Journal of the Orissa Mathematical Society ISSN:0975-2323
Volume 33, No.2(2014), p.55-60

# NUMERICAL APPROXIMATION OF TWO DIMENSIONAL INTEGRAL OF AN ANALYTIC FUNCTION

*P.M.Mohanty and **M. Acharya
*Department of Mathematics, S.C.S.College,Puri
Email: pravat_kin@yahoo.co.in
** ITER, SOA University,Bhubaneswar
Email: milu_acharya@yahoo.com

**Abstract** A seventh degree rule involving a set of twenty nodes has been constructed for the numerical approximation of an integral of analytic function of two complex variables. The truncation error associated with the approximation has been analysed and estimate of the error has been obtained.

**Keywords and phrases** Analytic function, quadrature rule, degree of precision, truncation error.
**2000 Mathematics Subject Classification No** 65 D 30.

## 1. Introduction

Birkoff and Young(2), Lether(4), Tosic(7) and Senapati et al(6) have constructed quadrature rules for the numerical approximation of one dimensional integral of an analytic function which is given by the following equation.

$$J(\phi) = \int_L \phi(z)dz \tag{1}$$

where $\phi(z)$ is an analytic function and is a directed line segment from the point $z_0 - h$ to $z_0 + h$ lying inside the domain of analyticity of the function.

It is a well-known that the numerical treatment of functions of several complex variables is always a challenging problem. Some research work conducted for the numerical approximation of double integral of analytic function of two complex variables, given by

$$I(f) = \int_{L_2}\int_{L_1} f(z,\zeta)dzd\zeta \tag{2}$$

where $L_1$ and $L_2$ are directed line segments from $z_0 - h_1$ to $z_0 + h_1$ and $\zeta_0 - h_2$ to $\zeta_0 + h_2$ respectively, are due to Das et al(3), Acharya and Das(1) and Milovanovic et al(5).

Our object in the present paper is to generate a quadrature rule meant for the numerical approximation of the double integral $I(f)$ given by equation (2).

## 2. Formulation of the quadrature rule

The four point Gauss-Legendre rule of degree seven meant for the numerical evaluation of the integral of $g(x)$ over [-1,1] is given by

$$R_{4GL} = A\sum g(\pm p)g(\pm q) \tag{3}$$

where the coefficients and nodes are as follows

$$p = \left(\frac{15+\sqrt{120}}{35}\right)^{\frac{1}{2}}, q = \left(\frac{15-\sqrt{120}}{35}\right)^{\frac{1}{2}},$$

$$A = \frac{1-3q^2}{3p^2-3q^2}, B = \frac{3p^2-1}{3p^2-3q^2}. \tag{4}$$

Following the method suggested by Lether (2), the transformed four point seventh degree rule meant for the integral $J(\phi)$ is given by the following equation.

$$R_1(\phi) = Ah\{\phi(z_0+ph)+\phi(z_0-ph)\} + Bh\{\phi(z_0+ph)+\phi(z_0-ph). \tag{5}$$

Similarly, the quadrature rule of degree seven given by Tosic(3) meant for the integral $J(\phi)$ is given by the following equation

$$R_2(\phi) = A^{'}h\Phi(z_0) + B^{'}h\{\phi(z_0 + sh) + \phi(z_0 - sh)\}$$
$$+C^{'}h\{\phi(z_0 + ish) + \phi(z_0 - ish) \tag{6}$$

where

$$A^{'} = \frac{16}{15}, B^{'} = \frac{16}{15}\left(\frac{7}{5} + \sqrt{\frac{7}{3}}\right), C^{'} = \frac{1}{6}\left(\frac{7}{5} + \sqrt{\frac{7}{3}}\right), s = \left(\frac{3}{7}\right)^{\frac{1}{4}}. \tag{7}$$

For construction of the quadrature rule for the double integral $I(f)$, we consider the following set of nodes which is a Cartesian product set

$$S = \{z_0 \pm ph_1, z_0 \pm qh_1\} \times \{\zeta_0, \zeta_0 \pm sh_2, \zeta_0 \pm ish_2\}. \tag{8}$$

Let the quadrature rule meant for the double integral $I(f)$ with the set of nodes $S$ be prescribed as follows.

$$Q(f) = W_1 \sum f(z_0 \pm ph_1, \zeta_0) + W_2 \sum f(z_0 \pm qh_1, \zeta_0)$$
$$+W_3 \sum f(z_0 \pm ph_1, \zeta_0 \pm sh_2) + W_4 \sum f(z_0 \pm qh_1, \zeta_0 \pm sh_2)$$
$$+W_5 \sum f(z_0 \pm ph_1, \zeta_0 \pm ish_2) + W_6 \sum f(z_0 \pm qh_1, \zeta_0 \pm ish_2) \tag{9}$$

where $p$,$q$ and $s$ are given by equations (4) and (7) . The rule $Q(f)$ being symmetric is exact i.e. $I(f) = Q(f)$ whenever $f(z,\zeta)$ is a monomial $z^n, \zeta^m$zn m if at least one of $n$ and $m$ is an odd integer. To determine the coefficients $Wj$ in the rule $Q(f)$ we make the rule exact for monomials $z^n, \zeta^m$ where $(n,m) = (0,0), (2,0), (0,2), (4,0), (0,4)$ and $(2,2)$. Then we get the following system of linear equations :

$$\left.\begin{aligned} W_1 + W_2 + 2W_3 + 2W_4 + 2W_5 + 2W_6 &= 2h_1h_2 \\ W_1p^2 + W_2q^2 + 2W_3p^2 + 2W_4q^2 + 2W_5p^2 + 2W_6q^2 &= 2h_1h_2/3 \\ W_3s^2 + W_4s^2 - W_5s^2 - W_6s^2 &= h_1h_2/3 \\ W_1p^4 + W_2q^4 + 2W_3p^4 + 2W_4q^4 + 2W_5p^4 + 2W_6q^4 &= 2h_1h_2/5 \\ W_3s^4 + W_4s^4 + W_5s^4 + W_6s^4 &= h_1h_2/5 \\ W_3s^2p^2 + W_4s^2q^2 - W_5s^2p^2 - W_6s^2q^2 &= h_1h_2/9 \end{aligned}\right\} \tag{10}$$

Solving the above system of equations we get the following values :

$$\left.\begin{array}{r} W_1 = 0.371045168146617H, W_2 = 0.695621498520049H \\ W_3 = 0.169725639348713H, W_4 = 0.318195232593278H \\ W_5 = -0.007393378284568H, W_6 = -0.013860826990757H \end{array}\right\} \tag{11}$$

Where $H = h_1 h_2$.

**Theorem 1.** *The rule $Q(f)$ (given by equations (9) and (11)) has degree of precision seven.*

**Proof :** It is obvious that the rule $Q(f)$ has degree of precision at least five in view of the system of equations (10). Setting $f(z,\zeta) = z^n\zeta^m$ in $I(f) - Q(f)$ ,the value of the expression $I(f) - Q(f)$ is equal to zero for $(n,m) = (6,0),(4,2).(2,4)$ *and* $(0,6)$ if $W_j$'s are as prescribed in equation (11). Further setting $(n,m) = (0,8)$ it is noted that

$$I(f) - Q(f) = 0.101587301587302H \tag{12}$$

which is not equal to zero. This implies that the degree of precision of the rule $Q(f)$ meant for the double integral $I(f)$ is seven.

## 3. Analysis of truncation error

The truncation error associated with the rule $Q(f)$ meant for the numerical evaluation of the double integral $I(f)$ is given by

$$E(f) = I(f) - Q(f). \tag{13}$$

Let the function $f$ be analytic in the product domain $\Omega_1 \times \Omega_2$ where $\Omega_1, \Omega_2$ are discs in the complex plane $C$ centered at $z_0$ and $\zeta_0$ of radii $\rho_1$ and $\rho_2$ respectively such that $\rho_1 > |h_1|$ and $\rho_2 > |h_2|$ . The Taylor series expansion of the function about the point $(z_0, \zeta_0)$ is given by

$$f(z,\zeta) = \sum_{n=0}^{\infty} \frac{1}{n!}\left((z-z_0)\frac{\partial}{\partial z} + (\zeta-\zeta_0)\frac{\partial}{\partial \zeta}\right) f(z_0,\zeta_0) \tag{14}$$

where the partial derivatives have been evaluated at $(z_0,\zeta_0)$. Using the expansion given by equation (14) in equation (13) and considering the leading term we have in view of theorem 1 the following.

**Theorem 2.** *The truncation error associated with the quadrature rule* $Q(f)$ *satisfies the approximate equality :* $E(f) \approx h_1h_2\left\{K_1h_1^8f^{8+0} + K_2h_2^8f^{0+8}\right\}/8!$ *where* $f^{m+n}$ *denotes the* $(m{+}n)th$ *partial derivative* $\frac{\partial^{m+n}f}{\partial z^m\partial\zeta^n}$ *evaluated at* $(z_0,\zeta_0)$ *and* $K_1 = 0.23219954648526$ *,*$K_2 = 0.101587301587302$ *.*

**corollary :** *Assuming* $h_1 = h_2 = h$ *we have* $|E(f)| = O(|h|^{10})$.

## 4. Numerical experiments

For performing the numerical experiment we consider the following two integrals:

$$J_1 = \int_{L_2}\int_{L_1} \sin(z+\zeta)dzd\zeta, J_2 = \int_{L_4}\int_{L_3} \exp(z+\zeta)dzd\zeta \tag{15}$$

where $L_1, L_2, L_3, L_4$ are directed line segments from $1+i$ to

$$2+i$$

,from $(1+3i)/2$ to $(3+i)/2$, from $i/2$ to $i$ and from $i/2$ to $i/2$ respectively. The exact values, computed approximate values and the absolute errors have been presented in the table appended below.

Table 1

| Integral | Exat value | Approximation | $\lvert Error\rvert$ |
|---|---|---|---|
| $J_1$ | $-0.213886115280338$ -4.986937883574649i | -0.213886091731078 -4.986937685841362i | $1.9 \times 10^-7$ |
| $J_2$ | -0.569621807385846 -0.410973271278006i | -0.569621805896171 -0.410973270203228i | $1.8 \times 10^{-9}$ |

It is noteworthy that the error reduces if $|h_1|$ and $|h_2|$ are assigned smaller values. The rule yields computed value correct upto at least seven decimal places.

## References


[1] Acharya,B.P. and Das,R.N., Approximate evaluation of multiple complex integrals of analytic functions,*Computing*,30,279-283, 1983.

[2] Birkoff,G. and Young, D., Numerical quadrature of analytic and harmonic functions, *J.Math.Phys.*, 29, 217-221, 1950.

[3] Das, R.N., Padhy, S., and Acharya, B.P., Numerical quadrature of analytic functions of more than one variable, *J.Math.Phys.Sci.*, 15, 573-579, 1981.

[4] Lether, F.G., On Birkoff -Young quadrature of analytic functions, *J.Comp. Appl.Math.*, 2, 81-84, 1976.

[5] Milovanovic, G.V., Acharya, B.P., Pattanaik, T.N., On numerical evaluation of double integral of an analytic function of two complex variables, *BIT*, 26, 321-326, 1986.

[6] Senapati, S., Acharya, M. and Acharya, B.P., On Numerical quadrature of analytic functions, *J.(theo & appl.), Appl.Math.Sci.*, 5, 3411-3420, 2011.

[7] Tosic, D., A modification of the Birkoff -Young quadrature formula for analytical functions, *Univ.Beograd Publ.Elektrotehn.Fak.Ser.Mat.Fiz.*, 601-633, 73-77, 1978.