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A GENERALISATION OF THE CONCEPT OF BANACH LIMITS

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Abstract The object of the present paper is to generalise the concept of Banach limit by using weights to the sequence.

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1. Banach limits

Let l_∞ be the space of all bounded sequences $x = (x_n)_{n\geq 0}$ of real numbers which is a Banach space with the norm

$$\|x\|_c = \sup_{0\leq t\leq 2\pi} |x_n|.$$

A Banach limit [1] is any linear functional L defined on l_∞ such that

(i) $L(x) \geq 0$ if $x_n \geq 0$ for $n \geq 0$
(ii) $L(x) = L(\sigma x)$ where σ is a shift operator defined by $\sigma x = \sigma(x_0, x_1, \dots) = (x_1, x_2, \dots)$ and
(iii) $L(e) = 1$ where e is the vector defined by $e = (1, 1, 1, \dots)$.

Let p be a sublinear functional defined by

$$p(x) = \overline{\lim_{p}} \sup_{n} \frac{1}{p} \sum_{i=0}^{p} x_{i+n}. \tag{1}$$

By using the functional the existence of Banach limit has been proved by use of the famous Hahn-Banach Theorem (see Simmons[4], Das[2]). However Banach [1] and Lorentz [3] took a different functional which was somewhat difficult to handle. It has been demonstrated in Das[2], Simmons[4] that L is a Banach limit on l_∞ if and only if

$$-p(-x) \leq L(x) \leq p(x), x \in l_\infty \tag{2}$$

A sequence $x \in l_\infty$ is said to be almost convergent to the number s if and only if $L(x) = s$ that is , if the Banach limit of x is unique(see Lorentz[3]). In general, Banach limit is not unique. It follows from (2) that Banach limit of x is unique if and only if

$$p(x) = -p(-x) \tag{3}$$

But it is seen that (3) is equivalent to saying that

$$\lim_{p\to\infty} \frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} = s \tag{4}$$

uniformly in n (see Lorenz [3]). Let $\hat{c}$ denote the set of almost convergent sequences

2. Generalised Banach Limits

The object of the present paper is to introduce a generalisation of Banach limit in the following way. For any sequence $q_n \geq 0$, with $Q_n = q_0 + q_1 + \cdots + q_n > 0$ Define a new sublinear functional on l_∞ by

$$\psi(x) = \overline{\lim_{p}} \sup_{n} \frac{1}{Q_{n+p} - Q_{n-1}} \sum_{i=0}^{p} q_{i+n} x_{i+n} \tag{5}$$

provided that for $p \geq 0$,

$$Q_{n+p} - Q_{n-1} > 0 \tag{6}$$

It can be easily verified that ψ is sublinear functional on l_∞. Now by Hahn-Banach theorem there exist a linear functional ϕ on l_∞ such that

$$\phi(x) \leq \psi(x), \forall x \in l_\infty \tag{7}$$

As ϕ is linear and is sublinear, it follows from (7) that

$$-\psi(-x) \leq \phi(x) \leq \psi(x), \forall x \in l_\infty \tag{8}$$

where

$$-\psi(-x) = \underline{\lim}_p \inf_n \frac{1}{Q_{n+p} - Q_{n-1}} \sum_{i=0}^{p} q_{i+n} x_{i+n} \tag{9}$$

We now define the generalised Banach limit or $\psi-$ limit in the following manner.

Definition ϕ is called $\psi-$ limit on l_∞ if

$$\phi(x) \leq \psi(x), \forall x \in l_\infty$$

Note that

$x_n \geq 0 \Rightarrow \psi(x) \geq 0, -\psi(-x) \geq 0$

Hence is follows from (8) that $x \geq 0 \Rightarrow (x) \geq 0 \Rightarrow -\psi(-x) \geq 0 \Rightarrow \phi(x) \geq 0$

If $x = e = (1, 1, 1, \dots)$ then $\psi(e) = 1 = -\psi(-e)$ so that $\phi(e) = 1$. Thus a linear functional ϕ dominated by satisfies condition (i) and (ii) of Banach limit. $\psi-$ limit is a Banach limit if the linear functional ϕ defined by ψ is shift invariant, that is $\phi(\sigma x) = \phi(x)$.

Now

$$\psi(x - x) = \overline{\lim_p} sup_n \frac{1}{Q_{n+1} - Q_{n-1}} \sum_{i=0}^{p} q_{i+n} \left(x_{i+n+1} - x_{i+n} \right) \tag{10}$$

Note that, by Abel's transformation

$$\begin{aligned}
&\sum_{i=0}^{p} q_{i+n}\left(x_{i+n+1}-x_{i+n}\right) \\
= &\sum_{i=0}^{p-1}(q_{i+n}-q_{i+1+n})\sum_{r=0}^{i}(x_{r+n+1}-x_{r+n})+q_{p+n}\sum_{r=0}^{p}(x_{r+n+1}-x_{r+n}) \\
= &\sum_{i=0}^{p-1}(q_{i+n}-q_{i+1+n})(x_{r+n+1}-x_{r+n})+q_{p+n}(x_{p+n+1}-x_{n}) \\
\leq &\ 2\|x\|\sum_{i=0}^{p-1}|q_{i+n}-q_{i+1+n}|+2\|x\|q_{p+n} \qquad (11)
\end{aligned}$$

it follows from (10) and (11) that

$$\begin{aligned}
|\psi(\sigma x-x)| \ &\leq \ 2\|x\|\overline{\lim_{p}}\sup_{n}\frac{\sum_{p-1}^{i=0}|q_{i+n}-q_{i+1+n}|}{Q_{n+p}-Q_{n}} \\
&+ \ 2\|x\|\overline{lim_{p}}\sup_{n}\frac{q_{p+n}}{Q_{n+p}-Q_{n}} \qquad (12)
\end{aligned}$$

We are now in a situation to prove the following theorem .

Theorem 1. *Let q_n satisfy the following additional conditions :*

$$\frac{q_{p+n}}{Q_{n+p}-Q_{n}} \to 0 \ as\ p\to\infty, uniformly\ in\ n \qquad (13)$$

$$\frac{\sum_{i=0}^{p-1}|q_{i+n}-q_{i+1+n}|}{Q_{n+p}-Q_{n}} \to 0 \ as\ p\to\infty; uniformly\ in\ n \qquad (14)$$

Then the sublinear functional ψ generates the Banach limit ; that is any linear functional φ dominated by ψ is a Banach limit .

Proof: We have only to show that ϕ is shift invariant. It follows from (12) that $|(\sigma x-x)|=0$ by virtue of the hypothesis (13) and (14). Similarly we can show that $|(x-\sigma x)|\leq 0$. This prove that $\phi(\sigma x-x)\leq(\sigma x-x)=0$. Similarly $\phi(x-\sigma x)\leq 0$; this proves that $\phi(x)=\phi(\sigma x)$.

Thus ϕ is $\sigma-$ invariant . This completes the proof of the Theorem 1.

Now the question can be raised if ψ dominates Banach limit , that is , if , ϕ is a Banach limit , then can we show that $\phi\leq\psi$? To the end we prove

Theorem 2. *ψ dominates Banach limits .*

Proof: We first observe that

$$\frac{\sum_{i=0}^{p} q_{i+n}}{Q_{n+p} - Q_{n-1}} = 1$$

Hence

$$\begin{aligned}
\phi(x) &= \frac{\sum_{i=0}^{p} q_{i+n}}{Q_{n+p} - Q_{n-1}} \phi(x) \\
&= \frac{1}{Q_{n+p} - Q_{n-1}} \sum_{i=0}^{p} q_{i+n} \phi(\sigma^i x) \\
&= \phi\left(\frac{1}{Q_{n+p} - Q_{n-1}} \sum_{i=0}^{p} q_{i+n} x_{i+n} \right) \\
&\leq \sup_n \left[\frac{1}{Q_{n+p} - Q_{n-1}} \sum_{i=0}^{p} q_{i+n} x_{i+n} \right] \\
&\leq \overline{\lim_p} \sup_n \frac{1}{Q_{n+p} - Q_{n-1}} \sum_{i=0}^{p} q_{i+n} x_{i+n} = \psi(x)
\end{aligned}$$

The above steps follow by using the Banach limit properties of ϕ. This proves Theorem 2 . By using Theorem 1 as Theorem 2, we obtain

Theorem 3. *The sublinear functional both generates and dominates Banach limits provided that $\{q_i\}$ satisfies the conditions* (13) *and* (14)*.*

As a corollary to Theorem 3, we now obtain .

Theorem 4. *A sequence $x \in \infty$ is almost convergent if and only if*

$$\frac{q_n x_n + q_{n+1} x_{n+1} + \cdots + q_{n+p} x_{n+p}}{q_n + q_{n+1} + \cdots + q_{n+p}}$$

converges to a limit s as $p \to \infty$ uniformly in n , when q_n satisfies conditions (13) *and* (14)*.*

Proof Since the Banach limit ϕ is dominated and generated by ψ (by Theorem 3), it follows that all Banach limits φ are given by $\phi(x) \leq \psi(x)$ and this implies that $-\psi(-x) \leq \phi(x) \leq \psi(x)$ Now ϕ can take any value in the interval

$[-\psi(-x), \psi(x)]$ as generated by Hahn-Banach theorem, that is why Banach limit is not necessarily unique. But in the case $-\psi(-x) = \psi(x)$ then φ has no option except to take unique value. Thus given $x \in \infty$ has unique Banach limit if and only if $-\psi(-x) = \psi(x) = s$ and this is equivalent to the statement of Theorem 4 .

This completes the proof .

By taking $q_n = 1$ for all n , we obtain the following famous theorem of Lorentz.

Corollary 1 A sequence $x \in \infty$ is almost convergent to s if and only if $\frac{x_n+x_{n+1}+\cdots+x_{n+p}}{p+1} \to s$ as $n \to 1$ uniformly in n

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