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RANDOM TRIGONOMETRIC INTERPOLATION

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Abstract We show that the random trigonometric interpolation polynomial associated with the stochastic process of independent increment having the semi-table distribution converges in the mean to the stochastic integral.

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1. Introduction

Random trigonometric interpolation has been studied earlier in Context of image reconstruction with noise (cf. Dash and Pattanayak). In this they considered trigonometric interpolation polynomials

$$I_n(f) = \frac{2}{2n+1} \sum_{j=0}^{2n} f(x_j) D_{\dot{n}}(x - x_j) \tag{1}$$

Which can be rewritten as

$$I_n(x, f) = \sum_{k=-n}^{n} C_k^{(n)} e^{2\pi ikx} (\text{cf. Zygmund [.], p. 8 Vol II}) \tag{2}$$

What Dash et.al. (Loc.cit) considered was to study the random trigonometric polynomial

$$\sum_{k=-n}^{n} X_k C_k^{(n)} e^{2\pi ikx} \tag{3}$$

where $(X_k)_{k=-\infty}^{\infty}$ are random variables defined as

$$X_k = \int_0^1 e^{-2\pi ikt} dX(t)$$

Where $X(t)$ is a stable process with index $\alpha \in (1,2)$. They were able to show that (3) converges in the mean to a stochastical integral.

What we try in this work is to see if a similar result holds for a semistable process.

Let $X(t)$ be a stochastic process with independent increment $X(t_2) - X(t_1)$ having the characteristics function $e^{-|t_1-t_2|(c+\cos\log|u|)|u|^\alpha}$ for $f \in L^\alpha[a,b]$ where $1 < \alpha \le 2$ we can show that it is possible to define the stochastic integral $\int_a^b f(t)dX(t)$ which has the characteristics function
$e^{-|u|^\alpha \left(c\int_a^b |f(t)|^\alpha dt + \int_a^b \cos(\log|f(t)| + \log|u|)|f(t)|^\alpha dt\right)}$.

The polynomial corresponding to the periodic function $f(x)$ at the points.

$$x_j^{(n)} = x_0^{(n)} + \frac{2\pi j}{2n+1} \qquad (j = 0, 1, 2, 3, ..., 2n) \tag{4}$$

is called the nth interpolating polynomial of f.

The interpolating trigonometric polynomial coincides with the function f at these points is given by

$$I_n(x, f) = \frac{2}{2n+1} \sum_{j=0}^{2n} f(x_j) D_n(x - x_j) \tag{5}$$

Where D_n is the Direchlet Kernel given by

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} \tag{6}$$

We can re-arrange the item in (2) and write

$$I_n(x, f) = \sum_{k=-n}^{+n} C_k^{(n)} e^{2\pi ikx} \qquad \text{(cf Zygmund [4] p-8 vol.II)} \tag{7}$$

The coefficients $C_k^{(n)}$ can be expressed as Fourier Stieltjes integrals.
Now we can write

$$I_{n,v}(x,f) = \sum_{k=-v}^{v} C_k^{(n)} e^{2\pi ikx} \tag{8}$$

To get to our result we need two definitions, one Lemma (Chow and Teicher [1] p-285) and result (cf Zygmund [4] vol. 11, p-30).

Definition 1.1

A sequence of random variable X_n is said to converge in the mean to the random variable X if $\lim_{n\to\infty} E|X_n - X| = 0$.

Definition 1.2

A sequence of random variable X_n is said to converge probability to a random variable X if $\lim_{n\to\infty} P\{|X_n - X| \geq \epsilon\} = 0$ for every $\epsilon > 0$.

Lemma 1.3

For any random variable X with the characteristics function Ψ, the absolute moment of the random variable X is given by

$$E|X| = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1 - Real\Psi(t)}{t^2} dt$$

Again it is known (cf Zygmund [4] vol. 11 p-30) that for $f \in L^p[0, 2\pi] p > 1$,

$$\lim_{n\to\infty} \int_0^{2\pi} |I_{n,v}(t-u) - f(t-u)|^p dt = 0$$

2. Our Main Results

Theorem 2.1

The random trigonometric polynomial $\sum_{k=-n}^{+n} X_k C_K^{(n)} e^{2\pi ikx}$

where $X_k = \int_0^1 e^{-2\pi ikt} dX(t)$ and $X(t)$ is a stochastic process with independent increment having the semi-stable distribution of index α where $1 < \alpha \leq 2$ with the characteristics function $e^{-|t_1 - t_2|(c + \cos\log|u|)|u|^\alpha}$ converges in the mean to the stochastic integral $\int_0^1 f(t-u) dX(u)$ for $f \in L^\alpha[0,1]$.

Proof of Theorem 2.1

$$\begin{aligned}\sum_{k=-v}^{v} X_k C_k^{(n)} e^{2\pi ikt} &= \sum_{k=-v}^{v} C_k^{(n)} \int_0^1 e^{-2\pi iku} dX(u) e^{2\pi ikt} \\ &= \int_0^1 \sum_{k=-v}^{v} C_k^{(n)} e^{2\pi ik(t-u)} dX(u) \\ &= \int_0^1 I_{n,v}(t-u) dX(u)\end{aligned}$$

Now

$$\begin{aligned}&E\left|\int_0^1 I_{n,v}(t-u)dX(u) - \int_0^1 f(t-u)dX(u)\right|^\alpha \\ =&\frac{2}{\pi}\int_{-\infty}^{+\infty}\left(\frac{1-e^{-(c+\cos\log|u|)|u|^\alpha\int_0^1|I_{n,v}(t-u)-f(t-u)|^\alpha dt}}{u^2}\right)du \\ =&\frac{4}{\pi}\int_0^1\left(\frac{1-e^{-(c+\cos\log|u|)|u|^\alpha}\int_0^1|I_{n,v}(t-u)-f(t-u)|^\alpha dt}{u^2}\right)du \\ +&\frac{4}{\pi}\int_1^\infty\left(\frac{1-e^{-(c+\cos\log|u|)|u|^\alpha}\int_0^1|I_{n,v}(t-u)-f(t-u)|^\alpha dt}{u^2}\right)du \\ &(1-e^{-x} < x \text{ for every } x>0) \\ &\le \frac{4}{\pi}\int_0^1 (c+1)|u|^{\alpha-2}du\int_0^1|I_{n,v}(t-u)-f(t-u)|^\alpha dt \\ &+ \frac{4}{\pi}\int_1^\infty\left(\frac{1-e^{-(c+\cos\log|u|)|u|^\alpha\int_0^1|I_{nv}(t-u)-f(t-u)|^\alpha dt}}{u^2}\right)du \\ &= \frac{4}{\pi}\times\frac{(c+1)}{\alpha-1}\int_0^1|I_{n,v}(t-u)-f(t-u)|^\alpha du\end{aligned}$$

We know (cf Zygmund [4] vol. 11, p-30) that for $f \in L^p[0,2\pi], p>1$, $\lim_{n\to\infty}\int_0^1 |I_{n,v}(t-u)-f(t-u)|^p dt = 0.$
Hence the result follows. We can, with much less mechanism, prove

Theorem 2.2

Let f be any continuous function with modulus of continuity $0\left(\frac{1}{\log\delta^{-1}}\right)$. Let the nth interpolating polynomial of f be given by

$$I_n(x,f) = \sum_{k=-n}^{+n} C_k^{(n)} e^{2\pi ikx}$$

Then the random interpolating polynomial

$$\overline{I}_{n,v}(X) = \sum_{k=-v}^{v} C_k^{(n)} A_k e^{2\pi ikx}$$

with $A_k = \displaystyle\int_0^1 e^{-2\pi ikt} dX(t)$

where $X(t)$ is stochastic process with independent increment $X(t_2) - X(t_1)$ of index $\alpha \in (1,2]$ having the characteristics function $e^{-\int_a^b (c+\cos\log(|u||f(t)|))|f(t)|^\alpha dt|u|^\alpha}$ converges in probability to the stochastic integral $\displaystyle\int_0^1 f(t-u)dX(u)$.

Proof of Theorem 2.2

We know (cf Mishra and Samal [3]) that

$$P\left\{\left|\int_a^b f(t)dX(t)\right| \geq \epsilon\right\} \leq \frac{k}{\epsilon^\alpha}\int_a^b |f(t)|^\alpha dt$$

Now

$$\begin{aligned}
\bar{I}_{n,v} &= \sum_{k=-v}^{v} C_k^{(n)} A_k e^{2\pi ikt} \\
&= \sum_{k=-v}^{v} C_k^{(n)} \int_0^1 e^{-2\pi iku} dX(u) e^{2\pi ikt} \\
&= \int_0^1 \sum_{k=-v}^{v} C_k^{(n)} e^{2\pi ik(t-u)} dX(u) \\
&= \int_0^1 I_{n,v}(t-u) dX(u)
\end{aligned}$$

Now

$$P\left\{\left|\bar{I}_{n,v}(x)-\int_0^1 f(t-u)dX(u)\right|\geq\epsilon\right\}$$

$$=P\left\{\left|\int_0^1 I_{n,v}(t-u)dX(u)-\int_0^1 f(t-u)dX(u)\right|\geq\epsilon\right\}$$

$$=P\left\{\left|\int_0^1 (I_{n,v}(t-u)-f(t-u))dX(u)\right|\geq\epsilon\right\}\leq\frac{k}{\epsilon^\alpha}\int_0^1 |I_{n,v}(t-u)-f(t-u)|^\alpha du$$

Weknow (cf Zygmund [4] vol. 11 p-30)that for $f\in L^p[0,2\pi]p>1$,

$$\lim_{n\to\infty}\int_0^{2\pi}|I_{n,v}(t-u)-f(t-u)|^p\,du=0$$

Hence the result follows.

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