 17 & 18 \\ 8 & 16 \end{bmatrix}$$

are $a_{53} = 18$ and $a_{12} = 8$ since

$$\mathrm{offtr}(D_2) < \mathrm{tr}(D_2).$$

Hence the second simulated sum is

$$S_2 = a_{21} + a_{35} + a_{44} + (a_{53} + a_{12}) = 9 + 4 + 8 + (18 + 8) = 47.$$

In $SAP_3$, for first three rows the assignments selected in Table 3.7
are $a_{35} = 4$, $a_{41} = 6$ and $a_{54} = 12$. Last two assignments obtained from the decision matrix

$$D_3 = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 15 & 24 \end{bmatrix}$$

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_3$ | 12 | 9 | 4 | 4 | **4** |
| $A_4$ | **6** | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | **12** | 20 |
| $A_1$ | 13 | 8 | 16 | 18 | 19 |
| $A_2$ | 9 | 15 | 24 | 9 | 12 |

Table 3.7

are $a_{13} = 16$ and $a_{22} = 15$, since

$$\text{offtr}(D_3) < \text{tr}(D_3).$$

Hence the third simulated sum is

$$S_3 = a_{35} + a_{41} + a_{54} + (a_{13} + a_{22}) = 4 + 6 + 12 + (16 + 15) = 53.$$

In $SAP_4$, for first three rows the assignments selected in Table 3.8

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_4$ | **6** | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | **12** | 20 |
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | 9 | 15 | 24 | 9 | 12 |
| $A_3$ | 12 | 9 | 4 | 4 | 4 |

Table 3.8

are $a_{41} = 6$, $a_{54} = 12$ and $a_{12} = 8$. Last two assignments obtained from the decision matrix

$$D_4 = \begin{bmatrix} a_{23} & a_{25} \\ a_{33} & a_{35} \end{bmatrix} = \begin{bmatrix} 24 & 12 \\ 4 & 4 \end{bmatrix}$$

are $a_{25} = 12$ and $a_{33} = 4$ since

$$\text{offtr}(D_4) < \text{tr}(D_4).$$

Hence the fourth simulated sum is

$$S_4 = a_{41} + a_{54} + a_{12} + (a_{25} + a_{33}) = 6 + 12 + 8 + (12 + 4) = 42.$$

In $SAP_5$, for first three rows the assignments selected in Table 3.9

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_5$ | 15 | 17 | 18 | **12** | 20 |
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | **9** | 15 | 24 | 9 | 12 |
| $A_3$ | 12 | 9 | 4 | 4 | 4 |
| $A_4$ | 6 | 12 | 10 | 8 | 13 |

Table 3.9

are $a_{54} = 12$, $a_{12} = 8$, and $a_{21} = 9$. Last two assignments obtained from the decision matrix

$$D_5 = \begin{bmatrix} a_{33} & a_{35} \\ a_{43} & a_{45} \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 13 \end{bmatrix}$$

are $a_{35} = 4$ and $a_{43} = 4$ since

$$\text{offtr}(D_5) < \text{tr}(D_5).$$

Hence the fifth simulated sum is

$$S_5 = a_{54} + a_{12} + a_{21} + (a_{35} + a_{43}) = 12 + 8 + 9 + (4 + 10) = 43.$$

Thus, the optimal simulated sum is

$$S = \min\{S_1, S_2, S_3, S_4, S_5\} = \min\{43, 47, 53, 42, 43\} = 42 = S_4.$$

Hence, $SAP_4$ gives allocation of the assignments as follows:

$$A_4 \to B_1; A_5 \to B_4; A_1 \to B_2; A_2 \to B_5 \quad \text{and} \quad A_3 \to B_3.$$

**Remark 3.2.2.** Using Hungarian method, we get the final table of the minimized assignment problem given in Example 3.2.1 is

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 8 | **0** | 8 | 13 | 11 |
| $A_2$ | 0 | 3 | 12 | 0 | **0** |
| $A_3$ | 11 | 5 | **0** | 3 | 0 |
| $A_4$ | **0** | 3 | 1 | 2 | 4 |
| $A_5$ | 3 | 2 | 3 | **0** | 5 |

Table 3.10

where the allocation of the assignments are

$$A_1 \to B_2; A_2 \to B_5; A_3 \to B_3; A_4 \to B_1 \quad \text{and} \quad A_5 \to B_4.$$

The optimal solution is

$$a_{12} + a_{25} + a_{33} + a_{41} + a_{51} = 8 + 12 + 4 + 6 + 12 = 42$$

which equals to $S = S_4$ obtained by **DAS**-technique.

We have studied some other problems to give the clear understanding about the **DAS**-technique. The following example shows the existence of dominated column.

**Example 3.2.3.** *Let the cost matrix of the minimized assignment problem be given by :*

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | *13* | *8* | *16* | *18* | *19* |
| $A_2$ | *9* | *15* | *15* | *9* | *9* |
| $A_3$ | *6* | *9* | *4* | *6* | *4* |
| $A_4$ | *7* | *12* | *10* | *8* | *13* |
| $A_5$ | *15* | *17* | *18* | *12* | *20* |

Table 3.11

For $SAP_1$, the minimum element $a_{12} = 8$ is the first assignment for first row.

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | 9 | 15 | 15 | 9 | 9 |
| $A_3$ | 6 | 9 | 4 | 6 | 4 |
| $A_4$ | 7 | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | 12 | 20 |

Table 3.12

In second row, the minimum element is 9 but it is the tie case in first column, fourth column and fifth column, i.e.,

$$a_{21} = a_{24} = a_{25} = 9.$$

So we have to find the dominated column. To break the tie case, we first see the third row. Since in third row, the largest element is 6 but there also we find the tie case lie in second and fourth columns, as

$$a_{32} = a_{34} = 6.$$

Again to break the tie case, we see the fourth row. Since the fourth row contains the unique maximum element 8 lies in fourth column, so fourth column is the dominated column for the selection of the assignment in second row. Thus for second row the assignment is $a_{24} = 9$. Covering $a_{24}$, we get

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | 9 | 15 | 15 | **9** | 9 |
| $A_3$ | 6 | 9 | 4 | 6 | 4 |
| $A_4$ | 7 | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | 12 | 20 |

Table 3.13

In third row, the minimum element is 4 that lies in third and fifth columns, i.e., $a_{33} = a_{35} = 4$. To break the tie case, we observe the maximum element in the next row, i.e., fourth row which contains unique maximum element 13 that lies in fifth column, so is the dominated column. Thus for third row, the assignment is $a_{35} = 4$. Covering $a_{35}$, we get the next cost matrix

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | 9 | 15 | 15 | **9** | 9 |
| $A_3$ | 6 | 9 | 4 | 6 | **4** |
| $A_4$ | 7 | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | 12 | 20 |

Table 3.14

For last two assignments of $SAP_1$, we get the first decision cost matrix is

$$D_1 = \begin{bmatrix} a_{41} & a_{43} \\ a_{51} & a_{53} \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 18 \end{bmatrix}.$$

Since

$$\text{tr}(D_1) = \text{offtr}(D_1) = 25,$$

we get the assignments for fourth and fifth rows are $a_{41} = 7$ and $a_{53} = 18$ respectively, obtained using the index priority concept (since in $SAP_1$, fourth row comes first if we observe from top to bottom). Hence the first simulated sum $S_1$ is

$$S_1 = a_{12} + a_{24} + a_{35} + (a_{41} + a_{53}) = 8 + 9 + 4 + (7 + 18) = 46.$$

Using the concept of dominated column, we get the assignments of the other simulated assignment problems, i.e., $SAP_2$, $SAP_3$, $SAP_4$ and $SAP_5$ given in the following tables, i.e., Table 3.15, Table 3.16, Table 3.17 and Table 3.18 respectively.

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_2$ | 9 | 15 | 15 | **9** | 9 |
| $A_3$ | 6 | 9 | 4 | 6 | **4** |
| $A_4$ | **7** | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | **18** | 12 | 20 |
| $A_1$ | 13 | **8** | 16 | 18 | 19 |

$SAP_2$: Table 3.15

The second simulated sum is

$$S_2 = a_{24} + a_{35} + a_{41} + (a_{53} + a_{12}) = 9 + 4 + 7 + (18 + 8) = 46.$$

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_3$ | 6 | 9 | 4 | 6 | **4** |
| $A_4$ | **7** | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | **12** | 20 |
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | 9 | 15 | **15** | 9 | 9 |

$SAP_3$: Table 3.16

The third simulated assignment sum is

$$S_3 = a_{35} + a_{41} + a_{54} + (a_{12} + a_{23}) = 4 + 7 + 12 + (8 + 15) = 46.$$

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_4$ | **7** | 12 | 10 | 8 | 13 |
| $A_5$ | 15 | 17 | 18 | **12** | 20 |
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | 9 | 15 | 15 | 9 | **9** |
| $A_3$ | 6 | 9 | **4** | 6 | 4 |

$SAP_4$: Table 3.17

The fourth simulated assignment sum is

$$S_4 = a_{41} + a_{54} + a_{12} + (a_{25} + a_{33}) = 7 + 12 + 8 + (9 + 4) = 40$$

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_5$ | 15 | 17 | 18 | **12** | 20 |
| $A_1$ | 13 | **8** | 16 | 18 | 19 |
| $A_2$ | **9** | 15 | 15 | 9 | 9 |
| $A_3$ | 6 | 9 | 4 | 6 | **4** |
| $A_4$ | 7 | 12 | **10** | 8 | 13 |

$SAP_5$: Table 3.18

and the fifth simulated assignment sum is

$$S_5 = a_{54} + a_{12} + a_{21} + (a_{35} + a_{43}) = 12 + 8 + 9 + (4 + 10) = 43.$$

Thus, the optimal solution is

$$S = \min\{S_1, S_2, S_3, S_4, S_5\} = \min\{46, 46, 46, 40, 43\} = 40 = S_4.$$

Thus the optimal solution is $S = S_4 = 40$ and the allocation of the assignments are

$$A_4 \to B_1; A_5 \to B_4; A_1 \to B_2; A_2 \to B_5 \quad \text{and} \quad A_3 \to B_3.$$

The following example shows the existence of tie case of simulated sums.

**Example 3.2.4.** *The cost matrix of the minimized assignment problem is given by :*

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_1$ | *1* | *4* | *6* | *3* |
| $A_2$ | *9* | *7* | *10* | *9* |
| $A_3$ | *4* | *5* | *11* | *7* |
| $A_4$ | *8* | *7* | *8* | *5* |

Table 3.19

The selected assignments of $SAP_1$ are given in Table 20:

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_1$ | **1** | 4 | 6 | 3 |
| $A_2$ | 9 | **7** | 10 | 9 |
| $A_3$ | 4 | 5 | 11 | **7** |
| $A_4$ | 8 | 7 | **8** | 5 |

$SAP_1$: Table 3.20

The first simulated assignment sum is

$$S_1 = a_{11} + a_{22} + (a_{34} + a_{43}) = 1 + 7 + (7 + 8) = 23.$$

The selected assignments of $SAP_2$ are given in Table 3.21:

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_2$ | 9 | **7** | 10 | 9 |
| $A_3$ | **4** | 5 | 11 | 7 |
| $A_4$ | 8 | 7 | **8** | 5 |
| $A_1$ | 1 | 4 | 6 | **3** |

$SAP_2$: Table 3.21

The second simulated assignment sum is

$$S_2 = a_{22} + a_{31} + (a_{43} + a_{12}) = 7 + 4 + (8 + 3) = 22.$$

The selected assignments of $SAP_3$ are given in Table 3.22:

The third simulated assignment sum is

$$S_3 = a_{31} + a_{44} + (a_{13} + a_{22}) = 4 + 5 + (6 + 7) = 21.$$

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_3$ | **4** | 5 | 11 | 7 |
| $A_4$ | 8 | 7 | 8 | **5** |
| $A_1$ | 1 | 4 | **6** | 3 |
| $A_2$ | 9 | **7** | 10 | 9 |

$SAP_3$: Table 3.22

The selected assignments of $SAP_4$ are given in Table 3.23:

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_4$ | 8 | 7 | 8 | **5** |
| $A_1$ | **1** | 4 | 6 | 3 |
| $A_2$ | 9 | 7 | **10** | 9 |
| $A_3$ | 4 | **5** | 11 | 7 |

$SAP_4$: Table 3.23

The fourth simulated assignment sum is

$$S_4 = a_{44} + a_{11} + (a_{23} + a_{32}) = 5 + 1 + (10 + 5) = 21.$$

Thus the optimal solution is

$$S = \min\{S_1, S_2, S_3, S_4\} = \min\{23, 22, 21, 21\} = 21 = \{S_3, S_4\}.$$

By the concept of index priority, we have the optimal solution is

$$S = S_3 = 21.$$

Hence the allocation of the assignments of the $AP$ are

$$A_3 \to B_1; A_4 \to B_4; A_1 \to B_3 \quad \text{and} \quad A_2 \to B_2.$$

**Note 3.2.5.** For the minimized assignment problem (loss matrix), if the sum of row minimums of the $AP$ is equal to sum of the simulated assignments ($S_k$), then the assignments of ($SAP_k$) are the exact assignments of the problem.

3.3. **Algorithm for Minimized Assignment Problem.** In the following algorithm, the computational algorithm of minimized assignment problem is given as follows:

**Algorithm 3.3.1.**

(1) Set initial optimal solution is $S = \bar{S}$ where $\bar{S}$ is the sum of row minimums of the $AP$.

(2) Find first group of assignments from $SAP_1$ and compute their simulated sum $S_1$. If $\bar{S} = S_1$, then $S = S_1$ and assignments of $AP$ are the assignments of $SAP_1$, otherwise go to next step.

(3) Find second group of assignments from $SAP_2$ and compute their simulated sum $S_2$. If $\bar{S} = S_2$, then $S = S_2$ and assignments of $AP$ are the assignments of $SAP_2$, otherwise go to next step.

(4) If $\bar{S} \neq S_i$ for any $i$, then continuing the above process, find set of all the simulated sums $\{S_i : i \in I\}$ where $I = \{1, 2, \cdots, n\}$ is the indexed set.

(5) Finally find $S = \min\{S_i : i \in I\} = S_k$ where $k$ is index priority element and hence write the allocations of the assignments from $SAP_k$.

## 4. DAS-technique for Maximized Assignment Problem

In this section, we represent the objective is to maximize the profit. To solve this, we represent **DAS**-techniques.

4.1. **Method-I.** In maximized assignment problem (i.e., profit matrix), the following steps are to be used.

($i$) Convert to minimized assignment problem (i.e., loss matrix) by subtracting all the elements from the maximum element of the given maximized assignment problem and find the assignments by DAS-Technique.

($ii$) Find the position of the assignments in the simulated assignment problem having the optimal simulated sum $S$.

($iii$) Select the actual assignments in the maximized assignment problem.

4.2. **Method-II.** Since the minimized assignment problem is the dual problem of maximized assignment problem, so taking dual relations in the **DAS**-technique, the problem can be solved, that is, replace the term minimum by maximum, Minimin by Maximax which means maximum of the maximums, and for decision matrix, the relation "$\leq$" will be replaced by "$\geq$". The dominated column concept will be changed to dominating column concept.

We have given the procedures to find the first simulated sum ($S_1$) of the $SAP_1$ from the cost matrix of original maximized assignments problem (i.e., profit matrix) are given below. The cost matrix of $SAP_1$ is given by

| | $B_1$ | $B_2$ | $\ldots$ | $B_n$ |
|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | $\ldots$ | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\ldots$ | $a_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $A_n$ | $a_{n1}$ | $a_{n2}$ | $\ldots$ | $a_{nn}$ |

Table 4.1

The steps are given as follows.

(1) Select the maximum element of the first row as $a_{1k}$, $1 \leq k \leq n$ to get the first feasible assignment, then draw the horizontal and vertical lines to cover the first row and $k^{\text{th}}$ column.

(2) Select the maximum element of the second row as $a_{2j}$, $1 \leq j \leq n, j \leq k$ to get the second feasible assignment, then cover the horizontal and vertical lines to cover the second row and $j^{\text{th}}$ column. Continue this process to find all the feasible assignments until getting of the decision matrix.

(3) For tie case, select the feasible assignment from a particular row using the dominating column concept. To find the dominating column, the steps are given as follows.

$(i)$ Let the tie case arises in $i^{\text{th}}$ row, then find the minimum element from the corresponding column elements of $(i+1)^{\text{th}}$ row. The column which contains the unique minimum element from $(i+1)^{\text{th}}$ row that corresponds to tie elements of the $i^{\text{th}}$ row is *dominated column*, otherwise go to step $(ii)$. If the dominating column is $l^{th}$ column, then the assignment for $i^{th}$ row is $a_{il}$. Cover the corresponding row and column of $a_{il}$.

$(ii)$ The column which contains the unique minimum element from $(i+2)^{\text{th}}$ row that corresponds to tie elements of the $(i+1)^{\text{th}}$ row is dominating column, and so on. Continue this process to get the dominated column, then cover the row and columns of the assignment for $i^{th}$ row.

(4) Cover the rows and columns of the selected assignments from top to bottom.

(5) Using step 1 to step 4, find $(n-2)$ numbers of assignments from the first $(n-2)$ rows (top to bottom) of the simulated assignment problem. Finally find the last two assignments from its corresponding decision matrix whose elements are uncovered elements of last two rows.

(6) Find the simulated sum, i.e., the summation of all feasible assignments of the $SAP$.

(7) Using the process from 1 to 5, find all the simulated sums, i.e., $S_1$, $S_2$, $S_3, \cdots, S_n$ of the simulated assignment problems $SAP_1$, $SAP_2$, $SAP_3$, $\cdots$, $SAP_n$ respectively, i.e.,

$$S_i = \text{Simulated sum of} \quad SAP_i, i = 1, 2, 3, \cdots, n.$$

(8) The optimal solution of the $AP$ is the maximum of all the simulated sums $S_1, S_2$, $S_3, \cdots, S_n$, i.e.,

$$S = \max_{1 \le i \le n} S_i = S_k \quad \text{(say)}.$$

(9) Allocate the feasible assignments corresponds to $S_k$.

(10) For tie case of simulated sums, i.e.,

$$S = \max_{1 \le i \le n} S_i = \{S_{k_1}, S_{k_2}, \cdots, S_{k_m}\}$$

where $k_1 < k_2 < \cdots < k_m$, take the first indexed simulated sum $S_{k_1}$ by index priority concept) and allocate the assignments corresponds to $S_{k_1}$.

4.3. **Examples of Maximized Assignment Problem.** The Example 4.3.1 showing that in the maximized assignment problem if sum of the row maximums is equal to sum of the simulated assignments ($S_k$) of the $k^{th}$ simulated problem ($SAP_k$), $1 \le k \le n$, then the assignments of the ($SAP_k$) are the assignments of the original $AP$. We find the optimal solution of the maximized assignment problem using the both methods of the **DAS**-technique and are equal.

**Example 4.3.1.** *Let the cost matrix of the maximized assignment problem be given by:*

| | *D* | *E* | *F* |
|---|---|---|---|
| *A* | *15* | *10* | *9* |
| *B* | *9* | *11* | *15* |
| *C* | *10* | *12* | *8* |

Table 4.2

Method-I. It is known that the maximized assignment problem can be coverted to minimized assignment problem by subtracting all the elements from the maximum elements of the $AP$. In the maximized assignment problem, the maximum element is 15. Subtracting all the elements from 15, we get the minimized assignment problem as

It is to be noted that *sum of row minimums* in Table 4.3 is

$$\bar{s} = a_{11} + (a_{23} + a_{32}) = 0 + (0 + 3) = 3.$$

| | D | E | F |
|---|---|---|---|
| A | 0 | 5 | 6 |
| B | 6 | 4 | 0 |
| C | 5 | 3 | 7 |

Table 4.3

The selected assignments of $SAP_1$ are given in Table 4.4:

| | D | E | F |
|---|---|---|---|
| A | **0** | 5 | 6 |
| B | 6 | 4 | **0** |
| C | 5 | **3** | 7 |

$SAP_1$: Table 4.4

The first simulated sum is

$$S_1 = a_{11} + (a_{23} + a_{32}) = 0 + (0 + 3) = 3.$$

The selected assignments of $SAP_2$ are given in Table 4.4:

| | D | E | F |
|---|---|---|---|
| B | 6 | 4 | **0** |
| C | 5 | **3** | 7 |
| A | **0** | 5 | 6 |

$SAP_2$: Table 4.5

The second simulated sum is

$$S_2 = a_{23} + (a_{32} + a_{11}) = 0 + (3 + 0) = 3.$$

The selected assignments of $SAP_3$ are given in Table 4.6:

| | D | E | F |
|---|---|---|---|
| C | 5 | **3** | 7 |
| A | **0** | 5 | 6 |
| B | 6 | 4 | **0** |

$SAP_3$: Table 4.6

The third simulated sum is

$$S_3 = a_{32} + (a_{11} + a_{23}) = 3 + (0 + 0) = 3.$$

Since $S_1 = S_2 = S_3 = 3$, by index priority concept optimal solution is $S_1$. Hence

$$A \to D; B \to F \quad \text{and} \quad C \to E.$$

The optimal solution of the maximized assignment problem is 15+15+12 = 42.
Method-II. This method solves the maximized assignment problem directly where the relations are dual of the relations used in the minimized assignment problem.
It is to be noted that *sum of row maximums* in Table 4.2 is

$$\bar{S} = a_{11} + (a_{23} + a_{32}) = 15 + (15 + 12) = 42.$$

The selected assignments (i.e., maximum element) of $SAP_1$ are given in Table 4.4:

| | D | E | F |
|---|---|---|---|
| A | **15** | 10 | 9 |
| B | 9 | 11 | **15** |
| C | 10 | **12** | 8 |

$SAP_1$: Table 4.7

where the first simulated sum is

$$S_1 = a_{11} + (a_{23} + a_{32}) = 15 + (15 + 12) = 42.$$

Since $S_1 = \bar{S}$, the optimal solution is $S_1$. So the allocations are

$$A \to D; B \to F \quad \text{and} \quad C \to E.$$

**Note 4.3.2.** From Example 4.3.1 we can get $S_2 = S_3 = 42$. Hence by the index priority concept we have the optimal solution is $S_1$.

The unbalanced assignment model with $m$ workers and $n$ jobs where $m \neq n$ is also known as rectangular assignment problem. The following example showing the existence of the optimal solution in rectangular assignment problem solved using **DAS**-technique.

**Example 4.3.3.** *The owner of a small machine shop has four mechanics available to each assign each jobs for the day. Five jobs are offered with expected profit for each mechanic on each job which are given as follows:* To

| Mechanics ↓ \ *Jobs* → | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
|---|---|---|---|---|---|
| $M_1$ | *62* | *78* | *50* | *101* | *82* |
| $M_2$ | *71* | *84* | *61* | *73* | *59* |
| $M_3$ | *87* | *92* | *111* | *71* | *81* |
| $M_4$ | *48* | *64* | *87* | *77* | *80* |

Table 4.8

find the assignment of the mechanics to the job such that the result is in maximum profit and to know the job declination, we introduce a dummy mechanic 5 with all elements 0, as follows.

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
|---|---|---|---|---|---|
| $M_1$ | 62 | 78 | 50 | 101 | 82 |
| $M_2$ | 71 | 84 | 61 | 73 | 59 |
| $M_3$ | 87 | 92 | 111 | 71 | 81 |
| $M_4$ | 48 | 64 | 87 | 77 | 80 |
| $M_5$ | 0 | 0 | 0 | 0 | 0 |

Table 4.9

Since the problem is maximized assignment problem, so we use the Maximax criterion. The first simulated assignment problem ($SAP_1$) is

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
|---|---|---|---|---|---|
| $M_1$ | 62 | 78 | 50 | **101** | 82 |
| $M_2$ | 71 | **84** | 61 | 73 | 59 |
| $M_3$ | 87 | 92 | **111** | 71 | 81 |
| $M_4$ | 48 | 64 | 87 | 77 | **80** |
| $M_5$ | **0** | 0 | 0 | 0 | 0 |

$SAP_1$: Table 4.10

has the first simulated sum $S_1$ is

$$S_1 = a_{14} + a_{22} + a_{33} + (a_{45} + a_{51}) = 101 + 84 + 111 + (80 + 0) = 376$$

where the elements given in the brackets are the assignments $a_{45} = 80$ and $a_{55} = 0$ obtained from first decision matrix

$$D_1 = \begin{bmatrix} a_{41} & a_{45} \\ a_{51} & a_{55} \end{bmatrix} = \begin{bmatrix} 48 & 80 \\ 0 & 0 \end{bmatrix}$$

since $\text{offtr}(D_1) > \text{tr}(D_1)$. The second simulated assignment problem ($SAP_2$) is

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
|---|---|---|---|---|---|
| $M_2$ | 71 | **84** | 61 | 73 | 59 |
| $M_3$ | 87 | 92 | **111** | 71 | 81 |
| $M_4$ | 48 | 64 | 87 | 77 | **80** |
| $M_5$ | **0** | 0 | 0 | 0 | 0 |
| $M_1$ | 62 | 78 | 50 | **101** | 82 |

$SAP_2$: Table 4.11

has the second simulated sum $S_2$ is

$$S_2 = a_{22} + a_{33} + a_{45} + (a_{51} + a_{14}) = 84 + 111 + 80 + (0 + 101) = 376$$

where the elements given in the brackets are the assignments $a_{51} = 0$ and $a_{14} = 101$ obtained from second decision matrix

$$D_2 = \begin{bmatrix} a_{51} & a_{54} \\ a_{11} & a_{14} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 62 & 101 \end{bmatrix}$$

since $\text{tr}(D_2) > \text{offtr}(D_2)$.

In the third simulated assignment problem ($SAP_3$)

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
|---|---|---|---|---|---|
| $M_3$ | 87 | 92 | **111** | 71 | 81 |
| $M_4$ | 48 | 64 | 87 | 77 | **80** |
| $M_5$ | **0** | 0 | 0 | 0 | 0 |
| $M_1$ | 62 | 78 | 50 | **101** | 82 |
| $M_2$ | 71 | **84** | 61 | 73 | 59 |

$SAP_3$: Table 4.12

the assignments for first row and second row are $a_{33} = 111$ and $a_{45} = 80$ respectively. In the third row all the elements are 0, so to find the dominating column. Here $J_1$ is the dominating column since the fourth row contains the unique minimum element $a_{41} = 62$. So the assignment for third row is $a_{51} = 0$. The third decision matrix is

$$D_3 = \begin{bmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{bmatrix} = \begin{bmatrix} 78 & 101 \\ 84 & 73 \end{bmatrix}.$$

Thus the last two assignments obtained from the decision matrix $D_3$ are $a_{14} = 101$ and $a_{22} = 84$ since $\text{offtr}(D_3) > \text{tr}(D_3)$. Hence the third simulated sum $S_3$ is

$$S_3 = a_{33} + a_{45} + a_{41} + (a_{14} + a_{22}) = 111 + 80 + 0 + (101 + 84) = 376.$$

In the fourth simulated assignment problem ($SAP_4$)

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
|---|---|---|---|---|---|
| $M_4$ | 48 | 64 | **87** | 77 | 80 |
| $M_5$ | **0** | 0 | 0 | 0 | 0 |
| $M_1$ | 62 | 78 | 50 | **101** | 82 |
| $M_2$ | 71 | **84** | 61 | 73 | 59 |
| $M_3$ | 87 | 92 | 111 | 71 | 81 |

$SAP_4$: Table 4.13

the assignments for first row is $a_{43} = 87$. In the second row, all the elements are 0, but in the fourth row $a_{41} = 62$ is the unique minimum element, so $J_1$ is the dominating column. Thus for second row $a_{51} = 0$ is the assignment. The assignment for third row is $a_{14} = 101$. Last two assignments obtained from the decision matrix

$$D_4 = \begin{bmatrix} a_{22} & a_{25} \\ a_{32} & a_{35} \end{bmatrix} = \begin{bmatrix} 84 & 59 \\ 92 & 81 \end{bmatrix}.$$

are $a_{22} = 84$ and $a_{35} = 81$ since $\text{tr}(D_4) > \text{offtr}(D_4)$. Hence the fourth simulated sum $S_4$ is

$$S_4 = a_{43} + a_{51} + a_{14} + (a_{22} + a_{35}) = 87 + 0 + 101 + (84 + 81) = 353$$

The fifth simulated assignment problem ($SAP_5$) is
In the first row, all the elements are 0, but in the second row $a_{13} = 50$ is the

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
|---|---|---|---|---|---|
| $M_5$ | 0 | 0 | **0** | 0 | 0 |
| $M_1$ | 62 | 78 | 50 | **101** | 82 |
| $M_2$ | 71 | **84** | 61 | 73 | 59 |
| $M_3$ | **87** | 92 | 111 | 71 | 81 |
| $M_4$ | 48 | 64 | 87 | 77 | **80** |

$SAP_5$: Table 4.14

unique minimum element, so $J_3$ is the dominating column. Thus for first row, $a_{53} = 0$ is the assignment. The assignments for second and third rows are $a_{14} = 101$ and $a_{22} = 84$ respectively. The last two assignments obtained from the decision matrix

$$D_5 = \begin{bmatrix} a_{31} & a_{35} \\ a_{41} & a_{45} \end{bmatrix} = \begin{bmatrix} 87 & 81 \\ 48 & 80 \end{bmatrix}.$$

are $a_{31} = 87$ and $a_{45} = 8$ since $\text{tr}(D_5) > \text{offtr}(D_5)$. Hence the fifth simulated sum $S_5$ is

$$S_5 = a_{53} + a_{14} + a_{22} + (a_{31} + a_{45}) = 0 + 101 + 84 + (87 + 80) = 352.$$

Thus optimal solution is $S = S_1 = 376$ taken by index priority concept, since

$$\max \{S_1, S_2, S_3, S_4, S_5\} = 376 = \{S_1, S_2, S_3\}.$$

Hence the allocation of the assignments are

$$M_1 \to J_4; M_2 \to J_2; M_3 \to J_3; M_4 \to J_5 \quad \text{and} \quad M_5 \to J_1.$$

Since the fifth mechanic is a dummy and job $J_1$ is assigned to the fifth mechanic, so this job is declined.

4.4. **Algorithm of Maximized Assignment Problem (Method-II).** The algorithm for the maximized assignment problem is given as follows.

(1) Find the sum of the row maximums $\bar{S}$ and the first simulated assignment sum $S_1$.

(2) If $S_1 = \bar{S}$, then the maximum elements are the assignments of the $AP$ otherwise find $S_2$, the second simulated assignment sum.

(3) If $S_2 = \bar{S}$, then the maximum elements are the assignments of the $AP$ otherwise compare $S_1$ and $S_2$. If $S_1 \geq S_2$, then local optimal solution $S$ is $S_1$ otherwise $S$ is $S_2$.

(4) Find all the simulated assignment sum of the $SAPs$ one after another and use the step 3 to find the local optimal solution $S$ for each case.

(5) Finally, if $\bar{S} \neq \{S_1, S_2, \cdots, S_n\}$ then

$$S = \max_{1 \leq i \leq n} S_i = \max\{S_1, S_2, \cdots, S_n\} = S_k \quad \text{(say)},$$

is the optimal assignment solution. Find the assignments corresponds to $S_k$.

(6) For tie case of simulated sums, i.e.,

$$S = \max_{1 \leq i \leq n} S_i = \{S_{k_1}, S_{k_2}, \cdots, S_{k_m}\}$$

where $k_1 < k_2 < \cdots < k_m$, we have $S = S_{k_1}$ by index priority concept. Allocate the assignments corresponds to $S_{k_1}$.

4.5. **Minimizing the order of assignment problem.** In the assignment problem, some assignments can be obtained very easily if

(1) *row minimum* = *column minimum* for minimized $AP$,

(2) *row maximum* = *column maximum* for maximized $AP$.

For tie case, select the first assignment to that element which is unique to both row and column, then cover its cell. Covering the assignments one by one, we can get a reduced assignment problem where the rest assignments can be obtained using **DAS**-technique. The reduced assignment problem contains the elements where the above two conditions fails according to type of the problem (minimize or maximize).

For instance we take the problem given in Example 3.2.1 and Example 4.3.3.

**Example 4.5.1.** Let the cost matrix of the minimized assignment problem be given by:

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | row minimum |
|---|---|---|---|---|---|---|
| $A_1$ | 13 | **8** | 16 | 18 | 19 | 8 |
| $A_2$ | 9 | 15 | 24 | 9 | 12 | 9 |
| $A_3$ | 12 | 9 | **4** | 4 | 4 | 4 |
| $A_4$ | **6** | 12 | 10 | 8 | 13 | 6 |
| $A_5$ | 15 | 17 | 18 | 12 | 20 | 12 |
| column minimum | 6 | 8 | 4 | 4 | 4 | |

Table 4.15

In the Table 4.15, the cells satisfying

$$\text{row minimum} = \text{column minimum}$$

are $a_{33} = 4$ (since it is unique among row minimums), $a_{41} = 6$ and $a_{12} = 8$. Last two assignments obtained from the decision matrix

$$D = \begin{bmatrix} a_{24} & a_{25} \\ a_{54} & a_{55} \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 12 & 20 \end{bmatrix}$$

are $a_{25} = 12$ and $a_{54} = 12$ since

$$\text{offtr}(D) < \text{tr}(D).$$

Hence the simulated sum is

$$S = a_{33} + a_{41} + a_{12} + (a_{25} + a_{54}) = 4 + 6 + 8 + (12 + 12) = 42$$

which is the optimal solution.

**Example 4.5.2.** We have considered the cost matrix of the maximized assignment problem given in Example 4.3.3. In the Table 4.16, the cells satisfying

$$\text{row maximum} = \text{column maximum}$$

are $a_{33} = 111$ and $a_{14} = 101$. The reduced assignment problem is given in Table 4.17: where the sum of rows maximums is $\bar{S} = 84 + 80 + 0 = 164$. Applying **DAS**-technique in Table 4.17, we get the first simulated sum is

$$S_1 = a_{22} + (a_{45} + a_{51}) = 84 + (80 + 0) = 164$$

| Jobs → / Mechanics ↓ | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | row maximum |
|---|---|---|---|---|---|---|
| $M_1$ | 62 | 78 | 50 | 101 | 82 | 101 |
| $M_2$ | 71 | 84 | 61 | 73 | 59 | 84 |
| $M_3$ | 87 | 92 | 111 | 71 | 81 | 111 |
| $M_4$ | 48 | 64 | 87 | 77 | 80 | 87 |
| $M_5$ | 0 | 0 | 0 | 0 | 0 | 0 |
| column maximum | 87 | 92 | 111 | 101 | 82 | |

Table 4.16

| | $J_1$ | $J_2$ | $J_5$ |
|---|---|---|---|
| $M_2$ | 71 | 84 | 59 |
| $M_4$ | 48 | 64 | 80 |
| $M_5$ | 0 | 0 | 0 |

Table 4.17

that equals to $\bar{S}$, so the assignments for Table 4.17 are $a_{22} = 84$, $a_{45} = 80$ and $a_{51} = 0$. Hence the assignments for Table 4.16 are

$$a_{33} = 111,\ a_{14} = 101,\ a_{22} = 84,\ a_{45} = 80 \quad \text{and} \quad a_{51} = 0.$$

The simulated sum is

$$S = a_{33} + a_{14} + [a_{22} + (a_{45} + a_{51})] = 111 + 101 + [84 + (80 + 0)] = 376$$

which equals to the optimal solution. The square bracket in $S$ indicates that the assignments are obtained from the reduced assignment problem.

## 5. Conclusion

The advantage of this method that we can get more than one group of assignment solutions to find the optimal solution at the same time. For decision theory, this method will help us to take a fair decision from more than one group of alternatives of the person-job assignment problem. Therefore, one can think to alternate the groups of persons for either case to find the same optimal solution to complete a particular job. Again, **DAS**-technique is also useful to solve the rectangular assignment problems which is the general form of assignment problem when we balance the rectangular assignment problems by inserting dummy rows or columns with zero costs.

**Acknowledgment**. The authors would like to thank the reviewer for his valuable suggestion.

## References

1. R. E. Burkard, B. Klinz and R. Rudolf, Perspectives of Monge properties in Optimazation, *Discrete Applied Mathematics* 70, 95 - 161, 1996.
2. R. E. Burkard and E. Çela, *Linear assignment problems and extensions. In: P.M. Pardalos and D.-Z. Du (Eds) Handbook of Combinatorial Optimization*, (Dortrecht: Kluwer Academic Publishers). 75 - 149, 1999.
3. R. E. Burkard, Selected topics on assignment problems, *Discrete Applied Mathematics.* 123, 257-302., 1999.
4. P. Butkovic, Max-algebra:the linear algebra of combinatorics?, *Liear Algebra and its application.* 367, 313 -335, 2003.
5. M. M. Flood, The Travelling Salesman Problem, *Operations Research* 4, 61 - 75, 1956.
6. Jr. L. R. Ford, D. R. Fulkerson, A Simple Algorithm for Finding Maximal Network Flows and an Application to the Hitchcock Problem, *Canadian Journal of Mathematics* 9, 210 - 218, 1957.
7. H. W. Kuhn, The Hungarian Method for the Assignment Problem, *Naval Research Loqistics Quarterly*, 2, 83 - 97, 1955.
8. H. W. Kuhn, Variants of the Hungarian Method for Assignment Problem, *Naval Research Loqistics Quarterly*, 3, 253 - 258, 1956.
9. C. H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization - Algorithms and Complexity*, (Mineda, NY: Dover Publications). pp. 83, 1998.
10. H. A. Taha, *Operation Research; An introduction*, (Prentice-Hall of India Private ltd, New Delhi). 2005.