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Second order nondifferentiable multiobjective symmetric dual programs over cone with generalized $(K, F) - (\rho, \theta)$ convexity

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Abstract In this paper, a new class of second order $(K, F) - (\rho, \theta)$ pseudo convex functions is introduced with example. A pair of Wolfe type second order nondifferentiable symmetric dual programs over arbitrary cones with square root term is formulated. The duality results are established under second order $(K, F) - (\rho, \theta)$ pseudo convexity assumption.

Keywords and phrases Multiobjective programming, second order $(K, F)-$ (ρ, θ)- univex function, square root term, Schwartz inequality, efficient solution.
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1. Introduction

In mathematical program, a pair of primal and dual programs is called symmetric if the dual of the dual is the primal problem. The duality in linear programming is symmetric. It is not so in nonlinear programming in general. The first symmetric dual formulation for quadratic programming was proposed by Dorn [4]. Dantzig et al. [3] and Mond [16] studied symmetric duality in nonlinear programming by assuming the scalar function $f(x, y)$ to be convex in x and concave in y. The symmetric duality result was generalized by

Bazaraa and Goode [1] to arbitrary cones. Subsequently, Mond and Weir [17] presented a distinct pair of symmetric dual nonlinear programs which admits the relaxation of the convexity/concavity assumption to pseudo convexity/ pseudo concavity.

Mond [16] initiated second order symmetric duality of Wolfe type in nonlinear programming and proved the duality theorems under second order convexity. Mangasarian [14] discussed second order duality in nonlinear programming under inclusion condition. Mond [16, pp.93] and Mangasarian [14, pp.609] also indicated possible computational advantages of the second order dual over the first order dual. This motivated several authors [2, 5, 6, 12, 16, 21, 22] in this field. Yang et al. [22] studied second order multiobjective symmetric dual programs and established the duality relations under F-convexity assumptions. Mishra [15] formulated a pair of multiobjective second order symmetric dual nonlinear programs over arbitrary cones and established weak, strong, converse and self duality theorems under second order (strict) pseudo-invexity. Also Yang et al. [21] formulated a pair of Wolfe type second order nondifferentiable symmetric dual programs containing support function and presented the duality results under F convexity.

Recently, Gulati et al. [7] studied Wolfe and Mond-Weir type second order symmetric duality over arbitrary cones and proved the duality results under generalized bonvexity assumption. Gulati and Geeta [9] studied Mond-Weir type second order symmetric duality in multiobjective programming over cones and established duality results under pseudoinvexity/K-F convexity assumption. Gulati and Verma [8] formulated a pair of Wolfe type nondifferentiable multiobjective symmetric duality and established the duality results under invexity assumption. Gupta and Kailey [10] formulated a pair of Wolfe type second order nondifferentiable multiobjective symmetric dual programs in which objective function contains support function and proved the duality results under second order F-convexity assumption. Gupta and Kailey [11] presented second order multiobjective symmetric duality involving cone-bonvex functions. Saini and Gulati [19] presented a pair of Wolfe type nondifferentiable second order symmetric dual program over arbitrary cones under second order K-F-convexity assumption.

In this paper, motivated by Saini and Gulati [19], a new class of second order $(K,F)-(\rho,\theta)$- univex function pseudo convex/ second order $(K,F)-(\rho,\theta)$ strongly pseudo convex function is introduced with example. A pair of Wolfe type second order nondifferentiable multiobjective symmetric dual programs over arbitrary cone containing square root term of positive semidefinite quadratic form is formulated. The duality results are established under second order $(K,F)-(\rho,\theta)$- pseudo convex function.

2. **Notation and Preliminaries**

The following convention for vectors in will be used; Let R^n be n-dimensional Euclidean space and R^n_+ be the nonnegative orthant. For vectors x and y in R^n , we denote $x < y \Leftrightarrow x_i < y_i$ for $i = 1, 2, .., n$; $x \leq y \Leftrightarrow x_i \leq y_i$ for $i = 1, 2, .., n$.

Definition 2.1: A set A function C of R^n is called a cone, if for each $x \epsilon C$ and $\lambda \epsilon R, \lambda \geq 0$,we have Moreover, if $\lambda x \epsilon C$ is convex, then it is the convex cone.

Definition 2.2 The positive polar cone C^* of C is defined as

$$C^* = \{z \epsilon R^n | x^T z \geq 0, \forall x \epsilon C\}$$

Let $C_1 \subset R^n, C_2 \subset R^m$ and $K \subset R^k$ be closed convex cones with nonempty interiors. Let C_1^*, C_2^* and K^* be the positive polar cones of C_1, C_2 and K respectively.

Throughout this paper let $X \subseteq R^n$ and $Y \subseteq R^m$ are open and $X \times Y \subseteq R^n \times R^m$. Let $C_1 \times C_2 \subseteq X \times Y$.

A general multiobjective nonlinear programming problem can be expressed in the following form:

Primal (P)

$$Minimize\ f(x) = (f_1(x), f_2(x), ..., f_k(x))$$
$$Subject\ to\ -g(x)\ \epsilon Q,\ x\epsilon X,\ where$$

$f : R^n \to R^k, g : R^n \to R^m$, and Q is a closed convex cone with nonempty interior in R^m.

Let $X_0 = \{x\epsilon X : -g(x)\varepsilon Q\}$ be the set of all feasible solutions of (P). Further let K_0 denote the set $K\backslash\{0\}$.All the vectors will be considered as column vectors.

Definition 2.3 A point $\bar{x}\epsilon X_0$ is weakly efficient solution of (P), if there exist no $x\epsilon X_0$ such that $f(\bar{x}) - f(x)\epsilon\ intK$.

Definition 2.4 A point $\bar{x}\epsilon X_0$ is efficient solution of (P), if there exist no $x\epsilon X_0$ such that $f(\bar{x}) - f(x)\epsilon\ K_0$.

Definition 2.5 A functional $F : X \times X \times R^n$ is sublinear in its third argument if for all $(x, u)\epsilon X \times X$,

(i) $F(x, u; a_1 + a_2 \leq F(x, u; a_1) + F(x, u; a_2), \forall a_1, a_2\epsilon R^n$ and

(ii) $F(x, u; \alpha a) = \alpha F(x, u; a), \forall\alpha\epsilon R_+$ and $a\epsilon R$.

For notational convenience, we can write $F_{(x,u)}(a)$for $F(x, u; a)$.

Now, we are in position to give definition of second order $K - (F, \alpha, \rho, \theta)$ -pseudo convex function and second order strongly $K - (F, \alpha, \rho, \theta)$ pseudo convex function.

Definition 2.6 A twice differentiable function $f = (f_1, f_2, ..., f_k) : X \times Y \to R^k$ is said to be second order $(K, F) - (\rho, \theta)$ -pseudo convex function in the first variable at $u\epsilon X$ for fixed $v\epsilon Y$, if there exists $\theta_1 : X \times X \to R, \rho_i, i = 1, 2, ..., k$ and a sublinear functional $F : X \times X \times R^n \to R$ such that for all

$(x, u; p)\epsilon X \times X \times R^n$ we have,

$$\begin{pmatrix} F_{x,u}(\nabla_u f_1(u,v) + \nabla_{uu} f_1(u,v)p_1) + \rho_1 \theta_1^2(x,u), \\ ..., \\ F_{x,u}(\nabla_u f_k(u,v) + \nabla_{uu} f_k(u,v)p_k) + \rho_k \theta_1^2(x,u) \end{pmatrix} \epsilon K$$

$$\Rightarrow \begin{pmatrix} f_1(x,v) - f_1(u,v) + \frac{1}{2} p_1^T \nabla_{uu} f_1(u,v)p_1, \\ ..., \\ f_k(x,v) - f_k(u,v) + \frac{1}{2} p_k^T \nabla_{uu} f_k(u,v)p_k \end{pmatrix} \epsilon K$$

Definition 2.7 A twice differentiable function $f = (f_1, f_2, ..., f_k) : X \times Y \to R^k$ is said to be second order $(K, F) - (\rho, \theta)$ -strongly pseudo convex function at $u\epsilon X$ for fixed $v\epsilon Y$, if there exists $\theta_1 : X \times X \to R, \rho_i, i = 1, 2, ..., k$ and a sublinear functional $F : X \times X \times R^n \to R$ such that for all $(x, u; p)\epsilon X \times X \times R^n$ we have,

$$\begin{pmatrix} F_{x,u}(\nabla_u f_1(u,v) + \nabla_{uu} f_1(u,v)p_1) + \rho_1 \theta_1^2(x,u), \\ ..., \\ F_{x,u}(\nabla_u f_k(u,v) + \nabla_{uu} f_k(u,v)p_k) + \rho_k \theta_1^2(x,u) \end{pmatrix} \epsilon K$$

$$\Rightarrow \begin{pmatrix} f_1(x,v) - f_1(u,v) + \frac{1}{2} p_1^T \nabla_{uu} f_1(u,v)p_1, \\ ..., \\ f_k(x,v) - f_k(u,v) + \frac{1}{2} p_k^T \nabla_{uu} f_k(u,v)p_k \end{pmatrix} \epsilon K_0$$

Definition 2.8 A twice differentiable function $f = (f_1, f_2, ..., f_k) : X \times Y \to R^k$ is said to be second order $(K, G) - (\sigma, \theta)$ -pseudo convex function in the second variable at $y\epsilon Y$ for fixed $x\epsilon X$, if there exists $\theta_2 : X \times X \to R, \sigma_i, i = 1, 2, ..., k$ and a sublinear functional $G : Y \times Y \times R^m \to R$ such that for all $(v, u; q)\epsilon Y \times Y \times R^m$ we have,

$$\begin{pmatrix} G_{v,y}(\nabla_y f_1(x,y) + \nabla_{yy} f_1(x,y)q_1) + \sigma_1 \theta_2^2(v,y), \\ ..., \\ F_{x,u}(\nabla_u f_k(u,v) + \nabla_{uu} f_k(u,v)p_k) + \sigma_k \theta_2^2(x,u) \end{pmatrix} \epsilon K$$

$$\Rightarrow \begin{pmatrix} f_1(x,v) - f_1(x,y) + \frac{1}{2} q_1^T \nabla_{yy} f_1(x,y)q_1, \\ ..., \\ f_k(x,v) - f_k(x,y) + \frac{1}{2} q_k^T \nabla_{yy} f_k(x,y)q_k \end{pmatrix} \epsilon K$$

Definition 2.9 A twice differentiable function $f = (f_1, f_2, ..., f_k) : X \times Y \to R^k$ is said to be second order$(K, G) - (\sigma, \theta)$ -strongly pseudo convex function at $u\epsilon X$ for fixed $v\epsilon Y$,if there exists $\theta_2 : X \times X \to R, \sigma_i, i = 1, 2, ..., k$ and a sublinear functional $G : Y \times Y \times R^m \to R$ such that for all $(v, y; q)\epsilon Y \times Y \times R^m$ we have,

$$\begin{pmatrix} G_{v,y}(\nabla_y f_1(x, y) + \nabla_{yy} f_1(x, y)q_1\}) + \sigma_1\theta_2^2(v, y), \\ ..., \\ G_{v,y}(\nabla_y f_k(x, y) + \nabla_{yy} f_k(x, y)q_k) + \sigma_k\theta_2^2(v, y) \end{pmatrix} \epsilon K$$

$$\Rightarrow \begin{pmatrix} f_1(x, v) - f_1(x, y) + \frac{1}{2}q_1^T\nabla_{yy} f_1(x, y)q_1, \\ ..., \\ f_k(x, v) - f_k(x, y) + \frac{1}{2}q_k^T\nabla_{yy} f_k(x, y)q_k \end{pmatrix} \epsilon K_0$$

Lemma 2.1 *Let B be a positive semi definite matrix of order n. Then for all $x, w\epsilon R^n, x^T Bw \le (x^T Bx)^{\frac{1}{2}}(w^T Bw)^{\frac{1}{2}}$. The equality holds if $Bx = \lambda Bw$ for some $\lambda \ge 0$.*

3. Second order multiobjective duality

We consider the following pair of second order Wolfe type nondifferentiable multiobjective programming problem with k-objective:

Primal(SWP) $L(x, y, \lambda, w, p) =$

$$Minimize \begin{pmatrix} f_1(x, y) + (x^T B_1 x)^{\frac{1}{2}} - \frac{1}{2}\sum_{i=1}^k \lambda_i p_i^T(\nabla_{yy} f_i(x, y)p_i) \\ -y^T[\sum_{i=1}^k \lambda_i\nabla_y f_i(x, y) + \sum_{i=1}^k \lambda_i\nabla_{yy} f_i(x, y)p_i], \\ ..., \\ f_k(x, y) + (x^T B_k x)^{\frac{1}{2}} - \frac{1}{2}\sum_{i=1}^k \lambda_i p_i^T(\nabla_{yy} f_i(x, y)p_i) \\ -y^T[\sum_{i=1}^k \lambda_i\nabla_y f_i(x, y) + \sum_{i=1}^k \lambda_i\nabla_{yy} f_i(x, y)p_i] \end{pmatrix}$$

Subject to

$$-\sum_{i=1}^k \lambda_i[\nabla_y f_i(x, y) - D_i w_i + \nabla_{yy} f_i(x, y)p_i]\epsilon C_2^*, \tag{3.1}$$

$$w_i^T D_i w_i \le 1, i = 1, 2, ..., k, \tag{3.2}$$

$$x\epsilon C_1, w_i\epsilon R^m, \tag{3.3}$$

$$\lambda\ \epsilon\ int\ K^*, \sum_{i=1}^k \lambda_i = 1 \tag{3.4}$$

Dual(SWD) $M(u,v,\lambda,z,q)=$

$$Maximize\begin{pmatrix} f_1(u,v)-(v^TD_1v)^{\frac{1}{2}}-\frac{1}{2}\sum_{i=1}^k\lambda_iq_i^T(\nabla_{uu}f_i(u,v)q_i) \\ -u^T[\sum_{i=1}^k\lambda_i\nabla_uf_i(u,v)+\sum_{i=1}^k\lambda_i\nabla_{uu}f_i(u,v)q_i], \\ ..., \\ f_k(u,v)+(v^TD_kv)^{\frac{1}{2}}-\frac{1}{2}\sum_{i=1}^k\lambda_iq_i^T(\nabla_{uu}f_i(u,v)q_i) \\ -u^T[\sum_{i=1}^k\lambda_i\nabla_uf_i(u,v)+\sum_{i=1}^k\lambda_i\nabla_{uu}f_i(u,v)q_i] \end{pmatrix}$$

Subject to

$$\sum_{i=1}^k\lambda_i[\nabla_uf_i(u,v)+B_iz_i+\nabla_{uu}f_i(u,v)q_i]\epsilon C_1^*, \tag{3.5}$$

$$z_i^TB_iz_i\leq 1, i=1,2,...,k, \tag{3.6}$$

$$v\epsilon C_2, z_i\epsilon R^n, \tag{3.7}$$

$$\lambda\ \epsilon\ int\ K^*, \sum_{i=1}^k\lambda_i=1 \tag{3.8}$$

where

(1) $f=(f_1,f_2,...,f_k):R^n\times R^m\rightarrow R^k$ is thrice differentiable vector function,

(ii) C_1 and C_2 are closed convex cones in R^n and R^m with nonempty interiors,respectively,

(iii) C_1^* and C_2^* are positive polar cones of C_1 and C_2 respectively,

(iv) K is a closed convex cone in R^k with $intK\neq\phi$ and $R_+^k\subset K$,

(v) $q_i,z_i,i=1,2,...,k$ are vectors in R^n and $p_i,w_i,i=1,2,...,k$ are vectors in R^m.

(vi) $B=(B_1,B_2,...,B_k)$ and$D=(D_1,D_2,...,D_k)$, B_i and D_i are positive semidefinite matrices of order n and m respectively. Now we established the following theorem

Theorem 3.1 (Weak Duality) Let (x, y, λ, w, p) be feasible solution for the primal (SWP) and (u, v, λ, z, q) be feasible solution for the dual (SWD). Suppose there exist $\theta_1 : X \times X \to R, \theta_2 : Y \times Y \to R, \rho_i \epsilon R, \sigma_i \epsilon R, i = 1, 2, ..., k$ and sublinear functional $F : X \times X \times R^n \to R$ and $G : Y \times Y \times R^m$ satisfying
(1)$F_{x,u}(a) + \sum_{i=1}^{k} \lambda_i \rho_i \theta_1^2(x, u) - u^T a \geq 0, \forall (x, u) \epsilon C_1 \times C_2, a \epsilon C_1^2$,
(2)$G_{v,y}(b) + \sum_{i=1}^{k} \lambda_i \sigma_i \theta_2^2(v, y) - y^T b \geq 0, \forall (x, u) \epsilon C_1 \times C_2, a \epsilon C_1^2$,
Furthermore assume that for each i
(3) $f(., v) + (.)^T Bz$ is second order $(K, F) - (\rho, \theta)$-pseudo convex in the first variable at u for fixed v and
(4) $f(x, .) + (.)^T Dw$ is second order $(K, F) - (\rho, \theta)$ -pseudo concave in the second variable at y for fixed x .
Then $Inf(SWP) - Sup(SWD) \epsilon K$
. **Proof:** Since (u, v, λ, z, q) is feasible solution for (SWD), from dual constraint (3.5) we have $a = \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u, v) + B_i z_i + \nabla_{uu} f_i(u, v) q_i] \epsilon C_1^*$.
Since $u \epsilon C_1$, we have

$$u^T a = u^T \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u, v) + B_i z_i + \nabla_{uu} f_i(u, v) q_i] \geq 0. \quad (3.9)$$

So, hypothesis (1) in lieu of (3.9), implies that

$$F_{x,u}(a) + \sum_{i=1}^{k} \lambda_i \rho_i \theta_1^2(x, u) \geq 0.$$

i.e.

$$F_{x,u}(\sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u, v) + B_i z_i + \nabla_{uu} f_i(u, v) q_i]) + \sum_{i=1}^{k} \lambda_i \rho_i \theta_1^2(x, u) \geq 0. \quad (3.10)$$

The sublinearity of F with respect to third argument and (3.10) gives

$$\sum_{i=1}^{k} \lambda_i [F_{x,u}(\nabla_u f_i(u, v) + B_i z_i + \nabla_{uu} f_i(u, v) q_i) + \rho_i \theta_1^2(x, u)] \geq 0. \quad (3.11)$$

Since $\lambda\epsilon\ intK^*$, the above inequality can be written as

$$\begin{pmatrix} F_{x,u}(\nabla_u f_1(u,v) + B_1 z_1 + \nabla_{uu} f_1(u,v)p_1) + \rho_1\theta_1^2(x,u), \\ ..., \\ F_{x,u}(\nabla_u f_k(u,v) + B_k z_k + \nabla_{uu} f_k(u,v)p_k) + \rho_k\theta_1^2(x,u) \end{pmatrix} \epsilon K. \tag{3.12}$$

So, second order $(K,F)-(\rho,\theta)$-pseudo convexity of $f_i(.,v)+(.)^T B_i z_i$ at u for fixed v implies that

$$\begin{pmatrix} f_1(x,v) + x^T B_1 z_1 - f_1(u,v) - u^T B_1 z_1 + \frac{1}{2} q_1^T \nabla_{uu} f_1(u,v) q_1, \\ ..., \\ f_k(x,v) + x^T B_k z_k - f_k(u,v) - u^T B_k z_k + \frac{1}{2} q_k^T \nabla_{uu} f_k(u,v) q_k \end{pmatrix} \epsilon K.$$

For $\lambda\epsilon\ intK^*$,

$$\sum_{i=1}^{k} \lambda_i [f_i(x,v) + x^T B_i z_i - f_i(u,v) - u^T B_i z_i + \frac{1}{2} q_i^T \nabla_{uu} f_i(u,v) q_i] \geq 0. \tag{3.13}$$

Similarly (u,v,λ,w,q) is feasible solution for (SWD), so from primal constraint (3.1), we have

$$b = -\sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x,y) - D_i w_i + \nabla_{yy} f_i(x,y) p_i] \epsilon C_2^*.$$

So $y\epsilon C_2$ implies that

$$y^T b = -y^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x,y) - D_i w_i + \nabla_{yy} f_i(x,y) p_i] \geq 0. \tag{3.14}$$

Again hypothesis (2) in lieu (3.14), implies that

$$G_{v,y}(b) + \sum_{i=1}^{k} \sigma_i \theta_2^2(v,y) \geq 0.$$

i.e.

$$G_{v,y}(-\sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x,y) - D_i w_i + \nabla_{yy} f_i(x,y) p_i]) + \sum_{i=1}^{k} \sigma_i \theta_2^2(v,y) \geq 0$$

$$\sum_{i=1}^{k} \lambda_i [G_{v,y}(-[\nabla_y f_i(x,y) - D_i w_i + \nabla_{yy} f_i(x,y) p_i]) + \sigma_i \theta_2^2(v,y)] \geq 0. \tag{3.15}$$

Since $\lambda \in intK^*$, inequality (3.15) can be written as

$$\begin{pmatrix} G_{v,y}(-[\nabla_y f_1(x,y) - D_1 w_1 + \nabla_{yy} f_1(x,y)p_1]) + \sigma_1 \theta_2^2(v,y), \\ ..., \\ G_{v,y}(-[\nabla_y f_k(x,y) - D_k w_k + \nabla_{yy} f_k(x,y)p_k]) + \sigma_k \theta_2^2(v,y) \end{pmatrix} \in K. \quad (3.16)$$

From hypothesis (4) and (3.16), we obtained

$$\begin{pmatrix} -[f_1(x,v) - (v)^T D_1 w_1 - f_1(x,y) + (y)^T D_1 w_1 + \frac{1}{2} p_1^T \nabla_{yy} f_1(x,y)p_1], \\ ..., \\ -[f_k(x,v) - (v)^T D_k z_k - f_k(x,y) + (y)^T D_k w_k + \frac{1}{2} p_k^T \nabla_{yy} f_k(x,y)p_k] \end{pmatrix} \in K.$$

This implies that for $\lambda \in intK^*$

$$\sum_{i=1}^{k} \lambda_i[-f_i(x,v) + (v)^T D_i w_i + f_i(x,y) - (y)^T D_i w_i - \tfrac{1}{2} p_i^T \nabla_{yy} f_i(x,y)p_i] \geq 0. \quad (3.17)$$

Adding (3.13) and (3.17), we get

$$\sum_{i=1}^{k} \lambda_i[f_i(x,y) + (x)^T B_i z_i - (y)^T D_i w_i - \tfrac{1}{2} p_i^T \nabla_{yy} f_i(x,y)p_i)]$$

$$-\sum_{i=1}^{k} \lambda_i[f_i(u,v) - (v)^T D_i w_i + u^T B_i z_i - \tfrac{1}{2} q_i^T \nabla_{uu} f_i(u,v)q_i] \geq 0. \quad (3.18)$$

Now from Schwarz inequality, (3.3) and (3.7), we have,

$$x^T B_i z_i \leq (x^T B_i x)^{\frac{1}{2}} (z_i^T B_i z_i)^{\frac{1}{2}} \leq (x^T B_i x)^{\frac{1}{2}}, i = 1, 2, ..., k. \quad (3.19)$$

$$v^T D_i w_i \leq (v^T D_i v)^{\frac{1}{2}} (w_i^T D_i w_i)^{\frac{1}{2}} \leq (v^T D_i v)^{\frac{1}{2}}, i = 1, 2, ..., k. \quad (3.20)$$

Also from primal constraint (3.1), we have

$$-\sum_{i=1}^{k} \lambda_i[\nabla_y f_i(x,y) - D_i w_i + \nabla_{yy} f_i(x,y)p_i] \in C_2^*.$$

Form (3.14), we have

$$y^T(\sum_{i=1}^{k} \lambda_i[\nabla_y f_i(x,y) - D_i w_i + \nabla_{yy} f_i(x,y)p_i]) \leq 0$$

$$\Rightarrow -\sum_{i=1}^{k} \lambda_i[y^T D_i w_i] \leq -y^T(\sum_{i=1}^{k} \lambda_i[\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i]). \quad (3.21)$$

Similarly from dual constraint (3.5), we get

$$\sum_{i=1}^{k}\lambda_i[u^T B_i z_i] \geq -u^T(\sum_{i=1}^{k}\lambda_i[\nabla_u f_i(u,v)+\nabla_{uu} f_i(u,v)q_i]). \qquad (3.22)$$

Using (3.19), (3.20),(3.21) and (3.22) in (3.18), we obtain that

$$\sum_{i=1}^{k}\lambda_i\{[f_i(x,y)+(x^T B_i x)^{\frac{1}{2}}-y^T[\nabla_y f_i(x,y)+\nabla_{yy} f_i(x,y)p_i]-\tfrac{1}{2}p_i^T[\nabla_{yy} f_i(x,y)p_i]\}$$

$$-\sum_{i=1}^{k}\lambda_i\{f_i(u,u)-(v^T D_i v)^{\frac{1}{2}}-u^T[\nabla_u f_i(u,v)+\nabla_{uu} f_i(u,v)q_i]-\tfrac{1}{2}q_i^T[\nabla_{uu} f_i(u,v)q_i]\} \geq 0.$$

$$\Rightarrow Inf(SWP)-Sup(SWD)\in K.$$

In order to prove the strong duality theorem, we obtain the following Lemma established by Suneja et al. [19]. It gives Fritz-John type necessary optimality conditions for a weakly efficient solution of (P).

Lemma 3.1 If $\bar{x}$ is a weakly efficient solution of (P), then there exist $\bar{\mu}\in K^*, \bar{\beta}\in Q^*$, not both zero, such that $(x-\bar{x})^T[\bar{\mu}\nabla f(\bar{x})+\bar{\beta}\nabla g(\bar{x})]\geq 0, \forall x\in C$and $\bar{\beta}^T g(\bar{x})=0$.

Theorem 3.2(Strong duality) Let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ be weakly efficient solution of (SWP) such that

(i) $\nabla_{yy}(\sum_{i=1}^{k}\lambda_i f_i(\bar{x},\bar{y}))$ is nonsingular,

(ii) $\bar{p}_i\neq 0$ implies $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y(\nabla_{yy} f_i(\bar{x},\bar{y})\bar{p}_i)\bar{p}_i\neq 0$,

(iii) the vectors $\nabla_y f_1(\bar{x},\bar{y}),....,\nabla_y f_k(\bar{x},\bar{y})$ are linearly independent,

(iv) the vector $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y(\nabla_{yy} f_i(\bar{x},\bar{y})\bar{p}_i)\bar{p}_i\notin span\{\nabla_y f_1(\bar{x},\bar{y}),....,\nabla_y f_k(\bar{x},\bar{y})\}\backslash\{0\}$.

Then there exist $\bar{z}_i\in R^n$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{q}=0)$ is feasible for (SWD) and two objective values of (SWP) and (SWD) are equal. Furthermore, if the hypotheses of Theorem 3.1 are satisfied for all feasible solution of (SWP) and (SWD), then $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{q}=0)$ is an efficient solution of (SWD).

Proof: Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ is weakly efficient solution of (SWP), by Lemma 3.1, there exist $\bar{\mu} \in K^*, \bar{\beta} \in C_2, \bar{\gamma} \in R_+, \bar{\delta}_i \in R_+, \bar{z} \in R^n$ such that

$$(x-\bar{x})^T \begin{pmatrix} \sum_{i=1}^k \bar{\mu}_i[\nabla_x f_i(\bar{x},\bar{y}) + B_i\bar{z}_i] \\ +\sum_{i=1}^k \lambda_i(\nabla_{yx} f_i(\bar{x},\bar{y}))^T(\bar{\beta} - \bar{y}\sum_{i=1}^k \bar{\mu}_i) \\ +\sum_{i=1}^k \lambda_i \nabla_x(\nabla_{yy} f_i(\bar{x},\bar{y})\bar{p}_i)^T[\bar{\beta} - (\sum_{i=1}^k \bar{\mu}_i)(\bar{y} + \frac{1}{2}\bar{p}_i)] \end{pmatrix} \geq 0, \forall x \in C_1, \tag{3.23}$$

$$(y-\bar{y})^T \begin{pmatrix} [\bar{\mu} - \bar{\lambda}_i(\sum_{i=1}^k \bar{\mu}_i)]\nabla_y f_i(\bar{x},\bar{y}) \\ +\sum_{i=1}^k \lambda_i(\nabla_{yy} f_i(\bar{x},\bar{y}))^T(\bar{\beta} - \bar{y}\sum_{i=1}^k \bar{\mu}_i) \\ -(\sum_{i=1}^k \bar{\mu}_i)\sum_{i=1}^k \lambda_i(\nabla_{yy} f_i(\bar{x},\bar{y}))p_i \\ +(\sum_{i=1}^k \lambda_i \nabla_y(\nabla_{yy} f_i(\bar{x},\bar{y}))\bar{p}_i)^T[\bar{\beta} - (\sum_{i=1}^k \bar{\mu}_i)(\bar{y} + \frac{1}{2}\bar{p}_i)] \end{pmatrix} \geq 0, \tag{3.24}$$

for all $y \in R^m$.

$$(\lambda - \bar{\lambda})^T \left(\nabla_y f_i(\bar{x},\bar{y})^T(\bar{\beta} - \bar{y}\sum_{i=1}^k \bar{\mu}_i) + \delta e_k \right)$$

$$+ \begin{pmatrix} (\bar{\beta} - (\sum_{i=1}^k \bar{\mu}_i)(y^T + \frac{1}{2}\bar{p}_i)^T \nabla_{yy} f_1(\bar{x},\bar{y})\bar{p}_i, ..., \\ (\bar{\beta} - (\sum_{i=1}^k \bar{\mu}_i)(y^T + \frac{1}{2}\bar{p}_k)^T \nabla_{yy} f_k(\bar{x},\bar{y})\bar{p}_k \end{pmatrix}^T \geq 0, \tag{3.25}$$

for all $\lambda \in intK^*$ and $e_k = (1, 1, ..., 1) \in R^k$.

$$\sum_{i=1}^k \lambda_i(\nabla_{yy} f_i(\bar{x},\bar{y})(\beta - (\sum_{i=1}^k \bar{\mu}_i)(\bar{y} + \bar{p}_i)) = 0, \tag{3.26}$$

$$\bar{\beta}^T \sum_{i=1}^k \bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y}) - D_i\bar{w}_i + \nabla_{yy} f_i(\bar{x},\bar{y})\bar{p}_i) = 0, \tag{3.27}$$

$$x^T B_i z_i = (x^T B_i x)^{\frac{1}{2}}, i = 1, 2, ..., k, \tag{3.28}$$

$$\bar{\delta}(\sum_{i=1}^k \bar{\lambda}_i - 1) - 0, \tag{3.29}$$

$$(-D_i\bar{\beta} + \bar{\gamma} D_i\bar{w}_i) = 0, i = 1, 2, ..., k; \tag{3.30}$$

$$\bar{\gamma}(\bar{w}_i^T D_i\bar{w}_i - 1) = 0, i = 1, 2, ..., k; \tag{3.31}$$

$$\bar{z}_i^T B_i\bar{z}_i \leq 1, \tag{3.32}$$

$$(\alpha, \beta, \gamma, \delta) \geq 0, \tag{3.33}$$

$$(\alpha, \beta, \gamma, \delta) \neq 0, \tag{3.34}$$

Since $\nabla_{yy}(\sum_{i=1}^{k}\lambda_i f_i(\bar{x},\bar{y}))$ is nonsingular, (3.26) gives

$$\beta = (\sum_{i=1}^{k}\mu_i)(\bar{y}+\bar{p}_i). \tag{3.35}$$

From (3.24) and (3.25), we obtained

$$\begin{pmatrix} [\bar{\mu}-\bar{\lambda}_i(\sum_{i=1}^{k}\bar{\mu}_i)][\nabla_y f_i(\bar{x},\bar{y})+\sum_{i=1}^{k}\lambda_i(\nabla_{yy}f_i(\bar{x},\bar{y}))^T(\bar{\beta}-\bar{y}\sum_{i=1}^{k}\bar{\mu}_i) \\ -(\sum_{i=1}^{k}\bar{\mu}_i)\sum_{i=1}^{k}\lambda_i(\nabla_{yy}f_i(\bar{x},\bar{y})p_i \\ +\sum_{i=1}^{k}\lambda_i(\nabla_y(\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i))^T[\bar{\beta}-(\sum_{i=1}^{k}\bar{\mu}_i)(\bar{y}+\frac{1}{2}\bar{p}_i)] \end{pmatrix} = 0, \tag{3.36}$$

and

$$\left(\nabla_y f_i(\bar{x},\bar{y})^T(\bar{\beta}-\bar{y}\sum_{i=1}^{k}\bar{\mu}_i)+\delta e_k\right)$$

$$+\begin{pmatrix} (\bar{\beta}-(\sum_{i=1}^{k}\bar{\mu}_i)(y^T+\frac{1}{2}\bar{p}_i))^T\nabla_{yy}f_1(\bar{x},\bar{y})\bar{p}_i, ..., \\ (\bar{\beta}-(\sum_{i=1}^{k}\bar{\mu}_i)(y^T+\frac{1}{2}\bar{p}_k))^T\nabla_{yy}f_k(\bar{x},\bar{y})\bar{p}_k \end{pmatrix}^T = 0. \tag{3.37}$$

We claim that $\bar{\mu} \neq 0$.

To do so, suppose $\bar{\mu} = 0$. So $\sum_{i=1}^{k}\bar{\mu}_i = 0$.

Then (3.35) gives $\beta = 0$,which along with (3.37) yields $\bar{\delta}e_k = 0$ or $\bar{\delta} = 0$.

From (3.30) and (3.31), we have

$$\bar{\gamma} = \bar{\gamma}(\bar{w}_i^T D_i\bar{w}_i) = \bar{w}_i(\bar{\gamma}D_i\bar{w}_i) = \bar{w}_i(D_i\bar{\beta}) = 0.$$

Thus $(\bar{\mu},\bar{\beta},\bar{\gamma},\bar{\delta}) = 0$, this contradicts (3.34).

Hence

$$\bar{\mu} \neq 0. \tag{3.38}$$

Since $\bar{\mu} \in K^*$ and $R_+^k \subseteq K$ implies $K^* \subseteq R_+^k$, so we get

$$\bar{\mu} \geq 0 \ or \ \sum_{i=1}^{k}\bar{\mu}_i > 0. \tag{3.39}$$

From (3.35) and (3.36), we get

$$\begin{pmatrix} [\bar{\mu}-\bar{\lambda}_i(\sum_{i=1}^{k}\bar{\mu}_i)][\nabla_y f_i(\bar{x},\bar{y})+\sum_{i=1}^{k}\lambda_i(\nabla_{yy}f_i(\bar{x},\bar{y}))^T(\sum_{i=1}^{k}\bar{\mu}_i)\bar{p}_i \\ -(\sum_{i=1}^{k}\bar{\mu}_i)\sum_{i=1}^{k}\lambda_i(\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i \\ +\sum_{i=1}^{k}\lambda_i(\nabla_y(\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i))^T[(\sum_{i=1}^{k}\bar{\mu}_i)(\frac{1}{2}\bar{p}_i)] \end{pmatrix} = 0,$$

or

$$[\sum_{i=1}^{k}\lambda_i(\nabla_y(\nabla_{yy}f_i(\bar{x},\bar{y}))\bar{p}_i]\bar{p}_i = -\frac{2}{\sum_{i=1}^{k}\bar{\mu}_i}[\bar{\mu}-\bar{\lambda}_i(\sum_{i=1}^{k}\bar{\mu}_i)][\nabla_y f_i(\bar{x},\bar{y})]. \quad (3.40)$$

Now suppose $\bar{p}_i = 0, i = 1, 2, ...k$. Then hypothesis (ii) implies that $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y(\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i)\bar{p}_i \neq 0$, which in view of (3.40) contradicts hypothesis (iv).

Therefore

$$\bar{p}_i = 0, i = 1, 2, ...k. \quad (3.41)$$

Since the vectors $\nabla_y f_1(\bar{x},\bar{y}), ..., \nabla_y f_k(\bar{x},\bar{y})$ are linearly independent, (3.40) and (3.41) yield

$$\bar{\mu} = \bar{\lambda}_i(\sum_{i=1}^{k}\bar{\mu}_i). \quad (3.42)$$

From (3.35) and (3.42), we obtain

$$\beta = (\sum_{i=1}^{k}\mu_i)\bar{y}. \quad (3.43)$$

or

$$\bar{y} = \frac{\beta}{(\sum_{i=1}^{k}\mu_i)} \in C_2. \quad (3.44)$$

Again using (3.39) , (3.41), (3.42) and (3.43) in (3.23), we get

$$(x-\bar{x})^T(\sum_{i=1}^{k}\lambda_i[\nabla_x f_i(\bar{x},\bar{y}) + B_i\bar{z}_i]) \geq 0, \forall x \in C_1. \quad (3.45)$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$and so (3.45) implies that

$$(x)^T\sum_{i=1}^{k}\lambda_i[\nabla_x f_i + B_i z_i] \geq 0, \forall x \in C_1$$

i.e.

$$\sum_{i=1}^{k}\lambda_i[\nabla_x f_i + B_i z_i] \in C_1^*. \quad (3.46)$$

Thus from (3.32),(3.44) and (3.46), we obtain that $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{q} = 0)$ satisfies the dual constraints (3.5), (3.6), (3.7) and (3.8).

Thus $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{q} = 0)$ is feasible for (SWD).

Now letting $x = 0$ and $x = 2\bar{x}$ in (3.45), we get

$$\bar{x}^T(\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(\bar{x}, \bar{y}) + B_i \bar{z}_i]) = 0,$$

or using (3.28), we get

$$\bar{x}^T(\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(\bar{x}, \bar{y})) = -\bar{x}^T B_i \bar{z}_i = -(\bar{x}^T B_i \bar{x})^{\frac{1}{2}}. \tag{3.47}$$

Again (3.27) along with (3.39), (3.41) and (**??**) gives

$$\bar{y}^T(\sum_{i=1}^{k} \lambda_i[\nabla_y f_i(\bar{x}, \bar{y}) = \bar{y}^T D_i \bar{w}_i. \tag{3.48}$$

From (3.30) and (3.43,) we get

$$D_i \bar{y} = \frac{\bar{\gamma}}{(\sum_{i=1}^{k} \mu_i)} D_i \bar{w}_i \tag{3.49}$$

or

$$D_i \bar{y} = a D_i \bar{w}_i, where a = \frac{\gamma}{(\sum_{i=1}^{k} \bar{\mu}_i)} \geq 0. \tag{3.50}$$

Under this condition of the Schwarz inequality holds as equality. Therefore

$$\bar{y} D_i \bar{w}_i = (\bar{y}^T D_i \bar{y})^{\frac{1}{2}} (\bar{w}_i^T D_i \bar{w}_i)^{\frac{1}{2}}. \tag{3.51}$$

In case $\gamma > 0$, from (3.31), we get $\bar{w}_i^T D_i \bar{w} = 1$.

So (3.50) implies $\bar{y} D_i \bar{w}_i = (\bar{y}^T D_i \bar{y})^{\frac{1}{2}}$.

In case $\bar{\gamma} = 0$, from (3.49), we get $D_i \bar{y} = 0$ and so $\bar{y} D_i \bar{w}_i = 0 = (\bar{y}^T D_i \bar{y})^{\frac{1}{2}}$.

Thus in either case

$$\bar{y}^T D_i \bar{w}_i = (\bar{y}^T D_i \bar{y})^{\frac{1}{2}}. \tag{3.52}$$

So, from (3.48) and (3.52), we find

$$\bar{y}^T(\sum_{i=1}^{k} \lambda_i[\nabla_y f_i(\bar{x}, \bar{y}) = (\bar{y}^T D_i \bar{y})^{\frac{1}{2}}. \tag{3.53}$$

So using (3.41), (3.47) and (3.53), we conclude that the two objective values are equal, i.e. for every $i \in \{1, 2, ..., k\}$

$$f_i(x,y)+(x^T B_i x)^{\frac{1}{2}}-y^T[\sum_{i=1}^{k}\lambda_i \nabla_y f_i(x,y)] = f_i(u,v)-(v^T D_i v)^{\frac{1}{2}}-u^T[\sum_{i=1}^{k}\lambda_i \nabla_u f_i(u,v)]$$

i.e.

$$L(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0) = M(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{q} = 0). \tag{3.54}$$

Now we claim that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{q} = 0)$ is efficient solution of (SWD). If this would not be case, the there would exist a feasible solution $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{q} = 0)$such that

$$M(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{q} = 0) \leq M(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{q} = 0) \Rightarrow L(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{q} = 0) \leq M(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{q} = 0).$$

This is a contradiction to weak duality Theorem 3.1.

Hence $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{q} = 0)$is efficient solution of dual (SWD)

Theorem 3.3 (Converse duality theorem)) Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ be weakly efficient solution of (SWP) such that

(i) $\nabla_{uu}(\sum_{i=1}^{k} \lambda_i f_i(\bar{u}, \bar{v}))$ is nonsingular,

(ii) $\bar{q}_i \neq 0$ implies $\sum_{i=1}^{k} \bar{\lambda}_i \nabla_u (\nabla_{uu} f_i(\bar{u}, \bar{v}))\bar{q}_i)\bar{q}_i \neq 0$,

(iii) the vectors $\nabla_u f_1(\bar{u}, \bar{v}), ..., \nabla_u f_k(\bar{u}, \bar{v})$ are linearly independent.

(iv) the vector $\sum_{i=1}^{k} \bar{\lambda}_i \nabla_u (\nabla_{uu} f_i(\bar{u}, \bar{v}))\bar{q}_i)\bar{q}_i \notin span\{\nabla_u f_1(\bar{u}, \bar{v}), ..., \nabla_u f_k(\bar{u}, \bar{v})\} \backslash \{0\}$.

Then there exist $\bar{w}_i \in R^m$ such that $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is feasible for (SWD) and two objective values of (SWP) and (SWD) are equal. Furthermore, if the hypotheses of Theorem 3.1 are satisfied for all feasible solution of (SWP) and (SWD), then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is an efficient solution of (SWD).

Proof: The proof follows on lines of Theorem 3.2.

4. Special Cases

(i) If $B_i = D_i = 0, k = 1$; then the problem (SWP) and (SWD) can be reduced to the problem proposed by Gulati et al. [7] as follows:

Primal (WP):

Minimize $f(x, y) - y^T[\nabla_y f(x, y) + \nabla_{yy} f(x, y)p] - \frac{1}{2}p^T[\nabla_{yy} f(x, y)p]$

Subject to $-[\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i] \in C_2^*$,

$$x \in C_1,$$

Dual (WD):

Maximize $f(u,u) + u^T[\nabla_u f(u,v) + \nabla_{uu} f(u,v)q] - \frac{1}{2}q^T[\nabla_{uu} f(u,v)q]$

Subject to $\nabla_u f(u,v) + \nabla_{uu} f(u,v)q \in C_1^*$,

$$v \in C_2,$$

(ii) If k=1, $C_1 = R_+^n, C_2 = R_+^m, (x^T Bx)^{\frac{1}{2}} = s(x|C')$ and $(y^T Dy)^{\frac{1}{2}} = s(y|D')$, where $C' = \{Bx|x^T Bx \leq 1\}$,$D' = \{Dy|y^T Dy \leq 1\}$, then the problem (SWP) and (SWD) can be reduced to the problem proposed by Yang et al. [21]

Primal (WP):

Minimize $f(x,y)+s(x|C')-y^T[\nabla_y f(x,y)+\nabla_{yy} f(x,y)p]-\frac{1}{2}p^T[\nabla_{yy} f(x,y)p]$

Subject to $\nabla_y f(x,y) - z + \nabla_{yy} f(x,y)p \leq 0$,

$$x \geq 0, z \in D'.$$

Dual (WD):

Maximize $f(u,u)-s(v|D')-u^T[\nabla_u f(u,v)+\nabla_{uu} f(u,v)q]-\frac{1}{2}q^T[\nabla_{uu} f(u,v)q]$

Subject to $\nabla_u f(u,v) + \nabla_{uu} f(u,v)q \geq 0$,

$$v \geq 0, w \in C'.$$

5. Numerical Example

Let k=2, m=n-1,

$K = \{(x,y) : x \geq 0, y \geq 0\}, intK^* = \{(x,y) : x > 0, y > 0\}$,

$C_1 = R_+, C_2 = R_+ C_1^* = R_+$ and $C_2^* = R_+$.

Let $f = (f_1, f_2) : R \times R \to R^2$ be defined as $f(x,y) = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) = x^2 - x - y^2 + y, f_2(x,y) = e^{-x} - e^{-y}$.

Let $p_1, p_2 \in R, z_1, z_2, w_1, w_2 \in [0,1], B_1 = B_2 = D_1 = D_2 = 1$.

Then our problems primal and dual problem reduces to

Primal: SWP

Minimize $\begin{pmatrix} x^2 - y^2 + y + \lambda_1(2y^2 - y + 2yp_1 + p_1^2) + \lambda_2 e^{-y}(yp_2 - y + \frac{1}{2}p_2^2), \\ e^{-x} - e^{-y} + x + \lambda_1(2y^2 - y + 2yp_1 + p_1^2) + \lambda_2 e^{-y}(yp_2 - y + \frac{1}{2}p_2^2) \end{pmatrix}$

Subject to

$$\lambda_1(2y + w_1 + 2p_1 - 1) + \lambda_2(e^{-y}p_2 + w_2 - e^{-y}) \in C_2^*, \tag{5.1}$$

$$w_1^2 \leq 1, w_2^2 \leq 1, \tag{5.2}$$

$$x \in C_1, w_1, w_2 \in [0, 1], p_1, p_2 \in R, \tag{5.3}$$

$$\lambda \in intK^*, \lambda_1 + \lambda_2 = 1. \tag{5.4}$$

Dual: SWD

Maximize $\begin{pmatrix} u^2 - u - v^2 - \lambda_1(2u^2 - u + 2uq_1 - q_1^2) + \lambda_2 e^{-u}(uq_2 + q_2^2 - u), \\ e^{-u} - e^{-v} - v - \lambda_1(2u^2 - u + 2uq_1 - q_1^2) + \lambda_2 e^{-u}(uq_2 + q_2^2 - u) \end{pmatrix}$

Subject to

$$\lambda_1(2u + z_1 + 2q_1 - 1) + \lambda_2(-e^{-u} + z_2 + e^{-u}q_2) \in C_1^*, \tag{5.5}$$

$$z_1^2 \leq 1, z_2^2 \leq 1, \tag{5.6}$$

$$v \in C_2, z_1, z_2 \in [0, 1], q_1, q_2 \in R, \tag{5.7}$$

$$\lambda \in intK^*, \lambda_1 + \lambda_2 = 1. \tag{5.8}$$

Let there exist $\theta_1 : R \times R \to R$ and $\theta_2 : R \times R \to R$ defined as $\theta_1(x, u) = \sqrt{x^2 + u^2}$, $\theta_2(x, u) = \sqrt{v^2 + y^2}$ and $\rho_1 = 3, \rho_2 = -2$,$\sigma_1 = -2, \sigma_2 = 1$

Let $F : R \times R \times R \to R$ and $G : R \times R \times R \to R$ are the functional defined as $F_{x,u}(a) = F(x, u; a) = a(x^2 + u^2)$ and $G_{v,y}(b) = G(v, y; b) = b(v^2 + y^2)$ and satisfying

(1) $F_{x,u}(a) + \sum_{i=1}^{k} \lambda_i \rho_i \theta_1^2(x, u) - u^T a \geq 0, \forall (x, u) \in C_1 \times C_1, a \in C_1^*$, and

(2) $G_{v,y}(b) + \sum_{i=1}^{k} \lambda_i \sigma_i \theta_2^2(v, y) - y^T b \geq 0, \forall (v, y) \in C_2 \times C_2, b \in C_2^*$ respectively.

Clearly F and G are sublinear in their third argument.

Now, $a_1 = \nabla_u f_1(u,v) + z_1 + \nabla_{uu} f_1(u,v)q_1 = 2u + 2q_1 + z_1 - 1,$

$$a_2 = \nabla_u f_2(u,v) + \nabla_{uu} f_2(u,v)q_2 = -e^{-u} + e^{-u}q_2 + z_2.$$

$$b_1 = \nabla_y f_1(x,y) + w_1 + \nabla_{yy} f_1(x,y)p_2 = -2y + 1 + w_1 - 2p_1,$$

$$b_2 = \nabla_y f_2(x,y) + w_2 + \nabla_{yy} f_2(x,y)p_2 = e^{-y} - e^{-y}p_2 + w_2,$$

From (5.5) and hypothesis (1), we find

$$(F(x,u;a_1) + \rho_1\theta^2(x,u), F(x,u;a_2) + \rho_2\theta^2(x,u)) \in K$$

$\Rightarrow ((2u + 2p_1 + z_1 - 1 + 3)(x^2 + u^2), (-e^{-u} + e^{-u}p_2 + z_2 - 2)(x^2 + u^2)) \in K$

$\Rightarrow 2u + 2p_1 + z_1 - 1 + 3 \geq 0$ and $-e^{-u} + e^{-u}p_2 + z_2 - 2 \geq 0$

$\Rightarrow p_1 \geq -1 - u - \frac{1}{2}z_1$ and $p_2 \geq 1 + 2e^u - z_2e^u$

Now, $f_1(x,v) + xz_1 - f_1(u,v) - uz_1 + \frac{1}{2}p_1^T\nabla_{uu} f_1(u,v)p_1$

$$= x^2 - x - u^2 + u + z_1(x-u) + (p_1)^2$$

$$\geq x^2 - x - u^2 + u + z_1(x-u) + (1 + u + \tfrac{1}{2}z_1)^2$$

$$= x^2 - x + z_1x + 1 + 3u + \tfrac{1}{4}z_1^2 + z_1 \geq 0, \forall x, u \in R_+$$

and $f_2(x, v + xz_2 - f_2(u,v) - uz_2 + \frac{1}{2}p_2^T\nabla_{uu} f_2(u,v)p_2)$

$$= e^{-x} - e^{-u} + z_2(x-u) + \tfrac{1}{2}p_2^2e^{-u}$$

$$\geq e^{-x} - e^{-u} + \tfrac{1}{2}(1 + 2e^u - e^uz_2)^2e^{-u} + z_2(x-u)$$

$$= e^{-x} - \tfrac{1}{2}e^{-u} + e^u(4 + \tfrac{1}{2}z_2^2 - 2z_2) + 2 - z_2 + z_2(x-u) \geq 0,$$

$\forall x, u \in R_+$ and $z_1, z_2 \in [0,1]$.

So,

$(f_1(x,v) - f_1(u,v) + \frac{1}{2}p_1^T\nabla_{uu} f_1(u,v)p_1, f_2(x,v) - f_2(u,v) + \frac{1}{2}p_2^T\nabla_{uu} f_2(u,v)p_2)) \in K.$

Hence $f = (f_1, f_2) : R \to R^2$ is second order $(K,F) - (\rho,\theta)$ -pseudo convex at $u \in R_+$ for all $x \in R$ and fixed v.

Similarly, $f = (f_1, f_2) : R \to R^2$ is second order $(K,F) - (\rho,\theta)$ -pseudo concave at y for fixed x.

So, all the conditions of Theorem 3.1 are satisfied.

Again, from primal constraint (5.1) to (5.4), we observed that,

$(x = 1, y = 0, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}, w_1 = \frac{1}{2}, w_2 = \frac{1}{2}, p_1 = \frac{1}{2}, p_2 = 2)$is a feasible solution of (SWP)

and the value of the objective functions at this point is (4, 1.49)

Also, from dual constraint (5.5) to (6.8), we see that

$(u = 1, v = 0, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}, z_1 = \frac{1}{2}, z_2 = \frac{1}{2}, q_1 = \frac{1}{2}, q_2 = 2)$ is a feasible solution of (SWD) and the value of the objective functions at this point is $(0.0446, -0.587)$.

From the above discussion, we observe that $Inf(SWP) - Sup(SWD) \in K$.

Hence the duality results holds good.

6. Conclusion

In this paper, a new class of second order $(K, F) - (\rho,\ \theta)$ pseudo convex/ second order $(K, F) - (\rho,\ \theta)$ strongly pseudo convex function is introduced with example. A pair of Wolfe type second order nondifferentiable symmetric dual programs over arbitrary cone containing square root term of positive semidefinite quadratic form is formulated. The duality results are established under second order $(K, F) - (\rho,\ \theta)$-pseudo convexity assumption. A numerical example is given to substantiate the analysis. The results developed in this paper cane be further extended to fractional programming.

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