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Sandwich Results for p–Valent Functions Involving a Linear Operator

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Abstract

Making use of the principle of subordination, in the present paper we obtain the sharp subordination- and superordination-preserving properties of some convex combinations associated with a linear operator in the open unit disk. The *sandwich-type theorem* on the space of normalized analytic functions for these operators is also given, together with a few interesting special cases obtained for an appropriate choices of the parameters and the corresponding functions.

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1 Introduction

Let denote by $H(\mathrm{U})$ the space of all analytical functions in the unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$, and for $a \in \mathbb{C}$, $n \in \mathbb{N}^*$, we denote

$$H[a,n] = \{f \in H(\mathrm{U}) : f(z) = a + a_n z^n + \dots\}.$$

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If f and F are analytic functions in U, we say that f is subordinate to F, written $f(z) \prec F(z)$, if there exists a *Schwarz function* w, which (by definition) is analytic in U, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in$ U, such that $f(z) = F(w(z))$, $z \in$ U. Furthermore, if the function F is univalent in U, then we have the equivalence

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathrm{U}) \subset F(\mathrm{U}).$$

Letting $\varphi : \mathbb{C}^3 \times \overline{\mathrm{U}} \to \mathbb{C}$, $h \in H(\mathrm{U})$ and $q \in H[a,n]$, in [9] the authors determined conditions on φ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) \quad \text{implies} \quad q(z) \prec p(z),$$

for all p functions that satisfy the above superordination. Moreover, they found sufficient conditions so that the q function is the *largest* function with this property, called the *best subordinant* of this superordination.

Using the principle of subordination, Miller et al. [10] investigated some subordination theorems involving certain integral operators for analytic functions in U (see also [1, 11]). Moreover, Miller and Mocanu [9] considered the differential superordinations as the dual concept of differential subordinations (see also [2]).

If $A(p)$, with $p \in \mathbb{N}$, denotes the class of functions of the form

$$f(z) = a_p z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p},\ z \in \mathrm{U},\ a_p \neq 0,$$

which are analytic and p–valent in U, Liu [5] defined the linear operator $\mathscr{I}^s_{p,b} : A(p) \to A(p)$ by

$$\mathscr{I}^s_{p,b} f(z) = a_p\, z^p + \sum_{n=1}^{\infty} \left(\frac{1+b}{n+1+b} \right)^s a_{n+p}\, z^{n+p},\ z \in \mathrm{U}, \tag{1.1}$$

where all the powers are principal ones, and

$$b \in \mathbb{C} \setminus \mathbb{Z}^- = \mathbb{C} \setminus \{-1, -2, \ldots\},\ s \in \mathbb{C}.$$

It is easy to verify that

$$z(\mathscr{I}_{p,b}^{s+1}f(z))' = [p-(1+b)]\,\mathscr{I}_{p,b}^{s+1}f(z)+(1+b)\mathscr{I}_{p,b}^{s}f(z),\ z\in \mathrm{U}. \tag{1.2}$$

We note that:

$$(i) \qquad \mathscr{I}_{p,b}^{0}f(z)=f(z),\ f\in A(p);$$

$$(ii) \qquad \mathscr{I}_{p,p-1}^{-1}f(z)=a_p\,z^p+\sum_{n=1}^{\infty}\frac{n+p}{p}\,a_{n+p}\,z^{n+p}=\frac{zf'(z)}{p},\ f\in A(p);$$

$$(iii) \qquad \mathscr{I}_{1,0}^{-1}f(z)=a_1\,z+\sum_{n=1}^{\infty}(n+1)\,a_{n+1}\,z^{n+1}=zf'(z),\ f\in A(1).$$

Let denote by $A^{\tau}(p)$, with $p\in\mathbb{N}$ and $\tau>0$, the class of functions $f\in A(p)$ of the form

$$f(z)=a_pz^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p},\ z\in\mathrm{U},\quad \text{where}\quad |a_p|\geq\tau.$$

In the present paper we obtain some type of subordination and superordination preserving properties for the linear operators $\mathscr{I}_{p,b}^{s}$ defined by (1.1), and the corresponding *sandwich-type theorem*. Some examples, obtained for an appropriate choices of the parameters and the corresponding functions, are also given.

2 Preliminaries

To prove our main results, we will need the following definitions and lemmas presented in this section.

A function $L(z;t):\mathrm{U}\times[0,+\infty)\to\mathbb{C}$ is called *a subordination (or a Loewner) chain* if $L(\cdot;t)$ is analytic and univalent in U for all $t\geq 0$, and $L(z;s)\prec L(z;t)$ when $0\leq s\leq t$.

The next well-known lemma gives a sufficient condition so that the $L(z;t)$ function will be a subordination chain.

Lemma 2.1. *[12, p. 159] Let $L(z;t)=a_1(t)z+a_2(t)z^2+\ldots$, with $a_1(t)\neq 0$ for all $t\geq 0$ and $\lim_{t\to+\infty}|a_1(t)|=+\infty$. Suppose that $L(\cdot;t)$ is analytic in* U *for all $t\geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0,+\infty)$ for* *all* $z\in\mathrm{U}$. *If* $L(z;t)$ *satisfy*

$$\mathrm{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right]>0,\ z\in\mathrm{U},\ t\geq 0.$$

and

$$|L(z;t)|\leq K_0\,|a_1(t)|\,,\ |z|<r_0<1,\ t\geq 0$$

for some positive constants K_0 *and* r_0*, then* $L(z;t)$ *is a subordination chain.*

We denote by $K(\alpha)$, $\alpha<1$, the class of *convex functions of order* α in the unit disk U, not necessarily normalized, i.e.

$$K(\alpha)=\left\{f\in H(\mathrm{U}):f'(0)\neq 0,\mathrm{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>\alpha,\ z\in\mathrm{U}\right\}.$$

In particular, the class $K\equiv K(0)$ represents the class of *convex (and univalent) functions* in the unit disk (not necessarily normalized).

Lemma 2.2. *[6], [8, Theorem 2.3i, p. 35] Suppose that the function* $H:\mathbb{C}^2\to\mathbb{C}$ *satisfies the condition*

$$\mathrm{Re}\,H(is,t)\leq 0,$$

for all $s,t\in\mathbb{R}$ *with* $t\leq -n(1+s^2)/2$*, where n is a positive integer. If the function* $p(z)=1+p_nz^n+\dots$ *is analytic in* U *and*

$$\mathrm{Re}\,H(p(z),zp'(z))>0,\ z\in\mathrm{U},$$

then $\mathrm{Re}\,p(z)>0$*,* $z\in\mathrm{U}$.

The next result deals with the solutions of the Briot–Bouquet differential equation (2.1), and more general forms of the following lemma may be found in [7, Theorem 1].

Lemma 2.3. *[7] Let* $\beta,\gamma\in\mathbb{C}$ *with* $\beta\neq 0$ *and let* $h\in H(\mathrm{U})$*, with* $h(0)=c$*. If* $\mathrm{Re}\,[\beta h(z)+\gamma]>0$*,* $z\in\mathrm{U}$*, then the solution q of the differential equation*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z),\qquad(2.1)$$

with $q(0)=c$, is analytic in U and satisfies $\mathrm{Re}[\beta q(z)+\gamma]>0$, $z\in \mathrm{U}$. *Moreover, if $c\neq 0$, then the solution is given by*

$$q(z)=z^{\gamma}[H(z)]^{\beta c}\left(\beta\int_0^z [H(t)]^{\beta c}t^{\gamma-1}\,\mathrm{d}t\right)^{-1}-\frac{\gamma}{\beta}, \tag{2.2}$$

where

$$H(z)=z\exp\int_0^z \frac{h(t)-c}{ct}\,\mathrm{d}t. \tag{2.3}$$

(All powers are principal ones).

Remark that in [8, Theorem 3.2d], the assumption $\mathrm{Re}\,[\beta h(z)+\gamma]>0$, $z\in\mathrm{U}$, of the above lemma was replaced by a more general one, and it is also given the solution of (2.1) for $c=0$.

As in [9], let denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{\mathrm{U}}\setminus E(f)$, where

$$E(f)=\left\{\zeta\in\partial\mathrm{U}:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and such that $f'(\zeta)\neq 0$ for $\zeta\in\partial\mathrm{U}\setminus E(f)$.

Lemma 2.4. *[9, Theorem 7] Let $q\in H[a,1]$, let $\chi:\mathbb{C}^2\to\mathbb{C}$ and set $\chi(q(z),zq'(z))\equiv h(z)$. If $L(z;t)=\chi(q(z),tzq'(z))$ is a subordination chain and $p\in H[a,1]\cap\mathcal{Q}$, then*

$$h(z)\prec\chi(p(z),zp'(z)) \quad \textit{implies} \quad q(z)\prec p(z).$$

Furthermore, if $\chi(q(z),zq'(z))=h(z)$ has a univalent solution $q\in\mathcal{Q}$, then q is the best subordinant.

Like in [6] and [8], let $\Omega\subset\mathbb{C}$, $q\in\mathcal{Q}$ and n be a positive integer. The *class of admissible functions* $\Psi_n[\Omega,q]$ is the class of those functions $\psi:\mathbb{C}^3\times\mathrm{U}\to\mathbb{C}$ that satisfy the *admissibility* condition

$$\psi(r,s,t;z)\notin\Omega,$$

whenever $r=q(\zeta)$, $s=m\zeta q'(\zeta)$, $\mathrm{Re}\,\frac{t}{s}+1\geq m\,\mathrm{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right]$, $z\in\mathrm{U}$, $\zeta\in\partial\mathrm{U}\setminus E(q)$ and $m\geq n$. This class will be denoted by $\Psi_n[\Omega,q]$.

We write $\Psi[\Omega,q] \equiv \Psi_1[\Omega,q]$. For the special case when $\Omega \neq \mathbb{C}$ is a simply connected domain and h is a conformal mapping of U onto Ω, we use the notation $\Psi_n[h,q] \equiv \Psi_n[\Omega,q]$.

The following lemma is a key result in the theory of *sharp differential subordinations*:

Remark 2.1. *If* $\psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$*, then the above defined admissibility condition reduces to*

$$\psi(q(\zeta), m\zeta q'(\zeta); z) \notin \Omega,$$

when $z \in \mathrm{U}$*,* $\zeta \in \partial\mathrm{U} \setminus E(q)$ *and* $m \geq n$*.*

Lemma 2.5. *[6], [8] Let h be univalent in* U *and* $\psi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$*. Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2 q''(z); z) = h(z)$$

has a solution q, with $q(0) = a$*, and one of the following conditions is satisfied:*

(i) $q \in \mathcal{Q}$ *and* $\psi \in \Psi[h,q]$

(ii) *q is univalent in* U *and* $\psi \in \Psi[h,q_\rho]$*, for some* $\rho \in (0,1)$*, where* $q_\rho(z) = q(\rho z)$*, or*

(iii) *q is univalent in* U *and there exists* $\rho_0 \in (0,1)$ *such that* $\psi \in \Psi[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$*, where* $h_\rho(z) = h(\rho z)$ *and* $q_\rho(z) = q(\rho z)$*.*

If $p(z) = a + a_1 z + \ldots \in H(\mathrm{U})$ *and* $\psi(p(z), zp'(z), z^2 p''(z); z) \in H(\mathrm{U})$*, then*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z) \quad \textit{implies} \quad p(z) \prec q(z)$$

and q is the best dominant.

3 Main results

Unless otherwise mentioned, we assume throughout this paper that $b = b_1 + ib_2 \in \mathbb{C} \setminus \mathbb{Z}^-$, with b_1, $b_2 \in \mathbb{R}$, $s \in \mathbb{C}$, $p \in \mathbb{N}$, $\tau > 0$, and all the powers are principal ones.

We begin by proving the following subordination theorem:

Theorem 3.1. *Let $\alpha < 1$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, with $\operatorname{Re} b > -\alpha$. For a given function $g \in A^\tau(p)$ of the form $g(z) = b_p z^p + \sum\limits_{n=1}^{\infty} b_{n+p} z^{n+p}$, $z \in \mathrm{U}$, suppose that*

$$\operatorname{Re}\left[1 + \frac{z\phi''(z)}{\phi'(z)}\right] > -\delta,\ z \in \mathrm{U}, \tag{3.1}$$

where

$$\phi(z) = \frac{(1-\alpha)\mathscr{I}^s_{p,b} g(z) + \alpha \mathscr{I}^{s+1}_{p,b} g(z)}{z^{p-1}}, \tag{3.2}$$

and

$$\delta = \delta(\alpha; b) := \frac{(1-\alpha)^2 + |\alpha + b|^2 - \sqrt{\left[(1-\alpha)^2 + |\alpha+b|^2\right]^2 - 4(1-\alpha)^2(\alpha + \operatorname{Re} b)^2}}{4(1-\alpha)(\alpha + \operatorname{Re} b)}. \tag{3.3}$$

If $f \in A(p)$ such that

$$\frac{(1-\alpha)\mathscr{I}^s_{p,b} f(z) + \alpha \mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}} \prec \frac{(1-\alpha)\mathscr{I}^s_{p,b} g(z) + \alpha \mathscr{I}^{s+1}_{p,b} g(z)}{z^{p-1}},$$

then

$$F(z) := \frac{\mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}} \prec G(z) := \frac{\mathscr{I}^{s+1}_{p,b} g(z)}{z^{p-1}},$$

and the function G is the best dominant.

Proof. If we denote

$$\varphi(z) = \frac{(1-\alpha)\mathscr{I}^s_{p,b} f(z) + \alpha \mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}}$$

and

$$F(z) = \frac{\mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}}, \quad G(z) = \frac{\mathscr{I}^{s+1}_{p,b} g(z)}{z^{p-1}}, \tag{3.4}$$

then we need to prove that $\varphi(z) \prec \phi(z)$ implies $F(z) \prec G(z)$. Remark that the assumption $\varphi(z) \prec \phi(z)$ implies $|a_p| \le |b_p|$, that is $\left|f^{(p)}(0)\right| \le \left|g^{(p)}(0)\right|$.

Differentiating the second part of the relation (3.4), by using the identity (1.2) we have

$$\mathscr{I}^s_{p,b} g(z) = \frac{1}{1+b}\left[z^p G'(z) + b z^{p-1} G(z)\right],$$

and replacing the left-hand side of the above relation in (3.2) we get

$$(1+b)\phi(z) = (\alpha+b)G(z) + (1-\alpha)zG'(z), \tag{3.5}$$

and by differentiating the above relation we get

$$\phi'(z) = G'(z) + \frac{1-\alpha}{1+b}zG''(z).$$

Now, let consider the differential equation

$$q(z) + \frac{zq'(z)}{q(z)+\frac{\alpha+b}{1-\alpha}} = 1 + \frac{z\phi''(z)}{\phi'(z)} \equiv h(z). \tag{3.6}$$

It is easy to check that the values of δ given by (3.3) satisfies the inequality $0 < \delta \leq \frac{1}{2}$, whenever $\alpha < 1$ and $\operatorname{Re} b > -\alpha$. Consequently, the assumption (3.1) implies that the function ϕ is close-to-convex, hence univalent in the unit disk U. It follows that $\phi'(z) \neq 0$ for all $z \in \mathrm{U}$, and thus $h \in H(\mathrm{U})$.

From (3.1), using the assumptions $\alpha < 1$ and $b_1 = \operatorname{Re} b > -\alpha$, we have

$$\operatorname{Re}\left[h(z) + \frac{\alpha+b}{1-\alpha}\right] > -\delta + \frac{\alpha+\operatorname{Re} b}{1-\alpha} \geq 0,\ z \in \mathrm{U}$$

and by using Lemma 2.3 we conclude that the differential equation (3.6) has a solution $q \in H(\mathrm{U})$, with $q(0) = h(0) = 1$ and

$$\operatorname{Re}\left[q(z) + \frac{\alpha+b}{1-\alpha}\right] > 0,\ z \in \mathrm{U}.$$

On the other hand, according to the formulas (2.2) and (2.3), we deduce that the Briot-Bouquet differential equation (3.6) has the solution $q \in H(\mathrm{U})$, with $q(0) = 1$, given by

$$q(z) = \frac{z^{\frac{\alpha+b}{1-\alpha}}H(z)}{\displaystyle\int_0^z H(t)\,t^{\frac{\alpha+b}{1-\alpha}-1}\,\mathrm{d}t} - \frac{\alpha+b}{1-\alpha}, \quad \text{where} \quad H(z) = z\phi'(z).$$

Now, by using (3.6), the above relations implies that

$$q(z) = 1 + \frac{zG''(z)}{G'(z)},$$

where $q \in H(\mathrm{U})$ and $q(0) = 1$.

Now we will use Lemma 2.2 to prove that, under our assumption, the inequality

$$\mathrm{Re}\, q(z) > 0,\ z \in \mathrm{U}, \tag{3.7}$$

holds. Let us put

$$H(u,v) = u + \frac{v}{u + \frac{\alpha+b}{1-\alpha}} + \delta, \tag{3.8}$$

where δ is given by (3.3). From the assumption (3.1), according to (3.6), we obtain

$$\mathrm{Re}\, H(q(z), zq'(z)) > 0,\ z \in \mathrm{U}, \tag{3.9}$$

and we proceed to show that $\mathrm{Re}\, H(is,t) \leq 0$ for all $s,t \in \mathbb{R}$, with $t \leq -(1+s^2)/2$.

From (3.8), using the assumptions $\alpha < 1$ and $b_1 = \mathrm{Re}\, b > -\alpha$, we have

$$\mathrm{Re}\, H(is,t) = \mathrm{Re}\left(is + \frac{t}{is + \frac{\alpha+b}{1-\alpha}} + \delta \right) = \frac{t\,\frac{\alpha+b_1}{1-\alpha}}{\left|is + \frac{\alpha+b}{1-\alpha}\right|^2} + \delta \leq \frac{E(s)}{-2\left|is + \frac{\alpha+b}{1-\alpha}\right|^2},$$

where

$$E(s) = \left(\frac{\alpha + b_1}{1-\alpha} - 2\delta \right) s^2 - \frac{4b_2\delta}{1-\alpha} s - 2\delta \frac{|\alpha+b|^2}{(1-\alpha)^2} + \frac{\alpha+b_1}{1-\alpha},$$

and $b_2 = \mathrm{Im}\, b$. It is well-known that the second order polinomial function $E(s)$ is nonnegative for all $s \in \mathbb{R}$, if and only if

$$\Delta \leq 0 \quad \text{and} \quad \frac{\alpha+b_1}{1-\alpha} - 2\delta > 0, \tag{3.10}$$

where Δ is the discriminant of $E(s)$, i.e.

$$\Delta = -\frac{4(\alpha+b_1)}{(1-\alpha)^2} \left\{ 4(\alpha+b_1)\,\delta^2 - \frac{2\left[(1-\alpha)^2 + |\alpha+b|^2\right]}{1-\alpha}\,\delta + \alpha + b_1 \right\}.$$

We may easily check that the value of δ given by (3.3) is the greater one for which $\Delta \leq 0$. Since this value of δ satisfies the second part of the conditions (3.10), it follows that $\mathrm{Re}\, H(is,t) \leq 0$ for all $s,t \in \mathbb{R}$, with $t \leq -(1+s^2)/2$.

Form (3.9), according to Lemma 2.2, we deduce that the inequality (3.7) holds, hence $G \in K$, that is G is a convex (and univalent) function in the unit disk, hence the following well-known growth and distortion sharp inequalities (see [3]) are true:

$$\frac{r}{1+r} \le |G(z)| \le \frac{r}{1-r}, \text{ if } |z| \le r,$$
$$\frac{1}{(1+r)^2} \le |G'(z)| \le \frac{1}{(1-r)^2}, \text{ if } |z| \le r.$$

If we let

$$L(z;t) = \frac{\alpha+b}{1+b}\, G(z) + \frac{(1-\alpha)(1+t)}{1+b}\, zG'(z), \tag{3.11}$$

from (3.5) we have $L(z;0) = \phi(z)$. Denoting $L(z;t) = a_1(t)z + \dots$, then

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = \frac{\alpha+b+(1-\alpha)(1+t)}{1+b}\, G'(0) = \frac{\alpha+b+(1-\alpha)(1+t)}{1+b}\, b_p,$$

hence $\lim\limits_{t\to+\infty} |a_1(t)| = +\infty$, and because $\alpha < 1$ and $\operatorname{Re} b > -\alpha$ we obtain $a_1(t) \neq 0$, $\forall t \ge 0$.

From (3.11) we may easily deduce the equality

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = \operatorname{Re}\left[\frac{\alpha+b}{1-\alpha} + (1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right] = \frac{\alpha+\operatorname{Re} b}{1-\alpha} + (1+t)\operatorname{Re} q(z).$$

Using the inequality (3.7) together with the assumptions $\alpha < 1$ and $\operatorname{Re} b > -\alpha$, the above relation yields that

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0,\ \forall z \in \mathrm{U},\ \forall t \ge 0.$$

Since $g \in A^{\tau}(p)$, from the definition (3.11), for all $t \ge 0$ we have

$$\begin{aligned}\frac{|L(z;t)|}{|a_1(t)|} &\le \frac{|\alpha+b|\,|G(z)| + |1-\alpha||1+t|\,|zG'(z)|}{|b_p|\,|1+b+(1-\alpha)t|} \le \\ &\frac{|\alpha+b|\,|G(z)| + |1-\alpha||1+t|\,|zG'(z)|}{\tau\,|1+b+(1-\alpha)t|}.\end{aligned} \tag{3.12}$$

Using the right-hand sides of these inequalities in (3.12), we deduce that

$$\frac{|L(z;t)|}{|a_1(t)|} \le \frac{1}{\tau}\left[\frac{|\alpha+b|}{|1-\alpha|}\frac{r}{1-r}\,\varphi_1(t) + \frac{r}{(1-r)^2}\,\varphi_2(t)\right],\ |z| \le r,\ \forall t \ge 0, \tag{3.13}$$

where

$$\varphi_1(t) = \frac{1}{\left|t + \frac{1+b}{1-\alpha}\right|} \quad \text{and} \quad \varphi_2(t) = \frac{|t+1|}{\left|t + \frac{1+b}{1-\alpha}\right|}.$$

Since $\mathrm{Re}\,\frac{1+b}{1-\alpha} > 0$ whenever $\mathrm{Re}\,b > -\alpha$ and $\alpha < 1$, it follows

$$\left|t + \frac{1+b}{1-\alpha}\right| \geq \left|\frac{1+b}{1-\alpha}\right|, \ \forall t \geq 0,$$

hence

$$\varphi_1(t) \leq \left|\frac{1-\alpha}{1+b}\right|, \ t \geq 0. \tag{3.14}$$

Moreover, since $\mathrm{Re}\,\frac{1+b}{1-\alpha} > 1$ whenever $\mathrm{Re}\,b > -\alpha$ and $\alpha < 1$, we obtain

$$\frac{|t+1|}{\left|t + \frac{1+b}{1-\alpha}\right|} < 1, \ \forall t \geq 0,$$

hence

$$\varphi_2(t) < 1, \ t \geq 0. \tag{3.15}$$

Using the inequalities (3.14) and (3.15), from (3.13) we deduce that

$$\frac{|L(z;t)|}{|a_1(t)|} < \frac{1}{\tau}\left[\frac{r}{(1-r)^2} + \left|\frac{\alpha+b}{1+b}\right| \frac{r}{1-r}\right], \ |z| \leq r, \ \forall t \geq 0,$$

hence the second assumption of Lemma 2.1 holds, and according to this lemma we conclude that the function $L(z;t)$ is a subordination chain.

Now, by using Lemma 2.5, we will show that $F(z) \prec G(z)$. Without loss of generality, we can assume that ϕ and G are analytic and univalent in $\overline{\mathrm{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we could replace ϕ with $\phi_\rho(z) = \phi(\rho z)$ and G with $G_\rho(z) = G(\rho z)$, where $\rho \in (0,1)$. These new functions will have the desired properties and we would prove our result using part (iii) of Lemma 2.5.

With our above assumption, we will use part (i) of the Lemma 2.5. If we denote by $\psi(G(z), zG'(z)) = \phi(z)$, we only need to show that $\psi \in \Psi[\phi, G]$, i.e. ψ is an admissible function. Because

$$\psi(G(\zeta), m\zeta G'(\zeta)) = \frac{\alpha+b}{1+b} G(z) + \frac{(1-\alpha)(1+t)}{1+b} zG'(z) = L(\zeta;t),$$

where $m = 1 + t$, $t \geq 0$, since $L(z;t)$ is a subordination chain and $\phi(z) = L(z;0)$, it follows that

$$\psi(G(\zeta), m\zeta G'(\zeta)) \notin \phi(\mathrm{U}).$$

According to the Remark 2.1 we have $\psi \in \Psi[\phi, G]$, and using Lemma 2.5 we obtain that $F(z) \prec G(z)$ and, moreover, G is the best dominant. □

Remark 3.1. *Like we remarked in the proof of the theorem, it is easy to check that the values of δ given by* (3.3) *satisfies the inequality* $0 < \delta \leq \frac{1}{2}$, *whenever* $\alpha < 1$ *and* $\operatorname{Re} b > -\alpha$.

For the special case $b = 0$, $s = -1$ and $p = 1$, Theorem 3.1 reduces to:

Corollary 3.1. *Let* $0 < \alpha < 1$ *and for a given function* $g \in A^{\tau}(1)$ *suppose that the inequality* (3.1) *holds, where*

$$\phi(z) = (1-\alpha)zg'(z) + \alpha g(z), \tag{3.16}$$

and

$$\delta = \delta(\alpha;0) = \frac{(1-\alpha)^2 + \alpha^2 - \left|(1-\alpha)^2 - \alpha^2\right|}{4\alpha(1-\alpha)} = \begin{cases} \dfrac{\alpha}{2(1-\alpha)}, & \text{if} \quad 0 < \alpha \leq 1/2, \\ \dfrac{1-\alpha}{2\alpha}, & \text{if} \quad 1/2 \leq \alpha < 1. \end{cases} \tag{3.17}$$

If $f \in A(1)$ *such that*

$$(1-\alpha)zf'(z) + \alpha f(z) \prec (1-\alpha)zg'(z) + \alpha g(z),$$

then

$$f(z) \prec g(z),$$

and the function g *is the best dominant.*

Now we will prove a dual of Theorem 3.1, in the sense that the subordinations are replaced by superordinations.

Theorem 3.2. *Let $\alpha < 1$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, with $\operatorname{Re} b > -\alpha$. For a given function $g \in A^\tau(p)$ of the form $g(z) = b_p z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}$, $z \in \mathrm{U}$, suppose that the function ϕ defined by* (3.2) *satisfies the condition* (3.1)*, with δ given by* (3.3).

Let $f \in A(p)$ such that $\dfrac{(1-\alpha)\mathscr{I}^s_{p,b} f(z) + \alpha \mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}}$ is univalent in U *and $\dfrac{\mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}} \in \mathscr{Q}$. Then,*

$$\frac{(1-\alpha)\mathscr{I}^s_{p,b} g(z) + \alpha \mathscr{I}^{s+1}_{p,b} g(z)}{z^{p-1}} \prec \frac{(1-\alpha)\mathscr{I}^s_{p,b} f(z) + \alpha \mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}}$$

implies

$$G(z) := \frac{\mathscr{I}^{s+1}_{p,b} g(z)}{z^{p-1}} \prec F(z) := \frac{\mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}},$$

and the function G is the best subordinant.

Proof. Denoting

$$\varphi(z) = \frac{(1-\alpha)\mathscr{I}^s_{p,b} f(z) + \alpha \mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}}$$

and

$$F(z) = \frac{\mathscr{I}^{s+1}_{p,b} f(z)}{z^{p-1}}, \quad G(z) = \frac{\mathscr{I}^{s+1}_{p,b} g(z)}{z^{p-1}}, \tag{3.18}$$

then we need to prove that $\phi(z) \prec \varphi(z)$ implies $G(z) \prec F(z)$. Like in the proof of the previous theorem, we remark that the assumption $\phi(z) \prec \varphi(z)$ implies $|b_p| \le |a_p|$, that is $\left|g^{(p)}(0)\right| \le \left|f^{(p)}(0)\right|$.

If we differentiate the second part of the relation (3.18), using the identity (1.2) we obtain

$$\mathscr{I}^s_{p,b} g(z) = \frac{1}{1+b}\left[z^p G'(z) + b z^{p-1} G(z)\right].$$

Replacing the left-hand side of the above relation in (3.2) we have

$$\phi(z) = \frac{\alpha + b}{1+b} G(z) + \frac{1-\alpha}{1+b} z G'(z). \tag{3.19}$$

If we let $q(z) = 1 + \dfrac{zG''(z)}{G'(z)}$, like in the proof of Theorem 3.1 it follows that $q \in H(\mathrm{U})$ and the inequality (3.7) holds, i.e. $\operatorname{Re} q(z) > 0$ for all $z \in \mathrm{U}$.

Letting

$$L(z;t)=\frac{\alpha+b}{1+b}\,G(z)+\frac{(1-\alpha)t}{1+b}\,zG'(z), \tag{3.20}$$

from (3.19) we have $L(z;1)=\phi(z)$. Thus, $L(z;t)=a_1(t)z+\ldots$, and then

$$a_1(t)=\frac{\partial L(0;t)}{\partial z}=\frac{\alpha+b+(1-\alpha)t}{1+b}\,G'(0)=\frac{\alpha+b+(1-\alpha)t}{1+b}\,b_p,$$

hence $\lim\limits_{t\to+\infty}|a_1(t)|=+\infty$, and because $\alpha<1$ and $\operatorname{Re}b>-\alpha$ we obtain $a_1(t)\neq 0$, $\forall t\geq 0$.

From (3.20), a simple computation shows that

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right]=\operatorname{Re}\left[\frac{\alpha+b}{1-\alpha}+t\left(1+\frac{zG''(z)}{G'(z)}\right)\right]=\frac{\alpha+\operatorname{Re}b}{1-\alpha}+t\operatorname{Re}q(z).$$

Since we already mentioned that the inequality (3.7) holds, combining with the assumptions $\alpha<1$ and $\operatorname{Re}b>-\alpha$, the above relation implies that

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right]>0,\ \forall z\in\mathrm{U},\ \forall t\geq 0.$$

Also, for all $t\geq 0$ we have

$$\frac{|L(z;t)|}{|a_1(t)|}\leq\frac{|\alpha+b|\,|G(z)|+|1-\alpha||t|\,|zG'(z)|}{|b_p|\,|\alpha+b+(1-\alpha)t|}. \tag{3.21}$$

and from the right-hand sides of these inequalities in (3.12), since $g\in A^{\tau}(p)$ we obtain that

$$\frac{|L(z;t)|}{|a_1(t)|}\leq\frac{1}{\tau}\left[\frac{|\alpha+b|}{|1-\alpha|}\frac{r}{1-r}\,\varphi_1(t)+\frac{r}{(1-r)^2}\,\varphi_2(t)\right],\ |z|\leq r,\ \forall t\geq 0, \tag{3.22}$$

where

$$\varphi_1(t)=\frac{1}{\left|t+\frac{\alpha+b}{1-\alpha}\right|}\quad\text{and}\quad\varphi_2(t)=\frac{|t|}{\left|t+\frac{\alpha+b}{1-\alpha}\right|}.$$

Since $\operatorname{Re}\frac{1+b}{1-\alpha}>0$ for $\operatorname{Re}b>-\alpha$ and $\alpha<1$, it follows

$$\left|t+\frac{\alpha+b}{1-\alpha}\right|\geq\left|\frac{\alpha+b}{1-\alpha}\right|\quad\text{and}\quad|t|<\left|t+\frac{\alpha+b}{1-\alpha}\right|,\ \forall t\geq 0,$$

and thus

$$\varphi_1(t)\leq\left|\frac{1-\alpha}{\alpha+b}\right|,\quad \varphi_2(t)<1,\ t\geq 0.$$

Using the above inequalities together with (3.21) we deduce that

$$\frac{|L(z;t)|}{|a_1(t)|} < \frac{1}{\tau}\left[\frac{r}{1-r} + \frac{r}{(1-r)^2}\right],\ |z| \le r,\ \forall t \ge 0,$$

hence the second assumption of Lemma 2.1 holds. Now, from this lemma we obtain that the function $L(z;t)$ is a subordination chain.

Using the fact that (3.7) holds, since $G \in A$, we have that G is convex (univalent) in U. Thus, if we denote by $\chi(G(z), zG'(z)) = \phi(z)$, then $L(z;t) = \chi(q(z), tzq'(z))$, and the differential equation $\chi(G(z), zG'(z)) = \phi(z)$ has the univalent solution G.

According to Lemma 2.4, we conclude that $\phi(z) \prec \varphi(z)$ implies $G(z) \prec F(z)$, and furthermore, since G is a univalent solution of the differential equation $\chi(G(z), zG'(z)) = \phi(z)$, it follows that it is the best subordinant of the given differential superordination. □

Taking $b = 0$, $s = -1$ and $p = 1$ in Theorem 3.2, we obtain the next special case:

Corollary 3.2. *Let* $0 < \alpha < 1$ *and for a given function* $g \in A^{\tau}(1)$ *suppose that the function* ϕ *defined by* (3.16) *satisfies the condition* (3.1)*, with* δ *given by* (3.17)*.*

Let $f \in A(1)$ *such that* $(1-\alpha)zf'(z) + \alpha f(z)$ *is univalent in* U *and* $f \in \mathcal{Q}$*. Then,*

$$(1-\alpha)zg'(z) + \alpha g(z) \prec (1-\alpha)zf'(z) + \alpha f(z)$$

implies

$$g(z) \prec f(z),$$

and the function g is the best subordinant.

Combining the Theorem 3.2 with Theorem 3.1, we obtain the following *sandwich-type theorem*:

Theorem 3.3. *Let* $\alpha < 1$ *and* $b \in \mathbb{C} \setminus \mathbb{Z}^-$*, with* $\operatorname{Re} b > -\alpha$*. For the two given functions* $g_k \in A^{\tau_k}(p)$*,* $\tau_k > 0$ $(k = 1, 2)$*, suppose that*

$$Re\left[1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right] > -\delta,\ z \in \mathrm{U}, \quad (k = 1, 2),$$

where

$$\phi_k(z)=\frac{(1-\alpha)\mathscr{I}^{s}_{p,b}g_k(z)+\alpha\mathscr{I}^{s+1}_{p,b}g_k(z)}{z^{p-1}},\quad (k=1,2), \tag{3.23}$$

and δ *is given by* (3.3).

Let $f\in A(p)$ *such that* $\frac{(1-\alpha)\mathscr{I}^{s}_{p,b}f(z)+\alpha\mathscr{I}^{s+1}_{p,b}f(z)}{z^{p-1}}$ *is univalent in* U *and* $\frac{\mathscr{I}^{s+1}_{p,b}f(z)}{z^{p-1}}\in\mathscr{Q}$. *Then,*

$$\phi_1(z)=\frac{(1-\alpha)\mathscr{I}^{s}_{p,b}g_1(z)+\alpha\mathscr{I}^{s+1}_{p,b}g_1(z)}{z^{p-1}}\prec\frac{(1-\alpha)\mathscr{I}^{s}_{p,b}f(z)+\alpha\mathscr{I}^{s+1}_{p,b}f(z)}{z^{p-1}}\prec$$
$$\phi_2(z)=\frac{(1-\alpha)\mathscr{I}^{s}_{p,b}g_2(z)+\alpha\mathscr{I}^{s+1}_{p,b}g_2(z)}{z^{p-1}}$$

implies

$$G_1(z):=\frac{\mathscr{I}^{s+1}_{p,b}g_1(z)}{z^{p-1}}\prec F(z):=\frac{\mathscr{I}^{s+1}_{p,b}f(z)}{z^{p-1}}\prec G_2(z):=\frac{\mathscr{I}^{s+1}_{p,b}g_2(z)}{z^{p-1}}.$$

Moreover, the functions G_1 *and* G_2 *are respectively the best subordinant and the best dominant.*

The assumptions that the functions

$$\phi_3(z)=\frac{(1-\alpha)\mathscr{I}^{s}_{p,b}f(z)+\alpha\mathscr{I}^{s+1}_{p,b}f(z)}{z^{p-1}} \tag{3.24}$$

and

$$\Phi(z)=\frac{\mathscr{I}^{s+1}_{p,b}f(z)}{z^{p-1}} \tag{3.25}$$

need to be univalent in U are difficult to be checked. Thus, in the following *sandwich-type result* we will replace these assumptions by another sufficient conditions, that are more easy to be verified.

Corollary 3.3. *Let* $\alpha<1$ *and* $b\in\mathbb{C}\setminus\mathbb{Z}^-$, *with* $\operatorname{Re}b>-\alpha$. *For the given functions* $f\in A(p)$, $g_k\in A^{\tau_k}(p)$, $\tau_k>0$ $(k=1,2)$, *suppose that*

$$\operatorname{Re}\left[1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right]>-\delta,\ z\in\mathrm{U},\quad (k=1,2,3), \tag{3.26}$$

where ϕ_1, ϕ_2 *and* ϕ_3 *are defined by* (3.23) *and* (3.24) *respectively, and* δ *is given by* (3.3). *Then,*

$$\phi_1(z)=\frac{(1-\alpha)\mathscr{I}^{s}_{p,b}g_1(z)+\alpha\mathscr{I}^{s+1}_{p,b}g_1(z)}{z^{p-1}}\prec\phi_3(z)=\frac{(1-\alpha)\mathscr{I}^{s}_{p,b}f(z)+\alpha\mathscr{I}^{s+1}_{p,b}f(z)}{z^{p-1}}\prec$$
$$\phi_2(z)=\frac{(1-\alpha)\mathscr{I}^{s}_{p,b}g_2(z)+\alpha\mathscr{I}^{s+1}_{p,b}g_2(z)}{z^{p-1}}$$

implies

$$G_1(z) := \frac{\mathscr{I}_{p,b}^{s+1} g_1(z)}{z^{p-1}} \prec \Phi(z) = \frac{\mathscr{I}_{p,b}^{s+1} f(z)}{z^{p-1}} \prec G_2(z) := \frac{\mathscr{I}_{p,b}^{s+1} g_2(z)}{z^{p-1}}.$$

Moreover, the functions G_1 and G_2 are respectively the best subordinant and the best dominant.

Proof. In order to prove our corollary, we have to show that the condition (3.26) for $k = 3$ implies the univalence of the functions ϕ_3 and Φ defined by (3.24) and (3.25).

Since $0 < \delta \leq \frac{1}{2}$ from Remark 3.1, the condition (3.26) for $k = 3$ means that $\phi_3 \in K(-\delta) \subseteq K\left(-\frac{1}{2}\right)$, and from [4] it follows that ϕ_3 is a close-to-convex function in U, hence it is univalent in U. Furthermore, by using the same techniques as in the proof of Theorem 3.1 we can prove the convexity (univalence) of Φ and so the details may be omitted. Therefore, by applying Theorem 3.3 we obtain the desired result. □

The following special case of Corollary 3.3 is obtained for $b = 0$, $s = -1$ and $p = 1$:

Corollary 3.4. *Let $0 < \alpha < 1$ and for the given functions $f \in A(1)$, $g_k \in A^{\tau_k}(1)$, $\tau_k > 0$ $(k = 1, 2)$, suppose that the inequalities* (3.26) *hold, where*

$$\phi_1(z) = (1-\alpha)zg_1'(z) + \alpha g_1(z), \quad \phi_2(z) = (1-\alpha)zg_2'(z) + \alpha g_2(z),$$
$$\phi_3(z) = (1-\alpha)zf'(z) + \alpha f(z),$$

and δ is given by (3.17)*. Then,*

$$(1-\alpha)zg_1'(z) + \alpha g_1(z) \prec (1-\alpha)zf'(z) + \alpha f(z) \prec (1-\alpha)zg_2'(z) + \alpha g_2(z)$$

implies

$$g_1(z) \prec f(z) \prec g_2(z).$$

Moreover, the functions g_1 and g_2 are respectively the best subordinant and the best dominant.

Next, we will give an interesting special case of our main results, obtained for an appropriate choice of the function g and the corresponding parameters.

Thus, for $\alpha < 1$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, with $\operatorname{Re} b > -\alpha$, let consider the function $g \in A(1)$ defined by

$$g(z) = a_1 z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1},\ z \in \mathrm{U},$$

with

$$a_{n+1} = \frac{1}{n+1} \frac{n+1+b}{(1-\alpha)n+1+b} \left(\frac{n+1+b}{1+b}\right)^s \binom{-2(\delta+1)}{n},\ n \geq 1,$$

where δ is given by (3.3),

$$\binom{\sigma}{n} = \frac{\sigma(\sigma-1)\ldots(\sigma-n+1)}{n!},\ \sigma \in \mathbb{C},\ n \in \mathbb{N},$$

and all powers are principal ones. The coefficient $a_1 \in \mathbb{C} \setminus \{0\}$ is choose to be fixed, hence $g \in A^{\tau}(1)$ with $\tau = |a_1| > 0$.

If the function ϕ is defined by (3.2) with $p = 1$, then

$$\phi(z) = a_1 \frac{1-(1+z)^{-(2\delta+1)}}{2\delta+1} = a_1 z + \ldots,\ z \in \mathrm{U},$$

where the power is principal one, i.e.

$$(1+z)^{-(2\delta+1)}\Big|_{z=0} = 1.$$

A simple computation shows that

$$\operatorname{Re}\left[1 + \frac{z\phi''(z)}{\phi'(z)}\right] = \operatorname{Re} \frac{1-(2\delta+1)z}{1+z} > -\delta,\ z \in \mathrm{U},$$

and from Theorem 3.1 and Theorem 3.2 we obtain:

Example 3.1. *Let $\alpha < 1$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, with $\operatorname{Re} b > -\alpha$, and let δ be given by* (3.3).

1. If $f \in A(1)$ such that

$$(1-\alpha)\mathscr{I}_{1,b}^{s} f(z) + \alpha \mathscr{I}_{1,b}^{s+1} f(z) \prec a_1 \frac{1-(1+z)^{-(2\delta+1)}}{2\delta+1}, \quad (a_1 \neq 0),$$

then

$$\mathscr{I}_{1,b}^{s+1}f(z) \prec a_1 z + \sum_{n=1}^{\infty} \frac{1}{n+1}\frac{1+b}{(1-\alpha)n+1+b}\binom{-2(\delta+1)}{n} z^{n+1},$$

and the right-hand side function is the best dominant (the power is principal one).

2. If $f \in A(1)$ such that $(1-\alpha)\mathscr{I}_{1,b}^{s}f(z) + \alpha\mathscr{I}_{1,b}^{s+1}f(z)$ is univalent in U *and $\mathscr{I}_{1,b}^{s+1}f(z) \in \mathscr{Q}$, then*

$$a_1 \frac{1-(1+z)^{-(2\delta+1)}}{2\delta+1} \prec (1-\alpha)\mathscr{I}_{1,b}^{s}f(z) + \alpha\mathscr{I}_{1,b}^{s+1}f(z), \quad (a_1 \neq 0),$$

implies

$$a_1 z + \sum_{n=1}^{\infty} \frac{1}{n+1}\frac{1+b}{(1-\alpha)n+1+b}\binom{-2(\delta+1)}{n} z^{n+1} \prec \mathscr{I}_{1,b}^{s+1}f(z),$$

and the right-hand side function is the best subordinant (the power is principal one).

By similar reasons, for the above mentioned choice of the function g, the Theorem 3.3 reduces to the following *sandwich-type results*:

Example 3.2. *Let $\alpha < 1$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, with $\operatorname{Re} b > -\alpha$, and let $\delta_1, \delta_2 \leq \delta$ where δ is given by* (3.3)*.*

If $f \in A(1)$ such that $(1-\alpha)\mathscr{I}_{1,b}^{s}f(z) + \alpha\mathscr{I}_{1,b}^{s+1}f(z)$ is univalent in U *and $\mathscr{I}_{1,b}^{s+1}f(z) \in \mathscr{Q}$, then*

$$a_1 \frac{1-(1+z)^{-(2\delta_1+1)}}{2\delta_1+1} \prec (1-\alpha)\mathscr{I}_{1,b}^{s}f(z) + \alpha\mathscr{I}_{1,b}^{s+1}f(z) \prec \widetilde{a_1} \frac{1-(1+z)^{-(2\delta_2+1)}}{2\delta_2+1}, \ (a_1, \widetilde{a_1} \neq 0),$$

implies

$$a_1 z + \sum_{n=1}^{\infty} \frac{1}{n+1}\frac{1+b}{(1-\alpha)n+1+b}\binom{-2(\delta_1+1)}{n} z^{n+1} \prec \mathscr{I}_{1,b}^{s+1}f(z) \prec$$
$$\widetilde{a_1} z + \sum_{n=1}^{\infty} \frac{1}{n+1}\frac{1+b}{(1-\alpha)n+1+b}\binom{-2(\delta_2+1)}{n} z^{n+1}.$$

Moreover, the left-hand side functions and the right-hand side are, respectively, the best subordinant and the best dominant (all powers are principal ones).

For $b = 0$ and $s = -1$, the Example 3.2 gives us the next special case:

Example 3.3. *Let* $0<\alpha<1$ *and let* $\delta_1,\delta_2\leq\delta$ *where* δ *is given by* (3.17).

If $f\in A(1)$ *such that* $(1-\alpha)zf'(z)+\alpha f(z)$ *is univalent in* U *and* $f\in\mathcal{Q}$, *then*

$$a_1\,\frac{1-(1+z)^{-(2\delta_1+1)}}{2\delta_1+1}\prec(1-\alpha)zf'(z)+\alpha f(z)\prec\widetilde{a_1}\,\frac{1-(1+z)^{-(2\delta_2+1)}}{2\delta_2+1},\quad(a_1,\widetilde{a_1}\neq0),$$

implies

$$a_1 z+\sum_{n=1}^{\infty}\frac{1}{n+1}\,\frac{1}{(1-\alpha)n+1}\binom{-2(\delta_1+1)}{n}z^{n+1}\prec f(z)\prec$$
$$\widetilde{a_1}\,z+\sum_{n=1}^{\infty}\frac{1}{n+1}\,\frac{1}{(1-\alpha)n+1}\binom{-2(\delta_2+1)}{n}z^{n+1}.$$

Moreover, the left-hand side functions and the right-hand side are, respectively, the best subordinant and the best dominant (all powers are principal ones).

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Some Inequalities in Inner Product Spaces Related to Buzano's and Grüss' Results

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Abstract

Some inequalities in inner product spaces related to Buzano's and Grüss' results are given. Applications for discrete and integral inequalities are provided as well.

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1 Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|\,\|y\| \geq |\langle x, y\rangle| \text{ for any } x, y \in H. \tag{1.1}$$

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The equality case holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [4] (see also [19]) established the following refinement of (1.1):

$$\|x\| \|y\| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \geq |\langle x, y\rangle| \tag{1.2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \geq |\langle x, e\rangle \langle e, y\rangle| - |\langle x, y\rangle|$$

and by (1.2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \\ &\geq 2 |\langle x, e\rangle \langle e, y\rangle| - |\langle x, y\rangle|, \end{aligned}$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y\rangle|] \geq |\langle x, e\rangle \langle e, y\rangle| \tag{1.3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

In [5], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

Theorem 1.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.4}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma|. \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

For other Schwarz, Buzano and Grüss related inequalities in inner product spaces, see [1]-[3], [4]-[13], [17]-[20], [22]-[29], and the monographs [14], [15] and [16].

2 Main Results

The following results hold:

Theorem 2.1. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be an inner product space over the real or complex numbers field* $\mathbb{K}$*. If* $x, y, e, f \in H$ *with* $\|e\| = \|f\| = 1$*, then*

$$\begin{aligned} & \|x\| \|y\| - |\langle x, e \rangle \langle f, y \rangle| \\ & \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle - \langle x, f \rangle \langle f, y \rangle + \langle x, e \rangle \langle f, y \rangle \langle e, f \rangle| . \end{aligned} \tag{2.6}$$

Proof. Using Schwarz inequality we have

$$\|x - \langle x, e \rangle e\|^2 \|y - \langle y, f \rangle f\|^2 \geq |\langle x - \langle x, e \rangle e, y - \langle y, f \rangle f \rangle|^2 \tag{2.7}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

Since

$$\|x - \langle x, e \rangle e\|^2 = \|x\|^2 - |\langle x, e \rangle|^2 , \ \ \|y - \langle y, f \rangle f\|^2 = \|y\|^2 - |\langle y, f \rangle|^2$$

and

$$\langle x - \langle x, e \rangle e, y - \langle y, f \rangle f \rangle = \langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle - \langle x, f \rangle \langle f, y \rangle + \langle x, e \rangle \langle f, y \rangle \langle e, f \rangle ,$$

then by (2.7) we get

$$\begin{aligned} & \left(\|x\|^2 - |\langle x, e \rangle|^2\right) \left(\|y\|^2 - |\langle y, f \rangle|^2\right) \\ & \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle - \langle x, f \rangle \langle f, y \rangle + \langle x, e \rangle \langle f, y \rangle \langle e, f \rangle|^2 \end{aligned} \tag{2.8}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

Using the elementary inequality

$$(ac-bd)^2 \geq \left(a^2-b^2\right)\left(c^2-d^2\right)$$

that holds for any real numbers $a, b, c, d \in \mathbb{R}$, we have

$$(\|x\|\,\|y\| - |\langle x, e\rangle|\,|\langle y, f\rangle|)^2 \geq \left(\|x\|^2 - |\langle x, e\rangle|^2\right)\left(\|y\|^2 - |\langle y, f\rangle|^2\right) \tag{2.9}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

By Schwarz inequality for the pairs (x, e) and (y, f) we have

$$\|x\| \geq |\langle x, e\rangle| \text{ and } \|y\| \geq |\langle y, f\rangle|,$$

which shows that

$$\|x\|\,\|y\| - |\langle x, e\rangle|\,|\langle y, f\rangle| \geq 0,$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

Making use of (2.8) and (2.9) we get

$$\begin{aligned}(\|x\|\,\|y\| - |\langle x, e\rangle|\,|\langle y, f\rangle|)^2 \\ \geq |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle - \langle x, f\rangle\langle f, y\rangle + \langle x, e\rangle\langle f, y\rangle\langle e, f\rangle|^2\end{aligned} \tag{2.10}$$

and by taking the square root in (2.10) we get the desired result. ∎

Corollary 2.2. *With the assumptions of Theorem 2.1 and if $e \perp f$, i.e. $\langle e, f\rangle = 0$, then we have the inequality*

$$\|x\|\,\|y\| - |\langle x, e\rangle\langle f, y\rangle| \geq |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle - \langle x, f\rangle\langle f, y\rangle|. \tag{2.11}$$

Remark 2.3. *From the inequality (2.11) we have*

$$\begin{aligned}\|x\|\,\|y\| &\geq |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle - \langle x, f\rangle\langle f, y\rangle| + |\langle x, e\rangle\langle f, y\rangle| \\ &\geq |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle - \langle x, f\rangle\langle f, y\rangle \pm \langle x, e\rangle\langle f, y\rangle|\end{aligned} \tag{2.12}$$

By the triangle inequality we also have

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| \geq |\langle x,e\rangle\langle e,y\rangle + \langle x,f\rangle\langle f,y\rangle| - |\langle x,y\rangle|$$

and by the first inequality in (2.14) we get

$$\|x\|\,\|y\| \geq |\langle x,e\rangle\langle e,y\rangle + \langle x,f\rangle\langle f,y\rangle| - |\langle x,y\rangle| + |\langle x,e\rangle\langle f,y\rangle|,$$

which implies

$$\begin{aligned} \|x\|\,\|y\| + |\langle x,y\rangle| &\geq |\langle x,e\rangle\langle e,y\rangle + \langle x,f\rangle\langle f,y\rangle| + |\langle x,e\rangle\langle f,y\rangle| \\ &\geq |\langle x,e\rangle\langle e,y\rangle + \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle| \end{aligned} \tag{2.13}$$

for any $x,y,e,f \in H$ with $\|e\| = \|f\| = 1$ and $e \perp f$.

Corollary 2.4. *With the assumptions of Theorem 2.1 we have*

$$\|x\|\,\|y\| - |\langle x,e\rangle\langle f,y\rangle|\,(1 - |\langle e,f\rangle|) \geq |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| \tag{2.14}$$

and

$$\|x\|\,\|y\| + |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| \geq |\langle x,e\rangle\langle f,y\rangle|\,(|\langle e,f\rangle| + 1). \tag{2.15}$$

Indeed, by the triangle inequality we have

$$\begin{aligned} &|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle| \\ &\geq |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| - |\langle x,e\rangle\langle f,y\rangle\langle e,f\rangle| \end{aligned}$$

and by (2.6) we get (2.14).

By the triangle inequality we also have

$$\begin{aligned} &|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle| \\ &\geq |\langle x,e\rangle\langle f,y\rangle\langle e,f\rangle| - |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| \end{aligned}$$

and by (2.6) we get (2.15).

Remark 2.5. *With the assumptions of Theorem 2.1 and if* $|\langle e,f\rangle|=1,$ *then we have*

$$\|x\|\,\|y\| \geq |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| \tag{2.16}$$

and

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle|\right] \geq |\langle x,e\rangle\langle f,y\rangle|\,. \tag{2.17}$$

If we take $f=e$ *in (2.16) and (2.17), then we get the inequalities*

$$\|x\|\,\|y\| \geq |\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle| \tag{2.18}$$

and

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle|\right] \geq |\langle x,e\rangle\langle e,y\rangle| \tag{2.19}$$

for any $x,y,e\in H$ *with* $\|e\|=1.$

Using the triangle inequality we have

$$|\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle| \geq 2|\langle x,e\rangle\langle e,y\rangle| - |\langle x,y\rangle|$$

and by (2.18) we get

$$\|x\|\,\|y\| \geq |\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle| \geq 2|\langle x,e\rangle\langle e,y\rangle| - |\langle x,y\rangle|\,. \tag{2.20}$$

The inequality between the first and last term in (2.20) is equivalent to Buzano's inequality (1.3).

The following lemma holds, see [6]:

Lemma 2.6. *Let* a,x,A *be vectors in the inner product space* $(H,\langle\cdot,\cdot\rangle)$ *over* $\mathbb{K}$ *with* $a\neq A.$ *Then*

$$\operatorname{Re}\langle A-x,x-a\rangle \geq 0 \tag{2.21}$$

if and only if

$$\left\|x-\frac{a+A}{2}\right\| \leq \frac{1}{2}\|A-a\|\,. \tag{2.22}$$

Proof. Define

$$I_1 := \operatorname{Re}\langle A-x, x-a\rangle \text{ and } I_2 := \frac{1}{4}\|A-a\|^2 - \left\|x - \frac{a+A}{2}\right\|^2 .$$

A simple calculation shows that

$$I_1 = I_2 = \operatorname{Re}\left[\langle x,a\rangle + \langle A,x\rangle\right] - \operatorname{Re}\langle A,a\rangle - \|x\|^2$$

and thus, obviously, $I_1 \geq 0$ iff $I_2 \geq 0$ showing the required equivalence. ∎

The following corollary is obvious:

Corollary 2.7. *Let $x, e \in H$ with $\|e\| = 1$ and $\delta, \Delta \in \mathbb{K}$ with $\delta \neq \Delta$. Then*

$$\operatorname{Re}\langle \Delta e - x, x - \delta e\rangle \geq 0 \tag{2.23}$$

iff

$$\left\|x - \frac{\delta+\Delta}{2}\cdot e\right\| \leq \frac{1}{2}|\Delta - \delta| . \tag{2.24}$$

Remark 2.8. *If $H = \mathbb{C}$, then $\operatorname{Re}\left[(A-x)(\bar{x}-\bar{a})\right] \geq 0$ if and only if $\left|x - \frac{a+A}{2}\right| \leq \frac{1}{2}|A-a|$, where $a, x, A \in \mathbb{C}$. If $H = \mathbb{R}$, and $A > a$ then $a \leq x \leq A$ if and only if $\left|x - \frac{a+A}{2}\right| \leq \frac{1}{2}(A-a)$.*

The following lemma is of interest [6].

Lemma 2.9. *Let $x, e \in H$ with $\|e\| = 1$. Then one has the following representation*

$$\|x\|^2 - |\langle x,e\rangle|^2 = \inf_{\lambda\in\mathbb{K}} \|x - \lambda e\|^2 \geq 0. \tag{2.25}$$

Proof. Observe, for any $\lambda \in \mathbb{K}$, that

$$\begin{aligned}\langle x - \lambda e, x - \langle x,e\rangle e\rangle &= \|x\|^2 - |\langle x,e\rangle|^2 - \lambda\left[\langle e,x\rangle - \langle e,x\rangle \|e\|^2\right] \\ &= \|x\|^2 - |\langle x,e\rangle|^2 .\end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned}\left[\|x\|^2 - |\langle x,e\rangle|^2\right]^2 &= |\langle x-\lambda e, x - \langle x,e\rangle e\rangle|^2 \le \|x-\lambda e\|^2 \|x - \langle x,e\rangle e\|^2 \\ &= \|x-\lambda e\|^2 \left[\|x\|^2 - |\langle x,e\rangle|^2\right],\end{aligned}$$

giving the bound

$$\|x\|^2 - |\langle x,e\rangle|^2 \le \|x-\lambda e\|^2, \qquad \lambda \in \mathbb{K}. \tag{2.26}$$

Taking the infimum in (2.26) over $\lambda \in \mathbb{K}$, we deduce

$$\|x\|^2 - |\langle x,e\rangle|^2 \le \inf_{\lambda\in\mathbb{K}} \|x-\lambda e\|^2.$$

Since, for $\lambda_0 = \langle x,e\rangle$, we get $\|x-\lambda_0 e\|^2 = \|x\|^2 - |\langle x,e\rangle|^2$, then the representation (2.25) is proved. ∎

The following result also holds:

Theorem 2.10. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e,\ f \in H, \|e\| = \|f\| = 1$. If $\varphi,\gamma,\Phi,\Gamma$ are real or complex numbers and x,y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \ge 0, \ \ \operatorname{Re}\langle \Gamma f - y, y - \gamma f\rangle \ge 0 \tag{2.27}$$

hold, or, equivalently, the following assumptions

$$\left\| x - \frac{\varphi+\Phi}{2}e\right\| \le \frac{1}{2}|\Phi-\varphi|, \quad \left\| y - \frac{\gamma+\Gamma}{2}f\right\| \le \frac{1}{2}|\Gamma-\gamma| \tag{2.28}$$

are valid, then one has the inequality

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle| \le \frac{1}{4}|\Phi-\varphi|\,|\Gamma-\gamma|. \tag{2.29}$$

Proof. Using the inequality (2.8) and Lemma 2.9 we have

$$\begin{aligned}&|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle|^2 \\ &\le \left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(\|y\|^2 - |\langle y,f\rangle|^2\right) = \inf_{\lambda\in\mathbb{K}}\|x-\lambda e\|^2 \inf_{\eta\in\mathbb{K}}\|y-\eta f\|^2 \\ &\le \left\| x - \frac{\varphi+\Phi}{2}e\right\|^2 \left\| y - \frac{\gamma+\Gamma}{2}f\right\|^2 \le \frac{1}{4}|\Phi-\varphi|^2 \frac{1}{4}|\Gamma-\gamma|^2,\end{aligned} \tag{2.30}$$

which is equivalent to the desired inequality (2.29). ∎

Corollary 2.11. *With the assumptions of Theorem 2.10 and if* $e \perp f$*, then we have the simpler inequality*

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma| . \tag{2.31}$$

Remark 2.12. *If we take* $f = e$ *in Theorem 2.10, then we get the result from Theorem 1.1.*

3 Applications

Consider the Hilbert space $\mathbb{C}^n$ endowed with the inner product $\langle \cdot, \cdot\rangle_{\mathbf{p}} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

$$\langle \mathbf{x}, \mathbf{y}\rangle_{\mathbf{p}} := \sum_{j=1}^{n} p_j x_j \overline{y}_j,$$

where $\mathbf{p} = (p_1, ..., p_n)$ is a probability distribution, i.e. $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ and

$$\mathbf{x} = (x_1, ..., x_n) , \ \mathbf{y} = (y_1, ..., y_n) \in \mathbb{C}^n.$$

Assume that $\mathbf{e} = (e_1, ..., e_n) , \mathbf{f} = (f_1, ..., f_n) \in \mathbb{C}^n$ with

$$\sum_{j=1}^{n} p_j |e_j|^2 = \sum_{j=1}^{n} p_j |f_j|^2 = 1. \tag{3.32}$$

Then for any $\mathbf{x} = (x_1, ..., x_n) , \mathbf{y} = (y_1, ..., y_n) \in \mathbb{C}^n$ we have the inequality

$$\begin{aligned} &\left(\sum_{j=1}^{n} p_j |x_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j |y_j|^2\right)^{1/2} - \left|\sum_{j=1}^{n} p_j x_j \overline{e}_j \sum_{j=1}^{n} p_j f_j \overline{y}_j\right| \\ &\geq \left|\sum_{j=1}^{n} p_j x_j \overline{y}_j - \sum_{j=1}^{n} p_j x_j \overline{e}_j \sum_{j=1}^{n} p_j e_j \overline{y}_j\right. \\ &\left. - \sum_{j=1}^{n} p_j x_j \overline{f}_j \sum_{j=1}^{n} p_j f_j \overline{y}_j + \sum_{j=1}^{n} p_j x_j \overline{e}_j \sum_{j=1}^{n} p_j f_j \overline{y}_j \sum_{j=1}^{n} p_j e_j \overline{f}_j\right| . \end{aligned} \tag{3.33}$$

Moreover, if $\mathbf{e} = (e_1, ..., e_n), \mathbf{f} = (f_1, ..., f_n) \in \mathbb{C}^n$ satisfy the additional condition

$$\sum_{j=1}^{n} p_j e_j \overline{f}_j = 0, \tag{3.34}$$

then from (3.33) we get

$$\begin{aligned}
&\left(\sum_{j=1}^{n} p_j |x_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j |y_j|^2\right)^{1/2} - \left|\sum_{j=1}^{n} p_j x_j \overline{e}_j \sum_{j=1}^{n} p_j f_j \overline{y}_j\right| \\
&\geq \left|\sum_{j=1}^{n} p_j x_j \overline{y}_j - \sum_{j=1}^{n} p_j x_j \overline{e}_j \sum_{j=1}^{n} p_j e_j \overline{y}_j - \sum_{j=1}^{n} p_j x_j \overline{f}_j \sum_{j=1}^{n} p_j f_j \overline{y}_j\right|.
\end{aligned} \tag{3.35}$$

If we denote by $\mathcal{C}(0,1)$ the unit circle of radius 1 in $\mathbb{C}$, namely $\mathcal{C}(0,1) = \{z \in \mathbb{C} |\ |z| = 1\}$, then for $\mathbf{e} = (e_1, ..., e_n), \mathbf{f} = (f_1, ..., f_n) \in \mathbb{C}^n$ with $e_j, f_j \in \mathcal{C}(0,1)$ for any $j \in \{1, ..., n\}$ we have that the condition (3.32) holds true and therefore the inequality (3.33) is valid.

If we consider the nonnegative weights $w_j \geq 0, j \in \{1, ..., n\}$ with $W_n = \sum_{k=1}^{n} w_k > 0$ and if we assume that

$$\frac{1}{W_n} \sum_{j=1}^{n} w_j |e_j|^2 = \frac{1}{W_n} \sum_{j=1}^{n} w_j |f_j|^2 = 1 \tag{3.36}$$

then by (3.33) we get

$$\begin{aligned}
&\left(\frac{1}{W_n} \sum_{j=1}^{n} w_j |x_j|^2\right)^{1/2} \left(\frac{1}{W_n} \sum_{j=1}^{n} w_j |y_j|^2\right)^{1/2} \\
&- \left|\frac{1}{W_n} \sum_{j=1}^{n} w_j x_j \overline{e}_j \frac{1}{W_n} \sum_{j=1}^{n} w_j f_j \overline{y}_j\right| \\
&\geq \left|\frac{1}{W_n} \sum_{j=1}^{n} w_j x_j \overline{y}_j - \frac{1}{W_n} \sum_{j=1}^{n} w_j x_j \overline{e}_j \frac{1}{W_n} \sum_{j=1}^{n} w_j e_j \overline{y}_j\right. \\
&- \frac{1}{W_n} \sum_{j=1}^{n} w_j x_j \overline{f}_j \frac{1}{W_n} \sum_{j=1}^{n} w_j f_j \overline{y}_j \\
&\left.+ \frac{1}{W_n} \sum_{j=1}^{n} w_j x_j \overline{e}_j \frac{1}{W_n} \sum_{j=1}^{n} w_j f_j \overline{y}_j \frac{1}{W_n} \sum_{j=1}^{n} w_j e_j \overline{f}_j\right|.
\end{aligned} \tag{3.37}$$

Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$.

The most important power series with nonnegative coefficients that can be used to illustrate the above results are:

$$\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C},\ \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\ z\in D(0,1), \tag{3.38}$$
$$\ln\frac{1}{1-z}=\sum_{n=1}^{\infty}\frac{1}{n}z^n,\ z\in D(0,1),\ \cosh z=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n},\ z\in\mathbb{C},$$
$$\sinh z=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1},\ z\in\mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1), \tag{3.39}$$
$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1},\ z\in D(0,1),$$

$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),$$
$${}_2F_1(\alpha,\beta,\gamma,z):=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0$$
$$z\in D(0,1),$$

where Γ is *Gamma function.*

Proposition 3.1. *Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $0<p<R$, $u,v\in\mathcal{C}(0,1)$ and*

$x,y\in\mathbb{C}$ with $p\left|x\right|^{2},p\left|y\right|^{2}<R$ then we have the inequality

$$\left(\frac{f\left(p\left|x\right|^{2}\right)}{f\left(p\right)}\right)^{1/2}\left(\frac{f\left(p\left|y\right|^{2}\right)}{f\left(p\right)}\right)^{1/2}-\left|\frac{f\left(px\overline{u}\right)}{f\left(p\right)}\frac{f\left(pv\overline{y}\right)}{f\left(p\right)}\right|$$
$$\geq\left|\frac{f\left(px\overline{y}\right)}{f\left(p\right)}-\frac{f\left(px\overline{u}\right)}{f\left(p\right)}\frac{f\left(pu\overline{y}\right)}{f\left(p\right)}-\frac{f\left(px\overline{v}\right)}{f\left(p\right)}\frac{f\left(pv\overline{y}\right)}{f\left(p\right)}+\frac{f\left(px\overline{u}\right)}{f\left(p\right)}\frac{f\left(pv\overline{y}\right)}{f\left(p\right)}\frac{f\left(pu\overline{v}\right)}{f\left(p\right)}\right|. \quad (3.40)$$

Proof. If $u,v\in\mathcal{C}\left(0,1\right)$ then for any $n\geq 0$ we have $u^{n},v^{n}\in\mathcal{C}\left(0,1\right)$. Observe that for any $m\geq 1$ we have that

$$\frac{\sum_{n=0}^{m}a_{n}p^{n}\left|u^{n}\right|^{2}}{\sum_{n=0}^{m}a_{n}p^{n}}=\frac{\sum_{n=0}^{m}a_{n}p^{n}\left|v^{n}\right|^{2}}{\sum_{n=0}^{m}a_{n}p^{n}}=\frac{\sum_{n=0}^{m}a_{n}p^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}=1.$$

Using the inequality (3.37) we have

$$\left(\frac{\sum_{n=0}^{m}a_{n}p^{n}\left|x\right|^{2n}}{\sum_{n=0}^{m}a_{n}p^{n}}\right)^{1/2}\left(\frac{\sum_{n=0}^{m}a_{n}p^{n}\left|y\right|^{2n}}{\sum_{n=0}^{m}a_{n}p^{n}}\right)^{1/2} \quad (3.41)$$
$$-\left|\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(x\overline{u}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(v\overline{y}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}\right|$$
$$\geq\left|\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(x\overline{y}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}-\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(x\overline{u}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(u\overline{y}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}\right.$$
$$-\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(x\overline{v}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(v\overline{y}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}$$
$$\left.+\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(x\overline{u}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(v\overline{y}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}\frac{\sum_{n=0}^{m}a_{n}p^{n}\left(u\overline{v}\right)^{n}}{\sum_{n=0}^{m}a_{n}p^{n}}\right|.$$

Since all the series whose partial sums are involved in inequality (3.41) are convergent, then by letting $m\to\infty$ in (3.41) we get the desired result (3.40). ∎

Remark 3.2. *The inequality (3.40) can provide some particular inequalities of interest. For instance, if we take $f\left(z\right)=\exp\left(z\right),\ z\in\mathbb{C}$, then we get*

$$\exp\left[p\left(\frac{\left|x\right|^{2}+\left|y\right|^{2}}{2}-1\right)\right]-\left|\exp\left[p\left(x\overline{u}+v\overline{y}-2\right)\right]\right| \quad (3.42)$$
$$\geq\left|\exp\left[p\left(x\overline{y}-1\right)\right]-\exp\left[p\left(x\overline{u}+u\overline{y}-2\right)\right]-\exp\left[p\left(x\overline{v}+v\overline{y}-2\right)\right]\right.$$
$$\left.+\exp\left[p\left(x\overline{u}+v\overline{y}+u\overline{v}-3\right)\right]\right|$$

for any $p > 0, u, v \in \mathcal{C}(0,1)$ *and* $x, y \in \mathbb{C}$.

If we take $u = v = 1$, *then from (3.42) we get*

$$\exp\left[p\left(\frac{|x|^2+|y|^2}{2}-1\right)\right] - |\exp[p(x+\overline{y}-2)]| \tag{3.43}$$
$$\geq |\exp[p(x\overline{y}-1)] - \exp[p(x+\overline{y}-2)]|$$

for any $p > 0$ *and* $x, y \in \mathbb{C}$.

Moreover, if we take in (3.43) $x = \overline{y} = z \in \mathbb{C}$, *then we get*

$$\exp\left[p\left(|z|^2-1\right)\right] - |\exp[2p(z-1)]| \geq \left|\exp\left[p\left(z^2-1\right)\right] - \exp[2p(z-1)]\right| \tag{3.44}$$

for any $p > 0$ *and* $z \in \mathbb{C}$.

Consider $L^2[a,b]$ the Hilbert space of all complex valued functions f with $\int_a^b |f(t)|^2 dt < \infty$. The inner product is given by

$$\langle f, g\rangle_2 := \int_a^b f(t)\overline{g(t)}dt.$$

Assume that $h, k \in L^2[a,b]$ with

$$\int_a^b |h(t)|^2 dt = \int_a^b |k(t)|^2 dt = 1. \tag{3.45}$$

For instance, if $h(t) = \frac{1}{\sqrt{b-a}}\rho(t), k(t) = \frac{1}{\sqrt{b-a}}\varphi(t)$ with $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ for almost any $t \in [a,b]$, then $h, k \in L^2[a,b]$ and the condition (3.45) is satisfied.

Proposition 3.3. *Assume that* $h, k \in L^2[a,b]$ *with the property (3.45). Then for any* $f, g \in L^2[a,b]$ *we have the inequality*

$$\left(\int_a^b |f(t)|^2 dt\right)^{1/2}\left(\int_a^b |g(t)|^2 dt\right)^{1/2} - \left|\int_a^b f(t)\overline{h(t)}dt\int_a^b k(t)\overline{g(t)}dt\right| \tag{3.46}$$
$$\geq \left|\int_a^b f(t)\overline{g(t)}dt - \int_a^b f(t)\overline{h(t)}dt\int_a^b h(t)\overline{g(t)}dt\right.$$
$$-\int_a^b f(t)\overline{k(t)}dt\int_a^b k(t)\overline{g(t)}dt$$
$$\left.+\int_a^b f(t)\overline{h(t)}dt\int_a^b k(t)\overline{g(t)}dt\int_a^b h(t)\overline{k(t)}dt\right|.$$

The proof follows by Theorem 2.1 for the inner product $\langle\cdot,\cdot\rangle_2$.

Remark 3.4. *If $\rho(t),\varphi(t)\in\mathcal{C}(0,1)$ for almost any $t\in[a,b]$, then we have the following inequalities for integral means*

$$\left(\frac{1}{b-a}\int_a^b|f(t)|^2dt\right)^{1/2}\left(\frac{1}{b-a}\int_a^b|g(t)|^2dt\right)^{1/2}\tag{3.47}$$
$$-\left|\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\frac{1}{b-a}\int_a^b\varphi(t)\overline{g(t)}dt\right|$$
$$\geq\left|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt-\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\frac{1}{b-a}\int_a^b\rho(t)\overline{g(t)}dt\right.$$
$$-\frac{1}{b-a}\int_a^b f(t)\overline{\varphi(t)}dt\frac{1}{b-a}\int_a^b\varphi(t)\overline{g(t)}dt$$
$$\left.+\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\frac{1}{b-a}\int_a^b\varphi(t)\overline{g(t)}dt\frac{1}{b-a}\int_a^b\rho(t)\overline{\varphi(t)}dt\right|,$$

for any $f,g\in L^2[a,b]$.

If we take $\rho(t)=1$, $\varphi(t)=sgn\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$, then $\rho(t),\varphi(t)\in\mathcal{C}(0,1)$ for almost any $t\in[a,b]$ and since

$$\int_a^b\rho(t)\overline{\varphi(t)}=\int_a^b sgn\left(t-\frac{a+b}{2}\right)dt=0,$$

then we get from (3.47)

$$\left(\frac{1}{b-a}\int_a^b|f(t)|^2dt\right)^{1/2}\left(\frac{1}{b-a}\int_a^b|g(t)|^2dt\right)^{1/2}\tag{3.48}$$
$$-\left|\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)\overline{g(t)}dt\right|$$
$$\geq\left|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b\overline{g(t)}dt\right.$$
$$\left.-\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)\,dt\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)\overline{g(t)}dt\right|$$

for any $f,g\in L^2[a,b]$.

On making use of Theorem 2.10 one can state similar discrete and integral inequalities. However the details are not presented here.

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New Classes of Invariant Harmonic Convex Functions and Inequalities

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Abstract

In this paper, we introduce a new class of invariant harmonic convex functions with respect to any arbitrary non-negative function h, which is called invarinat h-harmonic convex function. Several Hermite-Hadamard type inequalities via invariant h-harmonic convex functions are obtained. Several new classes of invariant harmonic convex functions are discussed. results obtained in this paper continue to hold for these classes. Our results represent a significant improvement and refinement of the known results. Ideas and techniques of this paper may be a staring point for further research in different directions.

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1 Introduction

Convex functions have been generalized and extended in several directions using interesting and novel ideas. Several new classes of convex functions and convex sets have been introduced and investigated. The harmonic set was introduced by Shi et. al. [19]. It is worth mentioning that the weighted harmonic mean is used to define the harmonic set, which has applications in electrical circuit theory and other branches of sciences. Using the weighted harmonic means, one usually define the harmonic convex functions, which have appeared as significant and important generalization of convex functions. Anderson et al. [1] and Iscan [6] have investigated several properties of the harmonic convex functions. For recent generalizations and applications of harmonic convex functions, see [2, 7, 9, 11, 13, 14, 15, 16] and references therein.

Motivated and inspired by the ongoing research, we introduce and investigate a new class of harmonic convex functions, which is called invariant harmonic convex functions. It is shown that this new class includes several new classes of harmonic convex functions such as invariant harmonic P-functions, invariant harmonic tgs-convex functions and invariant harmonic Godunova-Levin convex functions as special cases. We obtain several new Hermite-Hadamard type inequalities for invariant h-harmonic convex functions, product of two invariant h-harmonic convex functions. One can obtain several new results for other classes of convex functions as special cases of our results. These results can be viewed as significant contributions of this area. Ideas of this paper may motivate further research.

2 Preliminaries

In this section, we recall some basic concepts and results.

Definition 1. [2, 7]. *Let X be a topological vector space. Let $K \subset X \setminus \{0\}$ be a set satisfying the following conditions. For $x,y \in K$, let $I[y,x]$ be a path joining y and x contained in K and the map*

$\gamma_{xy} : [0,1] \to I[y,x]$ be continuous. The set K said to be invariant harmonic convex set in a given direction $v \in X$, if the following harmonic convex combination properties are satisfied.

1. $y+tv \in K$ *for all* $t \in [0,1]$, $v \in X$ *and* $y \in K$.
2. $y+tv = \begin{cases} y, & \text{if } t=0; \\ x & \text{if } t=1, \end{cases}$ *if and only if,* $y+tv = \frac{x+y}{2}$ *for* $t = \frac{1}{2}$.
3. *For any* $z \in I[y,x] \subset K$, *we have* $z = y+tv = x+(1-t)v$.
4. $\frac{xy}{y+tv} \in I[y,x]$ *for all* $x,y \in K$.

Remark 1. *If $t = \frac{1}{2}$ and $v = x-y$, then invariant harmonically convex set becomes harmonically convex set.*

Definition 2. [7]. *For a given direction $v \in \mathbb{R}^n \setminus \{0\}$, a function f is said to be invariant harmonically convex function on the invariant harmonic convex set K, if*

$$f\left(\frac{xy}{y+tv}\right) \leq (1-t)f(x)+tf(y), \qquad \forall x,y \in K, t \in [0,1].$$

For $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{2y+v}\right) \leq \frac{f(x)+f(y)}{2}, \qquad \forall x,y \in K.$$

Definition 3. *Let $K \subset \mathbb{R}\setminus\{0\}$ be an invariant harmonically convex set. For a given direction $v \in \mathbb{R}^n \setminus \{0\}$, a function f is said to be invariant harmonically convex function on the invariant harmonically convex set K with respect to an arbitrary non-negative function $h : [0,1] \to \mathbb{R}$, if*

$$f\left(\frac{xy}{y+tv}\right) \leq h(1-t)f(x)+h(t)f(y), \qquad \forall x,y \in K, t \in (0,1).$$

For $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{2y+v}\right) \leq h\left(\frac{1}{2}\right)[f(x)+f(y)], \qquad \forall x,y \in K.$$

We would like to emphasize that for suitable and appropriate choice of the non-negative function h, one can obtain a various new classes of convex functions, which include, but not limited to s-invariant harmonic convex functions, invariant beta-harmonic convex functions and invariant P-harmonic convex functions. This clearly shows that the class of invariant h-harmonic convex functions is very general and unified ones.

The Euler Beta function is a special function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x,y > 0.$$

where $\Gamma(\cdot)$ is a Gamma function.

3 Main Results

In this section, we obtain the Hermite-Hadamard type integral inequalities for invariant harmonically convex functions.

Theorem 1. *For $v \in \mathbb{R} \setminus \{0\}$, let $K \subset \mathbb{R}^n \setminus \{0\}$ be a invariant harmonically convex set. For a given direction $v \in \mathbb{R}^n \setminus \{0\}$, let $f : K \to \mathbb{R}$ be an invariant harmonically convex function on the interior (K°) of K with respect to an arbitrary non-negative function $h : [0,1] \to \mathbb{R}$. Then*

$$f\left(\frac{2xy}{x+y}\right) \leq h\left(\frac{1}{2}\right)[f(x)+f(y)], \qquad \forall x,y \in K.$$

Proof. Let f be an invariant harmonically convex function with respect to an arbitrary non-negative function h. Then

$$f\left(\frac{xy}{y+tv}\right) \leq h(1-t)f(x)+h(t)f(y), \qquad \forall x,y \in K, t \in (0,1).$$

For $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{2y+v}\right) \leq h\left(\frac{1}{2}\right)[f(x)+f(y)].$$

Since $x = y + v$, we have

$$f\left(\frac{2xy}{x+y}\right) \leq h\left(\frac{1}{2}\right)[f(x)+f(y)],$$

This completes the proof. □

Theorem 2. *For $v, w \in \mathbb{R}^n \setminus \{0\}$, let $K \subset \mathbb{R}^n \setminus \{0\}$ be a invariant harmonically convex set. For $a, b \in K$ with $a < b$, let there exists vectors $v, w \in \mathbb{R}^n \setminus \{0\}$ with $v + w = 0$, such that*

$$a+tv = \begin{cases} a, & \text{if } t = 0; \\ b & \text{if } t = 1, \end{cases} \qquad \text{and} \qquad b+tw = \begin{cases} b, & \text{if } t = 0; \\ a & \text{if } t = 1. \end{cases}$$

Let $f : K \to \mathbb{R}$ be an invariant harmonically convex function on the interior (K°) of K with respect to an arbitrary non-negative function $h : [0,1] \to \mathbb{R}$. If $f \in L[a,b]$, then

$$\begin{aligned} &\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]^2} f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) - \frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \\ \leq\ & M(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t + N(a,b)\int_0^1 [h(t)]^2\mathrm{d}t, \end{aligned}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b) \tag{3.1}$$

$$N(a,b) = f(a)g(b) + f(b)g(a). \tag{3.2}$$

Proof. Let f, g be invariant harmonically convex functions with respect to an arbitrary non-negative function h. With $t = \frac{1}{2}$ and letting $x = \frac{ab}{a+tv}$ and $y = \frac{ab}{b+tw}$, we have

$$f\left(\frac{2ab}{a+b}\right) \leq h\left(\frac{1}{2}\right)\left[f\left(\frac{ab}{a+tv}\right) + f\left(\frac{ab}{b+tw}\right)\right].$$
$$g\left(\frac{2ab}{a+b}\right) \leq h\left(\frac{1}{2}\right)\left[g\left(\frac{ab}{a+tv}\right) + g\left(\frac{ab}{b+tw}\right)\right].$$

Consider

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \\ \leq\ & \left[h\left(\frac{1}{2}\right)\right]^2 \left[f\left(\frac{ab}{a+tv}\right) + f\left(\frac{ab}{b+tw}\right)\right]\left[g\left(\frac{ab}{a+tv}\right) + g\left(\frac{ab}{b+tw}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \left[h\left(\frac{1}{2}\right)\right]^2\left[f\left(\frac{ab}{a+tv}\right)g\left(\frac{ab}{a+tv}\right)+f\left(\frac{ab}{b+tw}\right)g\left(\frac{ab}{b+tw}\right)\right.\\
&\quad \left.+f\left(\frac{ab}{a+tv}\right)g\left(\frac{ab}{b+tw}\right)+f\left(\frac{ab}{b+tw}\right)g\left(\frac{ab}{a+tv}\right)\right]\\
&\le \left[h\left(\frac{1}{2}\right)\right]^2\left\{f\left(\frac{ab}{a+tv}\right)g\left(\frac{ab}{a+tv}\right)+f\left(\frac{ab}{b+tw}\right)g\left(\frac{ab}{b+tw}\right)\right.\\
&\quad +\left[h(t)f(a)+h(1-t)f(b)\right]\left[h(1-t)g(a)+h(t)g(b)\right]\\
&\quad \left.+\left[h(1-t)f(a)+h(t)f(b)\right]\left[h(t)g(a)+h(1-t)g(b)\right]\right\}\\
&= \left[h\left(\frac{1}{2}\right)\right]^2\left\{\int_0^1 f\left(\frac{ab}{a+tv}\right)g\left(\frac{ab}{a+tv}\right)\mathrm{d}t+\int_0^1 f\left(\frac{ab}{b+tw}\right)g\left(\frac{ab}{b+tw}\right)\mathrm{d}t\right.\\
&\quad \left.+2[f(a)g(a)+f(b)g(b)]\int_0^1 h(t)h(1-t)\mathrm{d}t+2[f(a)g(b)+f(b)g(a)]\int_0^1 [h(t)]^2\mathrm{d}t\right\}\\
&= 2\left[h\left(\frac{1}{2}\right)\right]^2\left[\frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x+M(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t+N(a,b)\int_0^1 [h(t)]^2\mathrm{d}t\right],
\end{aligned}
$$

which is the required result. □

Corollary 1. *Under the assumptions of Theorem 2 with $h(t)=t$, we have*

$$
\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)-\frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\\
\le\ & \frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),
\end{aligned}
$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) *and* (3.2)*, respectively.*

Corollary 2. *Under the assumptions of Theorem 2 with $h(t)=t^p(1-t)^q$ where $p,q>-1$, we have*

$$
\begin{aligned}
&2^{2p+2q-1}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)-\frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\\
\le\ & M(a,b)\beta(p+q+1,p+q+1)+N(a,b)\beta(2p+1,2q+1),
\end{aligned}
$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) *and* (3.2)*, respectively.*

Corollary 3. *Under the assumptions of Theorem 2 with $h(t)=t^s$, we have*

$$
\begin{aligned}
&2^{2s-1}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)-\frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\\
\le\ & M(a,b)\beta(s+1,s+1)+\frac{1}{2s+1}N(a,b),
\end{aligned}
$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) *and* (3.2)*, respectively.*

Theorem 3. *For $v \in \mathbb{R} \setminus \{0\}$, let $K \subset \mathbb{R}^n \setminus \{0\}$ be a invariant harmonically convex set. For $a,b \in K$ with $a < b$, let there exists a vector $v \in \mathbb{R} \setminus \{0\}$, such that*

$$a + tv = \begin{cases} a, & \text{if } t = 0; \\ b & \text{if } t = 1. \end{cases}$$

Let $f : K \to \mathbb{R}$ be an invariant harmonically convex function on the interior (K°) of K with respect to an arbitrary non-negative function $h : [0,1] \to \mathbb{R}$. If $f \in L[a,b]$, then

$$\begin{aligned} & \frac{ab}{v}\int_a^b h\left(\frac{ab-ax}{xv}\right)\frac{f(a)g(x)+g(a)f(x)}{x^2}\mathrm{d}x \\ & +\frac{ab}{v}\int_a^b h\left(1-\frac{ab-ax}{xv}\right)\frac{f(b)g(x)+g(b)f(x)}{x^2}\mathrm{d}x \\ \leq \quad & M(a,b)\int_0^1 [h(t)]^2\mathrm{d}t + N(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t + \frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x, \end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) *and* (3.2) *respectively.*

Proof. Let f,g be invariant harmonically convex functions with respect to an arbitrary non-negative function h. Then

$$\begin{aligned} f\left(\frac{ab}{a+tv}\right) &\leq h(t)f(a) + h(1-t)f(b), \qquad \forall a,b \in K, t \in [0,1]. \\ g\left(\frac{ab}{a+tv}\right) &\leq h(t)g(a) + h(1-t)g(b), \qquad \forall a,b \in K, t \in [0,1]. \end{aligned}$$

Now, using $\langle x_1 - x_2, x_3 - x_4 \rangle \geq 0, (x_1,x_2,x_3,x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned} & f\left(\frac{ab}{a+tv}\right)[h(t)g(a)+h(1-t)g(b)] + g\left(\frac{ab}{a+tv}\right)[h(t)f(a)+h(1-t)f(b)] \\ \leq \quad & [h(t)f(a)+h(1-t)f(b)][h(t)g(a)+h(1-t)g(b)] + f\left(\frac{ab}{a+tv}\right)g\left(\frac{ab}{a+tv}\right) \end{aligned}$$

and we obtain

$$\begin{aligned}
& g(a)h(t)f\left(\frac{ab}{a+tv}\right)+g(b)h(1-t)f\left(\frac{ab}{a+tv}\right) \\
& +f(a)h(t)g\left(\frac{ab}{a+tv}\right)+f(b)h(1-t)g\left(\frac{ab}{a+tv}\right) \\
\leq\ & [h(t)]^2 f(a)g(a)+[h(1-t)]^2 f(b)g(b)+h(t)h(1-t)[f(a)g(b)+f(b)g(a)] \\
& +f\left(\frac{ab}{a+tv}\right)g\left(\frac{ab}{a+tv}\right)
\end{aligned}$$

Integrating with respect to t over the interval $[0,1]$, we have

$$\begin{aligned}
& g(a)\int_0^1 h(t)f\left(\frac{ab}{a+tv}\right)\mathrm{d}t+g(b)\int_0^1 h(1-t)f\left(\frac{ab}{a+tv}\right)\mathrm{d}t \\
& +f(a)\int_0^1 h(t)g\left(\frac{ab}{a+tv}\right)\mathrm{d}t+f(b)\int_0^1 h(1-t)g\left(\frac{ab}{a+tv}\right)\mathrm{d}t \\
\leq\ & f(a)g(a)\int_0^1 [h(t)]^2\mathrm{d}t+f(b)g(b)\int_0^1 [h(1-t)]^2\mathrm{d}t \\
& +[f(a)g(b)+f(b)g(a)]\int_0^1 h(t)h(1-t)\mathrm{d}t \\
& +\int_0^1 f\left(\frac{ab}{a+tv}\right)g\left(\frac{ab}{a+tv}\right)\mathrm{d}t
\end{aligned}$$

This implies

$$\begin{aligned}
& \frac{ab}{v}\int_a^b h\left(\frac{ab-ax}{xv}\right)\frac{f(a)g(x)+g(a)f(x)}{x^2}\mathrm{d}x \\
& +\frac{ab}{v}\int_a^b h\left(1-\frac{ab-ax}{xv}\right)\frac{f(b)g(x)+g(b)f(x)}{x^2}\mathrm{d}x \\
\leq\ & M(a,b)\int_0^1 [h(t)]^2\mathrm{d}t+N(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t+\frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

which is the required result. □

Theorem 4. *For $v,w \in \mathbb{R}^n \setminus \{0\}$, let $K \subset \mathbb{R}^n \setminus \{0\}$ be a invariant harmonically convex set. For $a,b \in K$ with $a<b$, let there exists vectors $v,w \in \mathbb{R}^n \setminus \{0\}$ with $v+w=0$, such that*

$$a+tv=\begin{cases} a, & \text{if } t=0; \\ b & \text{if } t=1, \end{cases} \qquad \text{and} \qquad b+tw=\begin{cases} b, & \text{if } t=0; \\ a & \text{if } t=1. \end{cases}$$

Let $f : K \to \mathbb{R}$ be an invariant harmonically convex function on the interior (K°) of K with respect to an arbitrary non-negative function $h : [0,1] \to \mathbb{R}$. If $f \in L[a,b]$, then

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\frac{ab}{v}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x + g\left(\frac{2ab}{a+b}\right)\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\ \leq\ & h\left(\frac{1}{2}\right)\left[\frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x + M(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t \right. \\ & \left. +N(a,b)\int_0^1 [h(t)]^2\mathrm{d}t\right] + \frac{1}{2h(\frac{1}{2})} f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right). \end{aligned}$$

Proof. Let f,g be invariant harmonically convex functions with respect to an arbitrary non-negative function h. With $t = \frac{1}{2}$ and letting $x = \frac{ab}{a+tv}$ and $y = \frac{ab}{b+tw}$, we have

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) \leq h\left(\frac{1}{2}\right)\left[f\left(\frac{ab}{a+tv}\right) + f\left(\frac{ab}{b+tw}\right)\right]. \\ g\left(\frac{2ab}{a+b}\right) \leq h\left(\frac{1}{2}\right)\left[g\left(\frac{ab}{a+tv}\right) + g\left(\frac{ab}{b+tw}\right)\right]. \end{aligned}$$

Now, using $\langle x_1 - x_2, x_3 - x_4 \rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned} & h\left(\frac{1}{2}\right) f\left(\frac{2ab}{a+b}\right)\left[g\left(\frac{ab}{a+tv}\right) + g\left(\frac{ab}{b+tw}\right)\right] \\ & +h\left(\frac{1}{2}\right) g\left(\frac{2ab}{a+b}\right)\left[f\left(\frac{ab}{a+tv}\right) + f\left(\frac{ab}{b+tw}\right)\right] \\ \leq\ & \left[h\left(\frac{1}{2}\right)\right]^2 \left[f\left(\frac{ab}{a+tv}\right) + f\left(\frac{ab}{b+tw}\right)\right]\left[g\left(\frac{ab}{a+tv}\right) + g\left(\frac{ab}{b+tw}\right)\right] \\ & +f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \end{aligned}$$

$$\begin{aligned} =\ & \left[h\left(\frac{1}{2}\right)\right]^2 \left[f\left(\frac{ab}{a+tv}\right) g\left(\frac{ab}{a+tv}\right) + f\left(\frac{ab}{b+tw}\right) g\left(\frac{ab}{b+tw}\right) \right. \\ & \left. +f\left(\frac{ab}{a+tv}\right) g\left(\frac{ab}{b+tw}\right) + f\left(\frac{ab}{b+tw}\right) g\left(\frac{ab}{a+tv}\right)\right] \\ & +f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \\ \leq\ & \frac{1}{4}\left[f\left(\frac{ab}{a+tv}\right) g\left(\frac{ab}{a+tv}\right) + f\left(\frac{ab}{b+tw}\right) g\left(\frac{ab}{b+tw}\right) \right. \\ & +[h(t)f(a) + h(1-t)f(b)][h(1-t)g(a) + h(t)g(b)] \\ & \left. +[h(1-t)f(a) + h(t)f(b)][h(t)g(a) + h(1-t)g(b)]\right] + f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \end{aligned}$$

Integrating with respect to t over the interval $[0,1]$, we have

$$\begin{aligned}
& h\left(\frac{1}{2}\right) f\left(\frac{2ab}{a+b}\right) \int_0^1 \left[g\left(\frac{ab}{a+tv}\right) + g\left(\frac{ab}{b+tw}\right)\right] \mathrm{d}t \\
& + h\left(\frac{1}{2}\right) g\left(\frac{2ab}{a+b}\right) \int_0^1 \left[f\left(\frac{ab}{a+tv}\right) + f\left(\frac{ab}{b+tw}\right)\right] \mathrm{d}t \\
\leq \quad & \left[h\left(\frac{1}{2}\right)\right]^2 \left[\int_0^1 f\left(\frac{ab}{a+tv}\right) g\left(\frac{ab}{a+tv}\right) \mathrm{d}t \right. \\
& + \int_0^1 f\left(\frac{ab}{b+tw}\right) g\left(\frac{ab}{b+tw}\right) \mathrm{d}t \\
& + [f(a)g(a) + f(b)g(b)] \int_0^1 h(t)h(1-t)\mathrm{d}t \\
& \left. + \left[f(a)g(b) + f(b)g(a)\right] \int_0^1 [h(t)]^2 \mathrm{d}t \right] + f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right)
\end{aligned}$$

From the above inequality, it follows that

$$\begin{aligned}
& f\left(\frac{2ab}{a+b}\right) \frac{ab}{v} \int_a^b \frac{g(x)}{x^2} \mathrm{d}x + g\left(\frac{2ab}{a+b}\right) \frac{ab}{v} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x \\
\leq \quad & h\left(\frac{1}{2}\right) \left[\frac{ab}{v} \int_a^b \frac{f(x)g(x)}{x^2} \mathrm{d}x + M(a,b) \int_0^1 h(t)h(1-t)\mathrm{d}t \right. \\
& \left. + N(a,b) \int_0^1 [h(t)]^2 \mathrm{d}t \right] + \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right),
\end{aligned}$$

which is the required result. □

Corollary 4. *Under the assumptions of Theorem* 4 *with* $h(t) = t$*, we have*

$$\begin{aligned}
& f\left(\frac{2ab}{a+b}\right) \frac{ab}{v} \int_a^b \frac{g(x)}{x^2} \mathrm{d}x + g\left(\frac{2ab}{a+b}\right) \frac{ab}{v} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x \\
\leq \quad & \frac{ab}{2v} \int_a^b \frac{f(x)g(x)}{x^2} \mathrm{d}x + \frac{1}{12} M(a,b) \\
& + \frac{1}{6} N(a,b) + f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right).
\end{aligned}$$

where $M(a,b)$ *and* $N(a,b)$ *are given by* (3.1) *and* (3.2)*, respectively.*

Corollary 5. *Under the assumptions of Theorem* 4 *with* $h(t) = t^p(1-t)^q$ *where* $p,q > -1$, *we have*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\frac{ab}{v}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x + g\left(\frac{2ab}{a+b}\right)\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\ \leq \quad & \frac{1}{2^{p+q}}\left[\frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x + \frac{1}{6}M(a,b) \right. \\ & \left. + \frac{1}{3}N(a,b)\right] + 2^{p+q-1}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right). \end{aligned}$$

where $M(a,b)$ *and* $N(a,b)$ *are given by* (3.1) *and* (3.2), *respectively.*

Corollary 6. *Under the assumptions of Theorem* 4 *with* $h(t) = t^s$, *we have*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\frac{ab}{v}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x + g\left(\frac{2ab}{a+b}\right)\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\ \leq \quad & \frac{1}{2^{s}}\left[\frac{ab}{v}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x + \frac{1}{6}M(a,b \right. \\ & \left. + \frac{1}{3}N(a,b)\right] + 2^{s-1}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right). \end{aligned}$$

where $M(a,b)$ *and* $N(a,b)$ *are given by* (3.1) *and* (3.2), *respectively.*

4 Conclusion

We have introduced a new class of harmonic convex functions with respect to an arbitrary non-negative function h, which is called invariant h- harmonic convex functions. We have derived several new Hermite-Hadamard type integral inequalities for invariant h-harmonic functions and the product of two invariant h-harmonic convex functions.discussed some special cases. Some special cases are discussed, which can be obtained from our main results.

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Next Generation Newton-Type Computational Methods with Cubic Convergence Rates

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Abstract

Recently, Verma [3] introduced several new classes of hybrid-type Newton's methods, which are unprecedented to one of the most popular computational methods over three centuries. These models outperform most of the traditional Newton's methods and its variants in the literature. In this paper, we prove successfully the cubic convergence rate for next generation Newton type methods by an analytical method, which is quite consistent with our numerical calculations as well. We consider this next generation Newton type methods, for n = 0, 1, 2, $\cdots$,

$$x_{n+1} = x_n - \frac{6f(x_n)[f''(x_n)]^2}{6f'(x_n)[f''(x_n)]^2 - 3[f''(x_n)]^2 f(x_n) + f'''(x_n)[f(x_n)]^2},$$

where x_0 is an initial point and f is a real-valued continuously differentiable function.

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1 Introduction

Recently, Verma [2] introduced a class of new Newton-type methods, which outperform most of the traditional Newton's methods and its variants in the literature. The convergence rate is higher than three, while the computational cost is nominal when comparing with other computational methods in the literature. We have introduced several classes of hybrid Newton-type methods to the context of solving polynomial equations and beyond, which have convergence rates ranging from super-quadratic to three or more (based on analytical as well as numerical methods), while they work at lower computational cost. These are based on the principle of zooming the graphs of polynomial functions that are deformed after each derivative procedure, for example, the graph of a cubic function is deformed to a quadratic and then to a line. This principle unlike Newton's methods (which are based on the secant-tangent methods) outperforms the secant-tangent methods. Since our analysis and findings are based on including just four decimal places, the picture will be more clearer by including more decimal places. To the best our knowledge, the general framework for the higher-order derivative Newton-type methods and its applications is new, and there is no evidence of any other publication(s) available in the literature by applying the zooming principle.

In this paper, we investigate and establish the cubic convergence analysis for (1.1), which would be a major step for this method in terms of applications to other Newton-like methods and beyond (including interdisciplinary research). We consider this next generation Newton type methods, for n = 0, 1, 2, $\cdots$,

$$x_{n+1} = x_n - \frac{6f(x_n)[f''(x_n)]^2}{6f'(x_n)[f''(x_n)]^2 - 3[f''(x_n)]^2 f(x_n) + f'''(x_n)[f(x_n)]^2}, \tag{1.1}$$

where x_0 is an initial point and f is a real-valued continuously differentiable function with $f'(x_n)$, $f''(x_n)$ and $f'''(x_n)$ non-zero.

2 Cubic Convergence Analysis for Model (1.1)

In this section, we prove and examine the cubic convergence analysis for a class of next generation Newton type methods (1.1). We also compare our finding based on numerical calculations with other existing computational methods in the literature.

Theorem 1. *Consider the following equation*

$$f(x) = 0, \tag{2.2}$$

where f is a real-valued continuously differentiable function with $f'(x_n)$, $f''(x_n)$ and $f'''f(x_n)$ non-zero. Then the next generation Newton type model (1.1) has the cubic convergence rate.

Proof. Let a be a root of $f(x) = 0$, where f is thrice continuously differentiable in the neighborhood of a and x_n. Moreover, suppose that

$$K_0 := \mid -\frac{2f'(a)f'''(a) - 3[f''(a)]^2)}{12[f'(a)]^2} \mid$$

exists and is finite.

Then by Taylor's theorem, we have

$$0 = f(a) = f(x_n) + f'(x_n)(a - x_n) + \frac{1}{2}f''(x_n)(a - x_n)^2 + \frac{1}{6}f'''(\zeta)(a - x_n)^3, \tag{2.3}$$

or

$$0 = f(a) = f(x_n) + f'(x_n)(a - x_n) + \frac{1}{2}f''(\eta)(a - x_n)^2. \tag{2.4}$$

Multiply (2.2) by $2f'(x_n)$ and (2.3) by $f''(x_n)(a-x_n)$, and then subtract in the following manner:

$$0 = 2f(x_n)f'(x_n) + 2[f'(x_n)]^2(a-x_n) + f'(x_n)f''(x_n)(a-x_n)^2$$
$$+\frac{1}{3}f'(x_n)f'''(\zeta)(a-x_n)^3$$
$$-f''(x_n)(a-x_n)[f(x_n) + f'(x_n)(a-x_n) + \frac{1}{2}f''(\eta)(a-x_n)^2].$$

Thus, we have

$$0 = 2f(x_n)f'(x_n) + \{2[f'(x_n)]^2 - f(x_n)f''(x_n)\}(a-x_n)$$
$$+\{f'(x_n)f''(x_n) - f'(x_n)f''(x_n)\}(a-x_n)^2$$
$$+\{\frac{1}{3}f'(x_n)f'''(\zeta) - \frac{1}{2}f(x_n)f''(x_n)f''(\eta)\}(a-x_n)^3.$$

As we notice that the coefficient of $(a-x_n)^2$ is zero, we find that

$$0 = 2f(x_n)f'(x_n) + \{2[f'(x_n)]^2 - f(x_n)f''(x_n)\}(a-x_n)$$
$$+\{\frac{1}{3}f'(x_n)f'''(\zeta) - \frac{1}{2}f(x_n)f''(x_n)f''(\eta)\}(a-x_n)^3.$$

This implies that

$$a - x_n = -\frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} - \frac{2f'(x_n)f'''(\zeta) - 3f''(x_n)f''(\eta)}{6(2[f'(x_n)]^2 - f(x_n)f''(x_n))}(a-x_n)^3.$$

Next, applying (1.1), we arrive at

$$a - x_{n+1} = \frac{6f(x_n)[f''(x_n)]^2}{6f'(x_n)[f''(x_n)]^2 - 3[f''(x_n)]^2 f(x_n) + f'''(x_n)[f(x_n)]^2}$$
$$-\frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}$$
$$-\frac{2f'(x_n)f'''(\zeta) - 3f''(x_n)f''(\eta)}{6(2[f'(x_n)]^2 - f(x_n)f''(x_n))}(a-x_n)^3, \tag{2.5}$$

where ζ and η are real numbers lying between a and x_n.

Next, we find the limits of the expressions and the limit of coefficients of $(a-x_n)^3$, respectively, in (2.4) on the right hand side as $x_n \to a$ as follows:

$$\frac{6f(x_n)[f''(x_n)]^2}{6f'(x_n)[f''(x_n)]^2-3[f''(x_n)]^2f(x_n)+f'''(x_n)[f(x_n)]^2} \\ -\frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2-f(x_n)f''(x_n)} \to 0,$$

and

$$-\frac{2f'(x_n)f'''(\zeta)-3f''(x_n)f''(\eta)}{6(2[f'(x_n)]^2-f(x_n)f''(x_n))} \to -\frac{2f'(a)f'''(a)-3f''(a)f''(a)}{12[f'(a)]^2}$$

Now, if we take a little larger

$$K > \mid -\frac{2f'(a)f'''(a)-3[f''(a)]^2)}{12[f'(a)]^2} \mid,$$

i.e.,

$$K > K_0.$$

Then taking the absolute value of both sides of (2.4) and we can replace the absolute value coefficient by its upper bound for

$$\mid a-x_{n+1} \mid \leq K \mid a-x_n \mid^3 .$$

□

Example 1. *Next, we consider an example on comparison of method (1.1) with other existing Newton-type methods in the literature. Let* $f(x)=x^3-2x-5$, $f'(x)=3x^2-2$ *and* $f''(x)=6x$. *Let us start with* *initial point* $x_0 = 2$ *based on the change of the values of the function. Then for n = 0, we have*

$x_1 = 2.0951$ *(Verma (1.1))*

$x_1 = 2.10$ *(Newton's Method)*

$x_1 = 2.0947$ *(Householder's Method)*

$x_1 = 2.0945$ *(Halley's Method)*

Remark 1. *The convergence rate for the methods (1.1) can also be determined using (COC) or (ACOC) given by*

$$\xi_1 = ln\Big(\frac{| x_{n+1} - x^* |}{| x_n - x^* |}\Big)/ln\Big(\frac{| x_n - x^* |}{| x_{n-1} - x^* |}\Big)$$

and

$$\xi_2 = ln\Big(\frac{| x_{n+1} - x_n |}{| x_n - x_{n-1} |}\Big)/ln\Big(\frac{| x_n - x_{n-1} |}{| x_{n-1} - x_{n-2} |}\Big).$$

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Degree of Approximation of Fourier Series of Functions in Besov Space

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Abstract

This paper studies the degree of approximation of functions by Cesaro means of their Fourier series in Besov space.

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1 Definitions

Modulus of Continuity:

Let $A = R, R+, [a,b] \subset R$ or T (which usually taken to be R with identification of points modulo 2π).

The modulus of continuity $w(f,t) = w(t)$ of a function f on A can be defined as

$$w(t) = w(f,t) = \sup_{\substack{|x-y|\leq t \\ x,y\in A}} |f(x) - f(y)|, t \geq 0$$

Modulus of Smoothness:

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The kth order modulus of smoothness of a function $f : A \longrightarrow R$ is defined by

$$w_k(f,t) = \sup_{0<h\leq t} \{\{\sup |\Delta_h^k(f,x)| : x, x+kh \in A\}\}, t \geq 0 \tag{1.1}$$

$$where\ \Delta_h^k(f,x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+ih), k \in N \tag{1.2}$$

For $k = 1, w_1(f,t)$ is called the modulus of continuity of f. The function w is continuous at $t = 0$ if and only if f is uniformly continuous on A, that is $f \in \widetilde{C}(A)$. The k^{th} order modulus of smoothness of $f \in L_p(A), 0 < p < \infty$ or of $f \in \widetilde{C}(A)$, if$p = \infty$ is defined by

$$w_k(f,t)_p = \sup_{0<h\leq t} \|\Delta_h^k(f,.)\|_p, t \geq 0 \tag{1.3}$$

if $p \geq 1, k = 1$, then $w_1(f,t)_p = w(f,t)_p$ is a modulus of continuity (or integral modulus of continuity). If $p = \infty, k = 1$ and f is continous then $w_k(f,t)_p$ reduces to modulus of continuity $w_1(f,t)$ or $w(f,t)$.

Lipschitz Space:

If $f \in \widetilde{C}(A)$ and

$$w(f,t) = O(t^\alpha), 0 < \alpha \leq 1 \tag{1.4}$$

then we write $f \in Lip\alpha$. If $w(f,t) = o(t)$ as $t \to 0+$ (in particular (1.4) holds for $\alpha > 1$) then f reduces to a constant.

If $f \in L_p(A), 0 < p < \infty$ and

$$w(f,t)_p = O(t^\alpha), 0 < \alpha \leq 1 \tag{1.5}$$

then we write $f \in Lip(\alpha, p), 0 < p < \infty, 0 < \alpha \leq 1$. The case $\alpha > 1$ is of no interest as the function reduces to a constant, whenever

$$w(f,t)_p = o(t) \text{ as } t \to 0+ \tag{1.6}$$

We note that if $p = \infty$ and $f \in C(A)$ then Lip (α, p) class reduces to Lip α class.

Generalized Lipschitz Space:

Let $\alpha > 0$ and suppose that $k = [\alpha] + 1$. For $f \in L_p(A), 0 < p \leq \infty$, if

$$w_k(f,t)_p = O(t^{\alpha}), t > 0 \tag{1.7}$$

then we write

$$f \in Lip^*(\alpha, p), \alpha > 0, 0 < p \leq \infty \tag{1.8}$$

and say that f belongs to generalized Lipschitz space. The seminorm is then

$$|f|_{Lip^*(\alpha,L_p)} = \sup_{t>0} \left(t^{-\alpha} w_k(f,t)_p \right).$$

It is known ([4], p-52) that the space $Lip^*(\alpha, L_p)$ contains $Lip(\alpha, L_p)$. For $0 < \alpha < 1$ the spaces coincide, (for $p = \infty$, it is necessary to replace L_∞ by $\widetilde{C}$ of uniformly continuous function on A). For $0 < \alpha < 1$ and $p = 1$ the space $Lip^*(\alpha, L_p)$ coincide with $Lip\alpha$.

For $\alpha = 1, p = \infty$, we have

$$\text{Lip}(1, \widetilde{C}) = \text{Lip } 1 \tag{1.9}$$

but

$$\text{Lip}^*(1, \widetilde{C}) = Z \tag{1.10}$$

is the Zygmund space[8], which is characterised by (1.7) with $k = 2$.

H_α Space [3]:

For $0 < \alpha \leq 1$, let

$$H_\alpha = \{ f \in C_{2\pi} : w(f,t) = O(t^{\alpha}) \} \tag{1.11}$$

It is known [3] that H_α is a Banach space with the norm $\|.\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{t>0} t^{-\alpha} w(t), 0 < \alpha \leq 1 \tag{1.12}$$

$$\|f\|_0 = \|f\|_c$$

$$\text{and } H_\alpha \subseteq H_\beta \subseteq C_{2\pi}, 0 < \beta \leq \alpha \leq 1 \tag{1.13}$$

$H_{(\alpha,p)}$ **Space:**

For $0 < \alpha \leq 1$, let

$$H_{(\alpha,p)} = \{f \in L_p[0,2\pi] : 0 < p \leq \infty, w(f,t)_p = O(t^{\alpha})\} \tag{1.14}$$

and introduce the norm $\|.\|_{(\alpha,p)}$ as follows

$$\|f\|_{(\alpha,p)} = \|f\|_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, \qquad 0 < \alpha \leq 1 \tag{1.15}$$

$$\|f\|_{(0,p)} = \|f\|_p$$

It is known [3] that $H_{(\alpha,p)}$ is a Banach space for $p \geq 1$ and a complete p-normed space for $0 < p < 1$. Also

$$H_{(\alpha,p)} \subseteq H_{(\beta,p)} \subseteq L_p, \qquad 0 < \beta \leq \alpha \leq 1 \tag{1.16}$$

Note that $H_{(\alpha,\infty)}$ is the space H_α defined above.

For study of degree of approximation problems the natural way to proceed to consider with some restrictions on some modulus of smoothness as prescribed in H_α and $H_{(\alpha,p)}$ spaces. As we have seen above only a constant function satisfies Lipschitz condition for $\alpha > 1$. However for generalized Lipschitz class there is no such restriction on α. We required a finer scale of smoothness than is provided by Lipschitz class. For each $\alpha > 0$ Besov developed a remarkable technique for restricting modulus of smoothness by introducing a third parameter q (in addition to p and α) and applying α, q norms (rather than α, ∞ norms) to the modulus of smoothness $w_k(f,.)_p$ of f.

Besove Space:

Let $\alpha > 0$ be given and let $k = [\alpha] + 1$. For $0 < p, q \leq \infty$ the Besove space ([4], p.54) $B_q^\alpha(L_p)$ is defined as follows:

$$B_q^\alpha(L_p) = \{f \in L_p : |f|_{B_q^\alpha(L_p)} = \|w_k(f,.)\|_{(\alpha,q)} \text{ is finite}\}$$

where

$$\|w_k(f,.)\|_{\alpha,q} = \begin{cases} \left(\int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty \quad (1.17) \\ \sup_{t>0} t^{-\alpha} w_k(f,t)_p, & q = \infty \quad (1.18) \end{cases}$$

It is known ([4], p.55) that $\|w_k(f,.)\|_{\alpha,q}$ is a seminorm if $1 \leq p,q \leq \infty$ and a quasi-seminorm in other cases.

The Besov norm for $B_q^{\alpha}(L_p)$ is

$$\|f\|_{B_q^{\alpha}(L_p)} = \|f\|_p + \|w_k(f,.)\|_{(\alpha,q)} \tag{1.19}$$

It is known ([7], p.237) that for 2π-periodic function f, the integral $\int_0^{\infty}$ in (1.18) is replaced by $\int_0^{\pi}$.
We know ([4], p.56, [7], p.236) the following inclusion relations.

(I) For fixed α and p

$$B_q^{\alpha}(L_p) \subset B_{q_1}^{\alpha}(L_p), \qquad q < q_1$$

(II) For fixed p and q

$$B_q^{\alpha}(L_p) \subset B_q^{\beta}(L_p), \qquad \beta < \alpha$$

(III) For fixed α and q

$$B_q^{\alpha}(L_p) \subset B_q^{\alpha}(L_{p_1}), \qquad p_1 < p$$

Special Cases of Besov Space:

For $q = \infty, B_{\infty}^{\alpha}(L_p), \alpha > 0, p \geq 1$ is same as $Lip^*(\alpha, L_p)$ the generalized Lipschitz space and the corresponding norm $\|.\|_{B_{\infty}^{\alpha}(L_p)}$ is given by

$$\|f\|_{B_{\infty}^{\alpha}(L_p)} = \|f\|_p + \sup_{t>0} t^{-\alpha} w_k(f,t)_p \tag{1.20}$$

For every $\alpha > 0$ with $k = [\alpha] + 1$,

For the special case when $0 < \alpha < 1, B_{\infty}^{\alpha}(L_p)$ space reduces to $H_{(\alpha,p)}$ space due to Das, Ghosh and Ray[3] and the corresponding norm is given by

$$\|f\|_{B_{\infty}^{\alpha}(L_p)} = \|f\|_{(\alpha,p)} = \|f\|_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, 0 < \alpha < 1 \tag{1.21}$$

For $\alpha = 1$, the norm is given by

$$\|f\|_{B_{\infty}^{1}(L_p)} = \|f\|_p + \sup_{t>0} t^{-1} w_2(f,t)_p \tag{1.22}$$

Note that $\|f\|_{B^1_\infty(L_p)}$ is not same as $\|f\|_{(1,p)}$ and the space $B^1_\infty(L_p)$ includes the space $H(1,p), p \geq 1$. If we further specialize by taking $p=\infty$, $B^\alpha_\infty(L_\infty), 0<\alpha<1$, coincides with H_α space due to Prossodorf[6] and the norm is given by

$$\|f\|_{B^\alpha_\infty(L_\infty)} = \|f\|_\alpha = \|f\|_c + \sup_{t>0} t^\alpha w(f,t), 0<\alpha<1 \tag{1.23}$$

For $\alpha=1, p=\infty$, the norm is given by

$$\|f\|_{B^1_{\infty(L_\infty)}} = \|f\|_c + \sup_{t>0} t^{-1} w_2(f,t), \alpha=1 \tag{1.24}$$

which is different from $\|f\|_1$ and $B^1_\infty(L_\infty)$ includes the H_1 space.

2 Introduction

Let f be a 2π periodic function and $f \in L_p[0,2\pi], p \geq 1$. The Fourier series of f at x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x)$$

In the case $0<p<1$, we can still regard it as the Fourier series of f by further assuming $f(t)\cos nt$ and $f(t)\sin nt$ are integrable.

Prossdorff [6] first obtained the following on approximation of functions in H_α space using Fejer mean of Fourier series.

Theorem A:

Let $f \in H_\alpha (0<\alpha\leq 1)$ and $0\leq\beta<\alpha\leq 1$. Then

$$\|\sigma_n(f)-f\|_\beta = O(1)\begin{cases} \dfrac{1}{n^{\alpha-\beta}}, & 0<\alpha<1 \\ \left(\frac{\log n}{n}\right)^{1-\beta}, & \alpha=1, \end{cases}$$

where $\sigma_n(f)$ is the Fejer means of the Fourier series of f.The case $\beta=0$ of Theorem A is due to Alexists [1]. Chandra [2] obtained the generalisation of Theorem A in the Nörlund (N,p) and $(\overline{N},p)$ transform set up.

Later Mohapatra and Chandra [5] studied the problem by matrix means and obtained the above results as corollaries. Das, Ghosh and Ray[3] further generalised the work by studying the problem for functions in $H(\alpha, p)$ space $(0 < \alpha \leq 1, p \geq 1)$ by matrix means of their Fourier series in the generalised Hölder metric.

In the present work we propose to study the degree of approximation of functions in Besov space which is a generalisation of $H(\alpha, p)$ space.

We write

$$\phi_x(u) = f(x+u) + f(x-u) - 2f(x) \tag{2.1}$$

Let $S_n(x) = S_n(f;x)$ denote the nth partial sum of the Fourier series. It is known ([9], Vol-I, p.50) that

$$S_n(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \phi_x(u) D_n(u)du \tag{2.2}$$

where the Dirichlet's Kernel

$$D_n(u) = \frac{1}{2} + \sum_{k=0}^{n} \cos kx = \frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}} \tag{2.3}$$

Let $\sigma_n^{\gamma}(f;x)$ denote the Cesaro mean $(C, \gamma), \gamma > 0$ of the Fourier series. Then

$$\sigma_n^{\gamma}(f;x) = \frac{1}{A_n^{\gamma}} \sum_{k=0}^{n} A_{n-k}^{\gamma-1} S_k(f;x) \tag{2.4}$$

Where A_n^{γ} is given by the formula ([9] Vol-I, p.76)

$$\sum_{n=0}^{\infty} A_n^{\gamma} x^n = (1-x)^{-\gamma-1}, \gamma > -1, |x| < 1 \tag{2.5}$$

We know ([9], Vol.I, p.49) that

$$l_n^{\gamma}(x) = \sigma_n^{\gamma}(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \phi_x(u) K_n^{\gamma}(u)du \tag{2.6}$$

where,

$$K_n^{\gamma}(u) = \frac{1}{A_n^{\gamma}} \sum_{k=0}^{n} A_{n-k}^{\gamma-1} D_k(u) \tag{2.7}$$

3 Main Result

Theorem: Let $0 \leq \beta < \alpha < 2$. If $f \in B_q^\alpha(L_p), p \geq 1$ and $1 < q \leq \infty$, then

(i) For $0 < \gamma < 1$

$$\|l_n^\gamma(.)\|_{B_q^\beta(L_p)} = O(1) \begin{cases} \frac{1}{n^\gamma}, & \alpha - \beta - \frac{1}{q} > \gamma \\ \frac{1}{n^{\alpha-\beta-\frac{1}{q}}}, & \alpha - \beta - \frac{1}{q} < \gamma \\ \frac{(\log n)^{1-\frac{1}{q}}}{n^\gamma}, & \alpha - \beta - \frac{1}{q} = \gamma \end{cases}$$

(ii) For $\gamma \geq 1$

$$\|l_n^\gamma(.)\|_{B_q^\beta(L_p)} = O(1) \begin{cases} \frac{1}{n}, & \alpha - \beta - \frac{1}{q} > 1 \\ \frac{1}{n^{\alpha-\beta-\frac{1}{q}}}, & \alpha - \beta - \frac{1}{q} < 1 \\ \frac{(\log n)^{1-\frac{1}{q}}}{n}, & \alpha - \beta - \frac{1}{q} = 1 \end{cases}$$

Note: The case $0 < q \leq 1$, needs different argument, to be consider separately.

We need the following additional notations for the proof of the theorem.

$$\Phi(x,t,u) = \begin{cases} \phi_{x+t}(u) - \phi_x(u), & 0 < \alpha < 1 \\ \phi_{x+t}(u) - \phi_{x-t}(u) - 2\phi_x(u), & 1 \leq \alpha < 2 \end{cases} \tag{3.1}$$

For $k = [\alpha] + 1$, we have for $p \geq 1$

$$w_k(f,t)_p = \begin{cases} w_1(f,t)_p, & 0 < \alpha < 1 \\ w_2(f,t)_p, & 1 \leq \alpha < 2 \end{cases} \tag{3.2}$$

For $\gamma > 0$, we write

$$L_n^\gamma(x,t) = \begin{cases} l_n^\gamma(x+t) - l_n^\gamma(x), & 0 < \alpha < 1 \\ l_n^\gamma(x+t) + l_n^\gamma(x-t) - 2l_n^\gamma(x), & 1 \leq \alpha < 2 \end{cases} \tag{3.3}$$

Using (2.6) and (3.1) respectively for the expressions $l_n^\gamma(x)$ and $\Phi(x,t,u)$, we have

$$L_n^\gamma(x,t) = \frac{1}{\pi}\int_0^\pi \Phi(x,t,u)K_n^\gamma(u)du \tag{3.4}$$

Using definition of $w_k(f,t)_p$ and (3.3), we have

$$w_k(l_n^\gamma,t)_p = \|L_n^\gamma(.,t)\|_p \tag{3.5}$$

We need the following lemmas to prove the theorem.

4 Lemmas:

Lemma 1: Let $1 \leq p \leq \infty$ and $0 < \alpha < 2$. If $f \in L_p[0,2\pi]$, then for $0 < t,u \leq \pi$

(i) $\|\Phi(.,t,u)\|_p \leq 4w_k(f,t)_p$

(ii) $\|\Phi(.,t,u)\|_p \leq 4w_k(f,u)_p$

(iii) $\|\phi.(u)\|_p \leq 2w_k(f,u)_p$,

where $k = [\alpha]+1$.

Proof: Case $0 < \alpha < 1$.

Clearly $k = [\alpha]+1 = 1$. By virtue of (2.1)

$\Phi(x,t,u) = \phi_{x+t}(u) - \phi_x(u)$, can be written as

$$\Phi(x,t,u) = \begin{cases} \{f(x+t+u)-f(x+u)\}+\{f(x+t-u)-f(x-u)\} \\ -2\{f(x+t)-f(x)\} & (4.1) \\ \{f(x+t+u)-f(x+t)\}+\{f(x-u+t)-f(x+t)\} \\ -\{f(x+u)-f(x)\}-\{f(x-u)-f(x)\} & (4.2) \end{cases}$$

Applying Minkowoski's inequality to (4.1), we get for $p \geq 1$

$$\begin{aligned} \|\Phi(.,t,u)\|_p &\leq \|f(.+t+u)-f(.+u)\|_p + \|f(.+t-u)-f(.-u)\|_p + 2\|f(.+t)-f(.)\|_p \\ &\leq 4w_1(f,t)_p,\ 0<\alpha<1 \end{aligned}$$

Similarly applying Minkowoski's inequality to (4.2), we get for $p \geq 1$

$$\|\Phi(.,t,u)\|_p \leq 4w_1(f,u)_p.$$

Case $1 \leq \alpha < 2$.

Clearly $k = [\alpha]+1 = 2$. By virtue of (2.1)

$\Phi(x,t,u) = \phi_{x+t}(u) + \phi_{x-t}(u) - 2\phi_x(u)$, can be written as

$$\Phi(x,t,u) = \begin{cases} \{f(x+t+u)+f(x+t-u)-2f(x+t)\}+\{f(x-t+u)+f(x-t-u) \\ -2f(x-t)\}-2\{f(x+u)+f(x-u)-2f(x)\} & (4.3) \\ \{f(x+t+u)+f(x-t+u)-2f(x+u)\}+\{f(x+t-u)+f(x-t-u) \\ -2f(x-u)\}-2\{f(x+t)+f(x-t)-2f(x)\} & (4.4) \end{cases}$$

Applying Minkowski's inequality to (4.3), we obtain for $p \geq 1$

$$\begin{aligned}\|\Phi(.,t,u)\|_p &\leq \|f(.+t+u)+f(.+t-u)-2f(.+t)\|_p \\ &+\|f(.-t+u)+f(.-t-u)-2f(.-t)\|_p \\ &+2\|f(.+u)+f(.-u)-2f(.)\|_p \\ &\leq 4w_2(f,u)_p\end{aligned}$$

Using (4.4) and proceeding as above, we get

$$\|\Phi(.,t,u)\|_p \leq 4w_2(f,t)_p$$

this completes the proof of part(i) and 9ii). We omit the proof of (iii) as it is trivial.

Lemma 2: Let $0 < \alpha < 2$.Suppose that $0 \leq \beta < \alpha$. If $f \in B_q^\alpha(L_p), p \geq 1, 1 < q < \infty$, then for $\gamma > 0$

(i) $\displaystyle\int_0^\pi |K_n^\gamma(u)| \left(\int_0^u \frac{\|\Phi(.,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right)^{\frac{1}{q}} du = O(1)\left\{\int_0^\pi \left(u^{\alpha-\beta}|K_n^\gamma(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$

(ii) $\displaystyle\int_0^\pi |K_n^\gamma(u)| \left(\int_u^\pi \frac{\|\Phi(.,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right)^{\frac{1}{q}} du = O(1)\left\{\int_0^\pi \left(u^{\alpha-\beta+\frac{1}{q}}|K_n^\gamma(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$

Proof: Applying Lemma 1(i), we have

$$\begin{aligned}&\int_0^\pi |K_n^\gamma(u)| \left(\int_0^u \frac{\|\Phi(.,t,u)\|_p^q}{t^{\beta q+1}}dt\right)^{1/q} du \\ &= O(1)\int_0^\pi |K_n^\gamma(u)| \left\{\int_0^u \left(\frac{w_k(f,t)_p}{t^\alpha}\right)^q t^{(\alpha-\beta)q}\frac{dt}{t}\right\}^{1/q} du \\ &= O(1)\int_0^\pi |K_n^\gamma(u)|u^{\alpha-\beta}du\left\{\int_0^u \frac{w_k(f,t)_p}{t^\alpha}\frac{dt}{t}\right\}^{1/q} \\ &= O(1)\int_0^\pi |K_n^\gamma(u)|u^{\alpha-\beta}du\end{aligned}$$

by Second Mean Value theorem and by definition of Besov Space.

Applying Holders inequality

$$\begin{aligned}&= O(1)\left\{\int_0^\pi \left(|K_n^\gamma(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \left(\int_0^\pi 1^q du\right)^{\frac{1}{q}} \\ &= O(1)\left\{\int_0^\pi \left(|K_n^\gamma(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\end{aligned}$$

For the second part, applying Lemma 1(ii), we get

$$\int_0^{\pi} |K_n^{\gamma}(u)|du \left\{\int_u^{\pi} \frac{\|\Phi(.,t,u)\|_p^q}{t^{\beta q+1}}dt\right\}^{1/q}$$
$$= O(1)\int_0^{\pi} |K_n^{\gamma}(u)|w_k(f,u)_p du\left(\int_u^{\pi} \frac{dt}{t^{\beta q+1}}\right)^{1/q}$$
$$= O(1)\int_0^{\pi} |K_n^{\gamma}(u)|w_k(f,u)_p u^{-\beta}du$$
$$= O(1)\int_0^{\pi} \left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right)u^{\alpha-\beta+\frac{1}{q}}|K_n^{\gamma}(u)|du$$

Applying Holder's inequality

$$= O(1)\left\{\int_0^{\pi} \left(\frac{w_k(f,u)_p}{u^{\alpha}}\right)^q \frac{du}{u}\right\}^{\frac{1}{q}} \left\{\int_0^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}}|K_n^{\gamma}(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$
$$= O(1)\left\{\int_0^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}}|K_n^{\gamma}(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

by definition of Besov space.

Lemma 3: Let $0<\alpha<2$. Suppose that $0\le\beta<\alpha$. If $f\in B_q^{\alpha}(L_p), p\ge 1$ and $q=\infty$ then

$$\sup_{0<t\le u\le\pi} t^{-\beta}\|\Phi(.,t,u)\|_p = O(u^{\alpha-\beta}).$$

Proof: For $0<t\le u\le\pi$, applying Lemma 1(i), we have

$$\begin{aligned}\sup_{\substack{t\\0<t\le u\le\pi}} t^{-\beta}\|\Phi(.,t,u)\|_p &= \sup_{\substack{t\\0<t\le u\le\pi}} t^{\alpha-\beta}(t^{-\alpha}\|\Phi(.,t,u)\|_p)\\ &\le 4u^{\alpha-\beta}\sup_t(t^{-\alpha}w_k(f,t)_p)\\ &= O(u^{\alpha-\beta}), \text{by the hypothesis.}\end{aligned}$$

Next for $0<u\le t\le\pi$, applying Lemma 1(ii), we get

$$\begin{aligned}\sup_{\substack{t\\0<u\le t\le\pi}} t^{-\beta}\|\Phi(.,t,u)\|_p &\le 4w_k(f,u)_p \sup_{\substack{t\\0<u\le t\le\pi}} t^{-B}\\ &\le 4u^{\alpha-\beta}\sup_u(u^{-\alpha}w_k(f,u)_p)\\ &= O(u^{\alpha-\beta}), \text{by the hypothesis.}\end{aligned}$$

and this completes the proof.

Lemma 4: Let the (C,γ) Kernel $K_n^{\gamma}(u)$ of the Fourier series be defined as in (2.7). Then for $-1<\gamma<1, 0<u\leq\pi$

(i)

$$K_n^{\gamma}(u)=\begin{cases} O(n) \\ O(n^{-\gamma}u^{-1-\gamma}) \end{cases}$$

and for $\gamma\geq 1$ and $0<u\leq\pi$

(ii)

$$K_n^{\gamma}(u)=\begin{cases} O(n) \\ O(n^{-1}u^{-2}). \end{cases}$$

Proof Part(i) can be found in ([9], Vol.I,p.95). We omit the proof of (ii) as it can be proved by similar argument.

5 Proof of the Theorem: We first consider the case $1<q<\infty$. We have for $p\geq 1,\ 0\leq\beta<\alpha<2, \gamma>0$.

$$\|l_n^{\gamma}(.)\|_{B^{\beta}_{q(L_p)}}=\|l_n^{\gamma}(.)\|_p+\|w_k(l_n^{\gamma},.)\|_{(\beta,q)} \tag{5.1}$$

Applying Lemma 1(iii) in (2.6), we get

$$\begin{aligned}\|l_n^{\gamma}(.)\|_p &\leq \frac{1}{\pi}\int_0^{\pi}\|\phi.(u)\|_p|K_n^{\gamma}(u)|du \\ &\leq \frac{2}{\pi}\int_0^{\pi}|K_n^{\gamma}(u)|w_k(f,u)_p du \end{aligned} \tag{5.2}$$

At this stage, applying Holder's inequality, we have

$$\begin{aligned}\|l_n^{\gamma}(.)\|_p &\leq \frac{2}{\pi}\left\{\int_0^{\pi}(|K_n^{\gamma}(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\left\{\int_0^{\pi}\left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right)^q du\right\}^{\frac{1}{q}} \\ &= O(1)\left[\int_0^{\pi}(|K_n^{\gamma}(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du\right]^{1-\frac{1}{q}}, \text{by definition of Besov Space} \\ &= O(1)\left\{\int_0^{\frac{\pi}{n}}+\int_{\frac{\pi}{n}}^{\pi}\right\}^{1-\frac{1}{q}}\end{aligned}$$

$$=O(1)\left[\left\{\int_0^{\pi/n}\left(|K_n^{\gamma}(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}+\left\{\int_{\pi/n}^{\pi}\left(|K_n^{\gamma}(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\right]$$

by the inequality $(x+y)^r\leq x^r+y^r$ for $0<r\leq 1$

$$=O(1)[I+J], \quad \text{say} \tag{5.3}$$

We shall consider the cases $0<\gamma<1$ and $\gamma\geq 1$ separately. We first study the case $0<\gamma<1$, to proof first part of the theorem.

Using Lemma 4(i), we have for $0<\gamma<1$

$$\begin{aligned}
I &= \left\{\int_0^{\pi/n}\left(|K_n^\gamma(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O(n)\left\{\int_0^{\pi/n}u^{\frac{q}{q-1}(\alpha+\frac{1}{q})}du\right\}^{1-\frac{1}{q}}\\
&= O(n)\left(\int_0^{\pi/n}u^{\frac{q}{q-1}(\alpha+1)-1}du\right)^{1-\frac{1}{q}}\\
&= O\left(\frac{1}{n^\alpha}\right) \qquad (5.4)
\end{aligned}$$

Using the second estimate of Lemma 4(i), we have for $0<\gamma<1$

$$\begin{aligned}
J &= \left\{\int_{\pi/n}^{\pi}\left(|K_n^\gamma(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O\left(\frac{1}{n^\gamma}\right)\left\{\int_{\pi/n}^{\pi}u^{\frac{q}{q-1}\left(\alpha+\frac{1}{q}-\gamma-1\right)}du\right\}^{1-\frac{1}{q}}\\
&= O\left(\frac{1}{n^\gamma}\right)\left\{\int_{\pi/n}^{\pi}u^{\frac{q}{q-1}(\alpha-\gamma)-1}du\right\}^{1-\frac{1}{q}}\\
&= O\left(\frac{1}{n^\gamma}\right)\begin{cases}1, & \alpha>\gamma\\ n^{\gamma-\alpha}, & \alpha<\gamma\\ (\log n)^{1-\frac{1}{q}}, & \alpha=\gamma\end{cases} \qquad (5.5)
\end{aligned}$$

From (5.3), (5.4) and (5.5) , we get

$$\|l_n^\gamma(.)\|_p = O(1)\begin{cases}\frac{1}{n^\gamma}, & \alpha>\gamma\\ \frac{1}{n^\alpha}, & \alpha<\gamma\\ \frac{(\log n)^{1-\frac{1}{q}}}{n^\gamma}, & \alpha=\gamma\end{cases} \qquad (5.6)$$

Now

$$
\begin{aligned}
\|w_k(l_n^\gamma,.)\|_{\beta,q} &= \left\{\int_0^\pi \left(\frac{w_k(l_n^\gamma,t)_p}{t^\beta}\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\pi \left(\frac{\|L_n^\gamma(.,t)\|_p}{t^\beta}\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\pi \left(\int_0^\pi |L_n^\gamma(x,t)|^p dx\right)^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\pi \left\{\int_0^\pi \left|\frac{1}{\pi}\int_0^\pi \Phi(x,t,u)K_n^\gamma(u)du\right|^p dx\right\}^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right\}^{\frac{1}{q}}
\end{aligned}
$$

By repeated application of generalized Minkowski's inequality, we have

$$
\begin{aligned}
\|w_k(l_n^\gamma,.)\|_{(\beta,q)} &\le \frac{1}{\pi}\left[\int_0^\pi \left\{\int_0^\pi \left(\int_0^\pi |\Phi(x,t,u)|^p |K_n^\gamma(u)|^p dx\right)^{1/p} du\right\}^q \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}} \\
&= \frac{1}{\pi}\left[\int_0^\pi \left\{\int_0^\pi |K_n^\gamma(u)| \|\Phi(.,t,u)\|_p du\right\}^q \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}} \\
&\le \frac{1}{\pi}\int_0^\pi \left(\int_0^\pi |K_n^\gamma(u)|^q \|\Phi(.t,u)\|_p^q \frac{dt}{t^{\beta q+1}}\right)^{\frac{1}{q}} du \\
&= \frac{1}{\pi}\int_0^\pi |K_n^\gamma(u)|du \left\{\int_0^\pi \frac{\|\Phi(.,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}} \\
&= \frac{1}{\pi}\int_0^\pi |K_n^\gamma(u)|du \left\{\left(\int_0^u + \int_u^\pi\right) \frac{\|\Phi(.,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}} \\
&\le \frac{1}{\pi}\int_0^\pi |K_n^\gamma(u)|du \left\{\int_0^u \frac{\|\Phi(.,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}} \\
&+ \frac{1}{\pi}\int_0^\pi |K_n^\gamma(u)|du \left\{\int_u^\pi \frac{\|\Phi(.,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}}
\end{aligned}
$$

Applying Lemma 2, we have

$$
\begin{aligned}
\|w_k(l_n^\gamma,.)\|_{(\beta,q)} &= O(1)\left[\left\{\int_0^\pi \left(|K_n^\gamma(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} + \left\{\int_0^\pi (|K_n^\gamma(u)|u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\right] \\
&= O(1)[I' + J'], \text{say} \qquad (5.7)
\end{aligned}
$$

We now proceed to estimate $I^{'}$ and $J^{'}$ when $0 < \gamma < 1$. Writing

$$\begin{aligned}
I^{'} &= \left\{ \left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right] \left(|K_n^{\gamma}(u)| u^{\alpha-\beta} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
&\leq \left\{ \int_0^{\pi/n} (|K_n^{\gamma}(u)| u^{\alpha-\beta})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} + \left\{ \int_0^{\pi/n} (|K_n^{\gamma}(u)| \, |u^{\alpha-\beta}|)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
&= I_1^{'} + I_2^{'}, \quad \text{say} \qquad (5.8)
\end{aligned}$$

Applying Lemma 4(i), we get

$$\begin{aligned}
I_1^{'} &= O(n) \left\{ \int_0^{\pi/n} u^{\frac{q}{q-1}(\alpha-\beta)} du \right\}^{1-\frac{1}{q}} \\
&= O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}} \right) \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
I_2^{'} &= O\left(\frac{1}{n^{\gamma}} \right) \left\{ \int_{\pi/n}^{\pi} u^{\frac{q}{q-1}(\alpha-\beta-\gamma-1)} du \right\}^{1-\frac{1}{q}} \\
&= O(1) \begin{cases} \frac{1}{n^{\gamma}}, & \alpha-\beta-\frac{1}{q} > \gamma \\ \frac{1}{n^{\alpha-\beta-\frac{1}{q}}}, & \alpha-\beta-\frac{1}{q} < \gamma \\ \frac{(\log n)^{1-\frac{1}{q}}}{n^{\gamma}}, & \alpha-\beta-\frac{1}{q} = \gamma \end{cases} \qquad (5.10)
\end{aligned}$$

Collecting the results from (5.8), (5.9) and (5.10)

$$I^{'} = O(1) \begin{cases} \frac{1}{n^{\gamma}}, & \alpha-\beta-\frac{1}{q} > \gamma \\ \frac{1}{n^{\alpha-\beta-\frac{1}{q}}}, & \alpha-\beta-\frac{1}{q} < \gamma \\ \frac{(\log n)^{1-\frac{1}{q}}}{n^{\gamma}}, & \alpha-\beta-\frac{1}{q} = \gamma \end{cases} \qquad (5.11)$$

We have,

$$\begin{aligned}
J^{'} &= \left\{ \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right) (|K_n^{\gamma}(u)| u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
&\leq \left\{ \int_0^{\pi/n} (|K_n^{\gamma}(u)| u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} + \left\{ \int_{\pi/n}^{\pi} (|K_n^{\gamma}(u)| u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
&= (J_1^{'} + J_2^{'}), \quad \text{say} \qquad (5.12)
\end{aligned}$$

Applying Lemma 4(i), we get

$$\begin{aligned} J_1' &= O(n)\left(\int_0^{\pi/n} u^{\frac{q}{q-1}(\alpha-\beta+\frac{1}{q})}du\right)^{1-\frac{1}{q}} \\ &= O(n)\left(\int_0^{\pi/n} u^{\frac{q}{q-1}(\alpha-\beta+1)-1}du\right)^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{n^{\alpha-\beta}}\right) \end{aligned} \tag{5.13}$$

$$\begin{aligned} J_2' &= O\left(\frac{1}{n^{\gamma}}\right)\left\{\int_{\pi/n}^{\pi} u^{\frac{q}{q-1}(\alpha-\beta+\frac{1}{q}-\gamma-1)}du\right\}^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{n^{\gamma}}\right)\left\{\int_{\pi/n}^{\pi} u^{\frac{q}{q-1}(\alpha-\beta-\gamma)-1}du\right\}^{1-\frac{1}{q}}, \\ &= O(1)\begin{cases} \frac{1}{n^{\gamma}}, & \alpha-\beta>\gamma \\ \frac{1}{n^{\alpha-\beta}}, & \alpha-\beta<\gamma \\ \frac{(\log n)^{1-\frac{1}{q}}}{n^{\gamma}}, & \alpha-\beta=\gamma \end{cases} \end{aligned} \tag{5.14}$$

Collecting the results from (5.12), (5.13) and (5.14), we have

$$J' = O(1)\begin{cases} \frac{1}{n^{\gamma}}, & \alpha-\beta>\gamma \\ \frac{1}{n^{\alpha-\beta}}, & \alpha-\beta<\gamma \\ \frac{(\log n)^{1-1/q}}{n^{\gamma}}, & \alpha-\beta=\gamma \end{cases} \tag{5.15}$$

Collecting the results from (5.7), (5.16) and (5.15), we have

$$\|w_k(l_n^{\gamma},.)\|_{B_q^{\beta}(L_p)} = O(1)\begin{cases} \frac{1}{n^{\gamma}}, & \alpha-\beta-\frac{1}{q}>\gamma \\ \frac{1}{n^{\alpha-\beta-\frac{1}{q}}}, & \alpha-\beta-\frac{1}{q}<\gamma \\ \frac{(\log n)^{1-\frac{1}{q}}}{n^{\gamma}}, & \alpha-\beta-\frac{1}{q}=\gamma \end{cases} \tag{5.16}$$

From (5.6), (5.16) and (5.1), we get for $1<q<\infty,\ p\geq 1, 0\leq\beta<\alpha<2, 0<\gamma<1$

$$\|l_n^{\gamma}(.)\|_{B_q^{\beta}(L_p)} = O(1)\begin{cases} \frac{1}{n^{\gamma}}, & \alpha-\beta-\frac{1}{q}>\gamma \\ \frac{1}{n^{\alpha-\beta-\frac{1}{q}}}, & \alpha-\beta-\frac{1}{q}<\gamma \\ \frac{(\log n)^{1-\frac{1}{q}}}{n^{\gamma}}, & \alpha-\beta-\frac{1}{q}=\gamma \end{cases} \tag{5.17}$$

Now we consider the case when $q=\infty$.

$$\begin{aligned}\|w_k(l_n^\gamma,.)\|_{(\beta,\infty)} &= \sup_{t>0}\frac{\|L_n^\gamma(.,t)\|_p}{t^\beta}\\ &= \sup_{t>0}\frac{t^{-\beta}}{\pi}\left[\int_0^\pi\left|\int_0^\pi \Phi(x,t,u)K_n^\gamma(u)du\right|^p dx\right]^{1/p}\end{aligned}$$

Applying generalized Minkowski's inequality

$$\begin{aligned}\|w_k(l_n^\gamma,.)\|_{(\beta,\infty)} &\le \sup_{t>0}\frac{t^{-\beta}}{\pi}\int_0^\pi du\left\{\int_0^\pi |\Phi(x,t,u)|^p|K_n^\gamma(u)|^p dx\right\}^{1/p}\\ &= \sup_{t>0}\frac{t^{-\beta}}{\pi}\int_0^\pi |K_n^\gamma(u)|\|\Phi(.,t,u)\|_p du\\ &\le \frac{1}{\pi}\int_0^\pi |K_n^\gamma(u)|du\ \sup_{t>0} t^{-\beta}\|\Phi(.,t,u)\|_p\\ &= O(1)\int_0^\pi u^{\alpha-\beta}|K_n^\gamma|du \quad \text{(by Lemma 3)}\\ &= O(1)\left(\int_0^{\pi/n}+\int_{\pi/n}^\pi\right)u^{\alpha-\beta}|K_n^\gamma(u)|du\\ &= O(1)\left[\int_0^{\pi/n}u^{\alpha-\beta}|K_n^\gamma(u)|du+\int_{\pi/n}^\pi u^{\alpha-\beta}|K_n^\gamma(u)|du\right]\end{aligned}$$

Using lemma 4(i) for $K_n^\gamma(u)$, it can be shown that

$$\|w_k(l_n^\gamma,.)\|_{(\beta,\infty)} = O(1)\begin{cases}\frac{1}{n^\gamma}, & \alpha-\beta>\gamma\\ \frac{1}{n^{\alpha-\beta}}, & \alpha-\beta<\gamma\\ \frac{\log n}{n^\gamma}, & \alpha-\beta=\gamma\end{cases} \tag{5.18}$$

From (5.2), we have for $q=\infty$

$$\begin{aligned}\|l_n^\gamma(.)\|_p &\le \frac{2}{\pi}\int_0^\pi |K_n^\gamma(u)|w_k(f,u)_p du\\ &= O(1)\int_0^\pi |K_n^\gamma(u)|u^\alpha du \quad \text{(by the hypothesis)}\\ &= O(1)\left[\int_0^{\frac{\pi}{n}}|K_n^\gamma(u)|u^\alpha du+\int_{\frac{\pi}{n}}^\pi |K_n^\gamma(u)|u^\alpha du\right]\end{aligned}$$

Applying lemma 4(i), it can be shown that

$$\|l_n^\gamma(.)\|_p = O(1)\begin{cases}\frac{1}{n^\gamma}, & \alpha>\gamma\\ \frac{1}{n^\alpha}, & \alpha<\gamma\\ \frac{(\log n)}{n^\gamma}, & \alpha=\gamma\end{cases} \tag{5.19}$$

From (5.18) and (5.19) we have for $q = \infty$

$$\begin{aligned}\|l_n^\gamma(.)\|_{B_\infty^\beta(L_p)} &= \|l_n^\gamma(.)\|_p + \|w_k(l_n^\gamma,.)\|_{(\beta,\infty)} \\ &= O(1)\begin{cases} \frac{1}{n^\gamma}, & \alpha-\beta>\gamma \\ \frac{1}{n^{\alpha-\beta}}, & \alpha-\beta<\gamma \\ \left(\frac{\log n}{n^\gamma}\right), & \alpha-\beta=\gamma \end{cases}\end{aligned} \tag{5.20}$$

Collecting the results of (5.17) and (5.20), we have first part of the Theorem.

For second part of the theorem, proceeding as above and applying Lemma 4(ii) instead of Lemma 4(i) at appropriate steps.

We can obtain for $1 < q < \infty, p \geq 1, 0 \leq \beta < \alpha < 2$ and $\gamma \geq 1$

$$\|l_n^\gamma(.)\|_{B_q^\beta(L_p)} = O(1)\begin{cases} \frac{1}{n}, & \alpha-\beta-\frac{1}{q}>1 \\ \frac{1}{n^{\alpha-\beta-\frac{1}{q}}}, & \alpha-\beta-\frac{1}{q}<1 \\ \frac{(\log n)^{1-\frac{1}{q}}}{n}, & \alpha-\beta-\frac{1}{q}=1 \end{cases} \tag{5.21}$$

and for $q = \infty, p \geq 1, 0 \leq \beta < \alpha < 2$

$$\|l_n^\gamma(.)\|_{B_\infty^\beta(L_p)} = O(1)\begin{cases} \frac{1}{n}, & \alpha-\beta>1 \\ \frac{1}{n^{\alpha-\beta}}, & \alpha-\beta<1 \\ \frac{\log n}{n}, & \alpha-\beta=1 \end{cases} \tag{5.22}$$

The second part of the theorem follows from (5.21) and (5.22).

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Some Results on Self-Centeredness and Minimal Self-Centeredness of Power and Product of Graphs

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Abstract

A simple connected non-trivial graph G is said to be a self-centered graph if every vertex has the same eccentricity. A graph G is said to be a minimal self-centered graph if G is not a self-centered graph after deletion of an arbitrary edge. In this paper we study self-centeredness and minimal self-centeredness property of power of graphs and different type of product of cycle graphs, like cartesian, strong, co-normal, and tensor product.

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1 Introduction

In this paper we consider simple and undirected graphs. Central vertices of a graph are of great importance in facility location problem in a network. For basic definitions and terminologies we refer to [21]. For any two vertices u and v in a graph G, the length of a shortest $u-v$ path is known as the *distance* between u and v and is denoted by $d(u,v)$. The *eccentricity* of a vertex v in G, denoted by $e(v)$, is defined as the distance between v and a vertex farthest from v in G, i.e., $e(v) = \max\{d(v,u) : u \in V(G)\}$. The radius $rad(G)$ and diameter $diam(G)$ of graph G are respectively the minimum and maximum eccentricity of the vertices, i.e., $rad(G) = \min\{e(v) : v \in V(G)\}$ and $diam(G) = \max\{e(v) : v \in V(G)\}$. A graph G is called a self-centered graph if eccentricity of every vertex is the same. Further, G is called a d-self-centered graph if the eccentricity of every vertex is d. A self-centered graph G is said to be a minimal self-centered graph if it loses it self-centeredness property after removal of an arbitrary edge. Vertices having minimum eccentricity in a graph G are called central vertices of the graph G. Vertices having maximum eccentricity in a graph G are called peripheral vertices of the graph G.

For a graph G, the k^{th} power of G, denoted by G^k, is the graph on same set of vertices where any two vertices are adjacent if the distance between vertices (in G) is at most k. It may be noted that we have $1 \leq k \leq d$, where d is the diameter of the graph G, and for $k = d$, G^k is a complete graph.

For graphs $G_1, G_2, \ldots, G_n$, let $V(G) = \{(x_1, x_2, \ldots, x_n) : x_i \in V(G_i)\}$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two arbitrary elements in $V(G)$. Then G is called

- *Cartesian product* of $G_1, G_2, \ldots, G_n$, denoted by $G = G_1 \Box G_2 \Box \ldots \Box G_n$, whenever $x \sim y$ if and only if $x_i y_i \in E(G_i)$ for exactly one index i, $1 \leq i \leq n$, and $x_j = y_j$ for each index $j \neq i$.

- *Strong product* of $G_1, G_2, \ldots, G_n$, denoted by $G = G_1 \boxtimes \ldots \boxtimes G_n$, whenever $x \sim y$ if and only if $x_iy_i \in E(G_i)$ or $x_i = y_i$, for every i, $1 \le i \le n$.

- *Tensor product* of $G_1, G_2, \ldots, G_n$, denoted by $G = G_1 \otimes \ldots \otimes G_n$, whenever $x \sim y$ if and only if $x_iy_i \in E(G_i)$ for each i, $1 \le i \le n$.
- *Co-normal product* of $G_1, G_2, \ldots, G_n$, denoted by $G = G_1 * G_2 * \ldots * G_n$, whenever $x \sim y$ if and only if $x_i \sim y_i$ for some $i \in \{1, 2, \ldots, n\}$.

The term self-centered graph was introduced by Capobianco [5] for the graph with central ratio one, where central ratio of a graph G is $|C(G)| \div |V(G)|$ and $C(G)$ is the center of G. Self-centered graphs were broadly studied and surveyed by many authors, see [3], [4] and [10]. An algorithm was discussed by Janakiraman etc. [11] to construct self-centered graphs from trees. They also provided algorithms for the construction of self-centered graphs by adding edges to a given non-self-centered graph. Huilgol etc. [6] discussed about *cyclic edge extensions* and studied self-centeredness of cycle graphs by adding an edge. Klavžar et al. [12] introduced a new family of graphs as *r-almost self-centered* (r-ASC) graphs if there are exactly two vertices with eccentricity $r+1$ and the remaining vertices have eccentricity r. That is all but two vertices are central. They also defined *ASC index* of a graph G as the minimum number of vertices to be added to the graph G such that G is an induced subgraph of some ASC graph. Further, Klavžar et al. [12] discussed embedding of any graph into some ASC graph and proposed constructions for embedding of any graph into ASC graph of radius two. Later on, Klavžar et al. [13] defined *r-almost peripheral* (r-AP) graphs if there is exactly one vertex with eccentricity r and the remaining vertices have eccentricity $r+1$. That is exactly one central vertex and the remaining are peripheral vertices. Several demonstrations for embedding graphs into AP graphs of radius r is discussed by them. The *r-embedding index* $\Phi_r(G)$ is defined as the minimum number of vertices required to add to the graph G such that G is an induced subgraph of some r-AP graph. Recently, Singh and Panigrahi [17] discussed the self-centeredness of strong product, co-normal product, and lexicographic product of graphs. The same authors, Singh and Panigrahi in [18], checked the self-centeredness of tensor product of some special classes of graphs like cycles, wheel graphs etc. with themselves and other graphs.

Zoran Stanic [19] was the first to study minimal self-centeredness property of graphs. The author obtained all minimal self-centered graphs of order up to ten. In the year 1960, Ross and Harary [16] provided necessary and sufficient condition for a graph to be square of a tree. A. Mukhopadhayay [14] studied and discussed the square root of a given graph. Some other results related to hamiltonicity, subdivision and circumference of square of graphs are discussed by various authors, see [1, 2, 7, 8, 9, 15, 20]. In algebraic and spectral graph theory, power of graphs has been widely studied and surveyed. Self-centeredness and minimal self-centeredness property of power graphs and product of graphs were not studied before.

The paper is organized as follows. In section 2, we give some results regarding self-centeredness of power of graphs. In the section 3, we check minimal self-centeredness of square of cycles and then study the minimum number of edges to be deleted to destroy self-centeredness of C_n^2. Finally in section 4 we discuss minimal self-centeredness of different kind of product of cycles.

2 Self-Centeredness of Power of Graphs

In this section we study self-centeredness of power of graphs. Throughout this paper, let $u_0, u_1, u_2, \ldots, u_{n-1}$ be the vertices of C_n and C_n^2. For our convenience, let us call the edges u_0u_1, u_1u_2, …, $u_{n-1}u_0$ as the outer edges. Similarly we name the edges $u_0u_2, u_2u_4, u_4u_6 \ldots$ and $u_1u_3, u_3u_5,$
$u_5u_7, \ldots$ as the inner edges of C_n^2. It is easy to note that shortest path between any two vertices of C_n^k mainly consists of inner edges and degree of every vertex in C_n^k is $k+2$, where k is any positive integer.

In the following we prove that power of any self-centered graph is a self-centered graph.

Theorem 2.1. *Let G be a d-self-centered graph. Then G^k is a $\lceil d/k \rceil$-self-centered graph, where $1 \leq k \leq d$.*

Proof. Since G is a d-self-centered graph, for every vertex u there exists a vertex v such that $d(u, v) = d$, and let a path of length d between u and v be $P : u = v_1, v_2, \ldots, v_{d+1} = v$. Now in G^k, there exists a path P' which skips every vertex at a distance k in P starting from the vertex u which is $P' : u = v_1, v_{1+k}, \ldots, v_{1+mk}, v_{d+1} = v$, where $m = \lceil \frac{d}{k} \rceil - 1$ when k divides d, and $m = \lceil \frac{d}{k} \rceil$ otherwise. It may be noted that the length of this path P' is $\lceil d/k \rceil$ and hence the result. □

If G is not a self-centered graph but $diam(G) = 2$ then G^2, being a complete graph, is self-centered. In the next result we show that square of a non-self-centered graph may or may not be self-centered graph.

Theorem 2.2. *For any non-self-centered graph G with $diam(G) \geq 3$ the following hold:*

(i) G^2 is a self-centered graph if $diam(G)$ is even and $diam(G) - rad(G) = 1$.

(ii) G^2 is not a self-centered graph if $diam(G) - rad(G) \geq 2$.

Proof. It is known that if a graph is not a self-centered graph then it contains at least two non-central vertices. Since $diam(G) \geq 3$, there is no vertex x such that $deg(x) = n - 1$, where $x \in V(G)$ and $|V(G)| = n$. For if degree of vertex x is $n - 1$, $diam(G) = 2$, a contradiction. Let eccentricity of vertex x_i be e_i. Now, for every x_i there exists a vertex y_i such that $d(x_i, y_i) = e_i$ and $P_i : x_i = v_0 v_1 \ldots v_{e_i} = y_i$ is the shortest $x_i - y_i$ path of length e_i. Further, in G^2, a path of shortest length between x_i and y_i is given as $P'_i : v_0 v_2 \ldots v_{e_{i-2}} v_{e_i} = y_i$ when e_i is even and $P'_i : v_0 v_2 \ldots v_{e_{i-1}} v_{e_i} = y_i$ when e_i is is odd. The length of the paths P'_i is $\lceil e_i/2 \rceil$ and thus eccentricity of every vertex $x_i \in V(G^2)$ is $\lceil e_i/2 \rceil$. We can see that eccentricity of every vertex in G^2 is the same in (i) and not same in (ii), and hence the result. □

Corollary 2.1. *Let G^2 be the square of a graph G. Then we have following.*

(i) If G is an r-ASC graph with r even then G^2 is $(\frac{r}{2})$-self-centered graph.

(ii) If G is an r-AP graph, where r is odd, then G^2 is $(\frac{r+1}{2})$-self-centered graph.

3 Minimal Self-centeredness of Power of Graphs

The simplest example of a minimal self-centered graph is complete graph K_n, $n \geq 3$, where deletion of any edge from K_n results into a non-self-centered graph. It is easy to show that cycle C_n is a minimal self-centered graph as deletion of any edge gives path P_n and P_n is not a self-centered graph. However, square of a cycle is not a minimal self-centered graph which is the content of our next result.

Theorem 3.1. *For any cycle C_n, $n \geq 6$, the graph C_n^2 is not a minimal self-centered graph.*

Proof. From Theorem 2.1, we get that C_n^2 is a k-self-centered graph, where $k = \lceil \frac{d}{2} \rceil$ and $d = \lfloor \frac{n}{2} \rfloor$. We shall show that the graph C_n^2 is not a minimal self-centered graph. Let $x = u_0 \in V(C_n^2)$. Then there exists a vertex $y = u_d$ such that $d(x, y) = k$. First consider the case when n is even. We have $x = u_0u_2u_4 \ldots u_d = y$ and $x = u_0u_2 \ldots u_{d-1}u_d = y$ as the shortest $x - y$ paths of length k, when d is even and odd, respectively. After deletion of any edge from these two paths and using the fact that $d_{C_n}(u_0, u_i) = d_{C_n}(u_0, u_{n-i})$ for $i = 1, 2, \ldots \lfloor \frac{n}{2} \rfloor$, we get paths $x = u_0u_{n-2} \ldots u_d = y$ and $x = u_0u_1u_3 \ldots u_d = y$ of length k in C_n^2, when d is even and odd, respectively. This shows that the eccentricity of vertex u_0 remains unchanged after deleting any edge. Next, assume that both n and d are odd. Then for an arbitrary vertex u_0 in C_n^2, there exists a vertex u_d such that $d(u_0, u_d) = k$ and the shortest $u_0 - u_d$ path of length k given by $u_0u_2 \ldots u_{d-1}u_d$ for which there also exists an alternate $u_0 - u_d$ path given by $u_0u_1u_3 \ldots u_d$ of the length k. It may be noted that once any edge is deleted from any path, an alternate path of the same length maintains the eccentricity of the vertex u_0.

Finally consider the case when n is odd and d is even. Consider a vertex u_0. Then there exists a vertex u_d in C_n^2 such that $d(u_0, u_d) = k$, and the shortest $u_0 - u_d$ path of length k is $P : u_0u_2u_4 \ldots u_d$. After deleting any edge u_iu_{i+2} from P, we get a possible $u_0 - u_d$ path $u_0u_2 \ldots u_iu_{i+1}u_{i+2} \ldots u_d$ of length $k + 1$. Thus, the eccentricity of the vertex u_0 is incremented by one. It may be noted that this does not affect the eccentricity of the vertices u_j, where j is odd, because for u_j, the shortest path between u_j and the vertex farthest from u_j consists of all odd indexed vertices. Thus, after deletion of an edge u_iu_{i+2}, the resulting graph is not a self-centered graph. It may be noted that if outer edges (edges lying on C_n) are deleted then the graph still remains a self-centered graph, which does not satisfy the definition of minimal self-centered graph. □

It may be noted that a self-centered graph G has no pendant vertex [19]. On the contrary, let d-self-centered graph G has a pendant vertex, say the vertex a. Here, eccentricity of evert vertex in G is d. We will calculate eccentricity of the vertex a. Let b be the neighbour vertex of a. Then $e(a) = e(b) + 1 = d + 1$ which gives a contradiction.

In the next two theorems $N(G)$ denotes the minimum number of edges to be deleted from a graph G such that the resultant graph is not a self-centered graph. In the next result, we calculate the minimum number of edges to be deleted from C_n^2 such that the resulting graph is not a self-centered graph.

Theorem 3.2. *For any cycle C_n, $n \geq 6$, we have*

$$N(C_n^2) = \begin{cases} 1, & \text{if } n \text{ is odd and } d \text{ is even,} \\ 2, & \text{if } n \text{ is even and } d \text{ is even,} \\ 3, & \text{if } n \text{ is odd and } d \text{ is odd,} \\ 3, & \text{if } n \text{ is even and } d \text{ is odd.} \end{cases}$$

Proof. Let $u_0, u_1, \ldots, u_{n-1}$ be the vertices of C_n as well as of C_n^2. We know that C_n is d-self-centered and C_n^2 is k-self-centered graph, where $d = \lfloor \frac{n}{2} \rfloor$ and $k = \lceil \frac{d}{2} \rceil$, respectively. It is clear that in C_n^2, shortest path between any two vertices mainly consists of inner edges(edges do not lie on C_n). Now, we will consider following four cases.

Case 1. n is odd and d is even. In this case, for an arbitrary vertex $u_0 \in C_n^2$, there exist two vertices u_d and u_{d+1} such that $d(u_0, u_d) = k = d(u_0, u_{d+1})$. It may be easily seen that there is a unique shortest path between the vertices u_0 and u_d and let this path be $P : u_0u_2u_4 \ldots u_d$. Let us delete some edge u_iu_{i+2} from this path. Since the path P contains inner edges, the length of shortest $u_0 - u_d$ path is incremented by one after deletion of an edge u_iu_{i+1}. This changes the eccentricity of the vertex u_0 and thus in this case, C_n^2 is not a self-centered graph. If the deleted edge does not lie on the shortest path between any pair of vertices, then this will not affect the eccentricity of the vertex u_0. One can see that there is no change in the eccentricity of the vertices u_l, l is odd, as no vertices with even indices lie on the shortest path between the vertex u_l and the vertex farthest from u_l. Thus, in this case $N(C_n^2) = 1$.

Case 2. Both n and d are even. For the given value of n and d, for an arbitrary vertex u_0 in C_n^2 there exists exactly one vertex u_d such that $d(u_0, u_d) = k$. We have two $u_0 - u_d$ paths of length k given by $P_1 : u_0u_2u_4 \ldots u_d$ and $P_2 : u_0u_{n-2}u_{n-4} \ldots u_d$. If any two edges from these paths are deleted, the resultant graph is not a self-centered graph as eccentricity of the vertex u_0 is $k+1$, where as the eccentricity of the vertices with odd indices is k. It may also be noted that if the deleted edge lie on the shortest path between an even indexed vertex and its eccentric vertex, then the eccentricity of even indexed vertex is changed to $k+1$. This gives that $N(C_n^2) = 2$.

Case 3. Both n and d are odd. Consider an arbitrary vertex u_0 whose eccentric vertex is u_d. In this case, as the value of n (or d) increases, the number of shortest $u_0 - u_d$ paths also increases. We will find all such paths first. The simplest $u_0 - u_d$ path is $P : u_0u_2u_4 \ldots u_{d-1}u_d$ of the length k. The distance between every vertex in the path is two except the vertices u_{d-1} and u_d. Thus, shifting every vertex (except the last vertex u_d) to its neighbouring vertex, we get new $u_0 - u_d$ paths of length k given as $(u_0u_1u_3\dots u_d)$, $(u_0u_2u_3u_5\dots u_d)$, $(u_0u_2u_4u_5u_6\dots u_d)$ and so on. The number of these paths are k (including P). We also have an alternate $u_0 - u_d$ path given by $u_0u_{n-2}u_{n-4}\dots u_d$ of length k. Thus, total number of such $u_0 - u_d$ paths are $k+1$. If we delete distinct edges from each path then the eccentricity of u_0 will be changed. But as we need to find the minimum number of edges to be removed such that the resulting graph loses its property of self-centeredness, we get that if we delete three edges incident on a single vertex the resultant graph will be a graph with a pendant vertex and thus not a self-centered graph. Hence $N(C_n^2) = 3$.

Case 4. n is even and d is odd. Following the case 3 here, we have more than three shortest $u_0 - u_d$ paths, where u_0 and u_d are eccentric vertices. Let $u_0u_2u_4\dots u_d$ and $u_0u_{n-2}u_{n-4}\dots u_d$ be the paths of length k. Again shifting every vertex (except the last vertex u_d) to its neighbouring vertex in both the paths, we get $2k$ number of $u_0 - u_d$ paths. Since $2k \geq 3$, we can easily conclude that $N(C_n^2) = 3$. □

In the next result we calculate $N(C_n^k)$ for some particular values of k.

Proposition 3.1. *Let C_n be any cycle and C_n^k be the k^{th} power of C_n, where C_n is $d = \lfloor \frac{n}{2} \rfloor$-self-centered graph, and k divides d. Then*

$$N(C_n^k) = \begin{cases} 1, & \textit{if } n \textit{ is odd,} \\ 2, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. We know that C_n^k is a m-self-centered graph, where $m = \lceil \frac{d}{k} \rceil$. It is known that for a vertex u_0 there exists a vertex u_d such that $d(u_0, u_d) = m$, where $u_0, u_d \in V(C_n^k)$. When n is odd, then there exists exactly one $u_0 - u_d$ path given as $(u_0u_ku_{2k}\dots u_{d=km})$. When n is even, then we get two $u_0 - u_d$ paths of length m given by $(u_0u_ku_{2k}\dots u_{d=km})$ and $(u_0u_{n-k}u_{n-2k}\dots u_{d=km})$. It can be easily seen that deletion of one edge and two edges from $u_0 - u_d$ paths gives a non-self-centered graph when n is odd and even, respectively. Hence the result. □

4 Minimal Self-Centeredness of Product of Graphs

In this section, we check minimal self-centeredness of some product of cycle graphs, namely, cartesian product, strong product, and co-normal product.

Theorem 4.1. *[17] Let $G = C_{n_1} \Box C_{n_2} \Box \ldots \Box C_{n_m}$ be the cartesian product of m cycles, where $|V(C_{n_i})| = n_i$ and C_i is $d_i = \lfloor \frac{n_i}{2} \rfloor$-self-centered graph for $1 \leq i \leq m$. Then G is a d self-centered graph, where $d = \sum_{i=1}^{m} d_i$.*

Theorem 4.2. *Let $G = C_{n_1} \Box C_{n_2} \Box \ldots \Box C_{n_m}$ be the cartesian product of m cycles, where $|V(C_{n_i})| = n_i$ for $1 \leq i \leq m$. Then G is not a minimal self-centered graph.*

Proof. It is known that cycles C_{n_i} are d_i-self-centered, where $d_i = \lfloor \frac{n_i}{2} \rfloor$ for $1 \leq i \leq m$ and cartesian product of self-centered graphs is a self-centered graph [19]. Thus, G is $\sum_{i=1}^{m} d_i$-self-centered graph. We shall prove that removing any edge from the shortest path between any two vertices has no effect on the eccentricity of the vertices. Let $V(C_{n_i}) = \{u_{i0}, u_{i1}, \ldots, u_{i(n_i-1)}\}$. Consider the vertices u_{i0}. For every u_{i0}, there exist a vertex u_{id_i} such that $d(u_{i0}, u_{id_i}) = d_i$ in C_{n_i}. Let $P_i : u_{i0}u_{i1}u_{i2} \ldots u_{id_i}$ be the shortest $u_{i0} - u_{id_i}$ path of length d_i for $1 \leq i \leq m$. Let $x = (u_{10}, u_{20}, \ldots, u_{m0})$. Then there exists a vertex $y = \{u_{1d_1}, u_{2d_2}, \ldots, u_{md_m}\}$ such that $d(x, y) = \sum_{i=1}^{m} d_i$. Now using all the paths $P_i's$, we get the shortest $x - y$ path P : $(u_{10}, u_{20}, \ldots, u_{m0})(u_{11}, u_{20}, \ldots, u_{m0}) \ldots (u_{1d_1}, u_{20}, \ldots, u_{m0})$
$(u_{1d_1}, u_{21}, \ldots, u_{m0}) \ldots (u_{1d_1}, u_{2d_2}, \ldots, u_{m0}) \ldots\ldots (u_{1d_1}, u_{2d_2}, \ldots, u_{m1})(u_{1d_1}, u_{2d_2}, \ldots, u_{m2}) \ldots$
$(u_{1d_1}, u_{2d_2}, \ldots, u_{md_m})$, where the length of the path P is $\sum_{i=1}^{m} d_i$. If we delete any edge from the path P then we get an alternate $x - y$ path

$P' : (u_{10}, u_{20}, \ldots, u_{(m-1)0}, u_{m0})(u_{10}, u_{20}, \ldots, u_{(m-1)0}, u_{m1}) \ldots (u_{10}, u_{20}, \ldots, u_{(m-1)0}, u_{md_m})$
$(u_{10}, u_{20}, \ldots, u_{(m-1)1}, u_{md_m}) \ldots (u_{10}, u_{20}, \ldots, u_{(m-1)d_{m-1}}, u_{md_m}) \ldots (u_{11}, u_{2d_2}, \ldots,$

$u_{(m-1)d_{m-1}}, u_{md_m})$ of the length $\sum_{i=1}^{m} d_i$. We can see that after deleting an edge from the shortest path, the eccentricity of the vertex is not changed and hence the graph G is not a minimal self-centered graph. □

Theorem 4.3. *[17] Let $G = G_1 \boxtimes \ldots \boxtimes G_n$ be the strong product of graphs $G_1, G_2, \ldots, G_n$. Then G is d-self-centered graph if and only if for some $k \in \{1, \ldots, n\}$, G_k is d-self-centered graph and $diam(G_i) \leq d$ for every i, $1 \leq i \leq n$.*

In the next result we show that strong product of even cycles is not a minimal self-centered graph.

Theorem 4.4. *Let $G = C_{n_1} \boxtimes C_{n_2} \boxtimes \ldots \boxtimes C_{n_m}$ be the strong product of m even cycles, where $|V(C_{n_i})| = n_i$ and $n_1 \geq n_2 \geq \ldots \geq n_m$ for $1 \leq i \leq m$. Then G is not a minimal self-centered graph.*

Proof. Let $d = \max\{\frac{n_i}{2} : 1 \leq i \leq m\} = \frac{n_1}{2}$. It is known that strong product of self-centered graphs is a self-centered graph [17] and thus G is a d-self-centered graph. Let $V(C_{n_i}) = \{u_{i0}, u_{i1}, \ldots, u_{i(n_i-1)}\}$. For an arbitrary vertex $x = (u_{10}, u_{20}, \ldots, u_{m0})$, there exists a vertex $y = (u_{1d_1}, u_{2d_2}, \ldots, u_{md_m})$ such that $d(x, y) = d$. It may be noted that the vertex y is not unique. Let P_i be the $u_{i0} - u_{id_i}$ paths of length $\frac{n_i}{2}$, where $P_i : u_{i0}u_{i1}u_{i2} \ldots u_{id_i}$ for $1 \leq i \leq m$. In even cycles we have alternate $u_{i0} - u_{id_i}$ paths of length $\frac{n_i}{2}$ given by P'_i : $u_{i(n_i-1)}u_{i(n_i-2)} \ldots u_{id_i}$. Now the concatenation of the paths P_i gives the $x - y$ path of length d given by $P : (u_{10}, u_{20}, \ldots, u_{m0})(u_{11}, u_{21}, \ldots, u_{m1}) \ldots (u_{1d_m}, u_{2d_m}, \ldots, u_{md_m})$ $(u_{1(d_m+1)}, u_{2d_m}, \ldots, u_{md_m})(u_{1(d_m+2)}, u_{2d_m}, \ldots, u_{md_m}) \ldots (u_{1d_1}, u_{2d_m}, \ldots, u_{md_m})$. On the other hand, concatenation of the paths P'_i gives the path P' which is an alternate $x - y$ path of length d, where $P' : (u_{10}, u_{20}, \ldots, u_{m0})(u_{1(n_1-1)}, u_{2(n_2-1)}, \ldots, u_{m(n_m-1)}) \ldots (u_{1d_m}, u_{2d_m}, \ldots, u_{md_m})$ $(u_{1(d_m+1)}, u_{2d_m}, \ldots, u_{md_m})(u_{1(d_m+2)}, u_{2d_m}, \ldots, u_{md_m}) \ldots (u_{1d_1}, u_{2d_m}, \ldots, u_{md_m})$. As we get an alternate shortest $x - y$ path, deletion of any edge from either paths will not affect the eccentricity of the vertex x. This proves the result. □

In the lemma given below, we find the degree of an arbitrary vertex x in the co-normal product of finite number of cycles.

Lemma 4.1. *Let $G = C_{n_1} * C_{n_2} * \ldots * C_{n_m}$ be the co-normal product of m cycles and $x = (x_1, \ldots, x_m)$ be an arbitrary vertex in G. Then the degree of x is $2m + 2^m$.*

Proof. It may be noted that for any vertex u in a cycle, there are two vertices adjacent to u. Consider a vertex $x = (x_1, \ldots, x_m)$ in G. If we fix all the values of x_i in x but x_j, $i, j \in \{1 \ldots, m\}, i \neq j$, then there exists vertices $y = (y_1, \ldots, y_m)$ and $z = (z_1, \ldots, z_m)$ such that $y_i = x_i = z_i$ for every $i \neq j$ and y_j and z_j are adjacent to x_j in G_j. This gives $2m$ number of vertices which are adjacent to x. Also, we can see that since every vertex has two choices of adjacent vertices, so in this case we get 2^m number of vertices adjacent to x. Thus, total number of vertices adjacent to any arbitrary vertex x in G is $2m + 2^m$. □

We have following result related to self-centeredness of co-normal product of graphs.

Theorem 4.5. *[17] Let $G = G_1 * G_2 * \ldots * G_n$ be the co-normal product of graphs $G_1, G_2, \ldots, G_n$ with $|V(G_i)| = n_i$. Then the following hold:*

(i) *Let $G_i \neq K_1$ and $G_j = K_1$ for all $j \neq i$. Then G is d-self-centered graph if and only if G_i is d-self-centered graph.*

(ii) *Let there be at least two values of i such that $G_i \neq K_1$. Then G is 2-self-centered graph if and only if there exists an index l such that $\Delta(G_l) \neq n_l - 1$, where $\Delta(G)$ is the maximum degree of a vertex in G.*

Theorem 4.6. *Let $G = C_{n_1} * C_{n_2} * \ldots * C_{n_m}$ be the co-normal product of m cycles. Then G is not a minimal-self-centered graph.*

Proof. The proof may be obtained by applying Lemma 4.1 and proceeding in the similar manner as the proof of Theorem 4.2. □

Lemma 4.2. *Let $T = C_{n_1} \otimes C_{n_2} \otimes \ldots \otimes C_{n_m}$ be the tensor product of m distinct odd cycles and x be an arbitrary vertex of T. Let $n_k = \max\{n_i : 1 \leq i \leq m\}$ and $n_l = \max\{n_i \setminus n_k : 1 \leq i \leq m\}$. Then T is a n_l-self-centered graph. Also, number of vertices y such that $d(x, y) = l$ are $2(m-1) + 2^{m-1}$.*

Proof. Let $V(C_i) = n_i$ for $1 \leq i \leq m$. From [18], it is clear that T is a n_l-self-centered graph, where n_l is the second maximum of all n_i. Let $l = n_k$ (second maximum of n_i) for some $k \in \{1, \ldots, m\}$. Consider an arbitrary vertex $x = (x_1, \ldots, x_k, \ldots, x_m)$ and let us fix the vertices x_k and x_i except the one, say x_p, where $i \neq k$. Using the fact that every vertex has two neighbors in cycle graphs, we see that the x_p has two choices such that x_{p_1} and x_{p_2} are adjacent to x_p in C_{n_p}. Thus, we get two vertices $y = (x_1, \ldots, x_{p_1}, \ldots, x_k, \ldots, x_m)$ and $z = (x_1, \ldots, x_{p_2}, \ldots, x_k, \ldots, x_m)$ such that y and z are adjacent to x in T. Proceeding in the similar manner, we get $2(m-1)$ number of such vertices adjacent to x. Also, we can see that every x_i $(i \neq k)$ in x has two choices for the number of vertices adjacent to them in the respective cycle graph. Then by fundamental theorem of counting, total number of such vertices in 2^{m-1} and thus we proved that total number of vertices are $2(m-1) + 2^{m-1}$. It may be seen that we will get a $x_k - x_k$ walk of an odd length l by moving around the cycle C_k. And for any two adjacent vertices x_p and x_{p_1}, we can get $x_p - x_{p_1}$walk of odd length l by moving back and forth. □

Theorem 4.7. *Let $T = C_{n_1} \otimes C_{n_2} \otimes \ldots \otimes C_{n_m}$ be the tensor product of m odd cycles. Then T is not a minimal self-centered graph.*

Proof. The proof may be obtained by applying Lemma 4.2 and proceeding in the similar manner as the proof of Theorem 4.2. □

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A Study of Equilibrium Problems and Variational Inequality Problems on Hadamard Manifold

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Abstract

In this article we study equilibrium problems and variational inequality problems on Hadamard manifolds. Using the KKM technique, we establish the existence of solutions of the stated problems under the generalized monotonicity assumptions on the functions involved. We construct some examples to justify our work in Hadamard manifolds.

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1 Introduction

The theory of equilibrium problems and variational inequality problems has many important applications in many fields of mathematics such as optimization problems, fixed point problems, Nash equilibria problems, complementarity problems etc. In recent decades, many results concerned with the existence of

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solutions for equilibrium problems and variational inequality problems have been established, see for example ([2], [3], [10], [19]) and the references therein.

On the other hand, recent interests of a number of researchers are focused on extending some concepts and techniques of nonlinear analysis in Euclidean spaces to Riemannian manifolds. There are some advantages for a generalization of optimization methods from Euclidean spaces to Riemannian manifolds, because nonconvex and nonsmooth constrained optimization problems can be seen as convex and smooth unconstrained optimization problems from the Riemannian geometry point of view, see for example ([17], [13], [14]). Colao et al. [5] have constructed an example of an equilibrium problem on an Euclidean space which can not be solved by using the classical results known in vector spaces but the problem can be solved by rewriting it on a Riemannian manifold. Therefore, the extension of the concepts and techniques of the theory of equilibrium problems and variational inequality problems from Euclidean spaces to Riemannian manifolds is natural.

Németh [12] studied geodesic monotone vector fields, Wang et al. [18] studied monotone and accretive vector fields on Riemannian manifolds. Li et al. [7] extended maximal monotone vector fields from Banach spaces to Hadamard manifolds (simply connected complete Riemannian manifold with nonpositive sectional curvature). Németh [11] generalized some basic existence and uniqueness theorems of the classical theory of variational inequalities on Euclidean spaces to Hadamard manifolds. Li et al. [8] studied the variational inequality problems on Riemannian manifolds. Zhou and Huang [20] introduced the notion of the KKM mapping and proved a generalized KKM theorem on the Hadamard manifold. An existence result for equilibrium problems on Hadamard manifolds was first introduced by Colao et al. [5] where the equilibrium problem was associated to a monotone bifunction. Zhou and Huang [21] investigated the relationship between a vector variational inequality problem and a vector optimization problem on a Hadamard manifold. Tang et al. [16] introduced the proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. Li and Huang [9], studied the generalized vector quasi-equilibrium problems.

Motivated by the research work mentioned above, we study the existence of solutions of equilibrium problems and variational inequality problems on Hadamard manifolds by using KKM techniques under weaker assumptions used in ([5]) and [11]. We also introduce the existence of solutions of mixed equilibrium problems and mixed variational inequality problems on Hadamard manifolds.

2 Preliminaries

In this section, we recall some fundamental definitions, basic properties and notations needed for a comprehensive reading of this article. These can be found in any textbook on Riemannian geometry, for example ([15], [17]).

Let M be an n-dimensional connected manifold. We denote by T_xM the n-dimensional tangent space of M at x and by $TM = \cup_{x\in M} T_xM$, the tangent bundle of M. When M is endowed with a Riemannian metric $< .,. >$ on the tangent space T_xM with corresponding norm denoted by $\|.\|$, then M is a Riemannian manifold. The length of a piecewise smooth curve $\gamma : [a,b] \to M$ joining x to y such that $\gamma(a) = x$ and $\gamma(b) = y$, is defined by $L(\gamma) = \int_a^b \| \dot{\gamma}(t) \|_{\gamma(t)} \, dt$. Then for any $x, y \in M$ the Riemannian distance $d(x,y)$ which induces the original topology on M is defined by minimizing this length over the set of all curves joining x to y.

On every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X, Y on M. Let γ be a smooth curve in M. A vector field X is said to be parallel along γ if $\nabla_{\gamma'} X = 0$. If γ' itself is parallel along γ, we say that γ is a geodesic. A geodesic joining x to y in M is said to be a minimal geodesic if its length equals $d(x,y)$.

A Riemannian manifold is complete if for any $x \in M$ all geodesics emanating from x are defined for all $t \in \mathbb{R}$. By the Hopf-Rinow theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M,d) is a complete metric space and bounded closed subsets are compact.

Assuming that M is complete the exponential mapping $\exp_x : T_xM \to M$ is defined by $\exp_x v = \gamma_v(1)$, where γ_v is the geodesic defined by its position x and velocity v at x.

A Hadamard manifold is a simply connected complete Riemannian manifold with nonpositive sectional curvature. The exponential mapping exp and its inverse $\exp^{-1}$ are continuous on Hadamard manifold. Let M denote a finite dimensional Hadamard manifold.

Definition 1. *([14]) A subset K of M is said to be geodesic convex if and only if for any two points $x,y \in K$, the geodesic joining x to y is contained in K. That is if $\gamma : [0,1] \to M$ is a geodesic with $x = \gamma(0)$ and $y = \gamma(1)$, then $\gamma(t) \in K,\ for\ 0 \le t \le 1$.*

Definition 2. *([14]) A real-valued function $f : M \to \mathbb{R}$ defined on a geodesic convex set K is said to be geodesic convex if and only if for $0 \le t \le 1$,*

$$f(\gamma(t)) \le (1-t)f(\gamma(0)) + tf(\gamma(1)).$$

Definition 3. *([5]) For an arbitrary subset $C \subseteq M$ the minimal geodesic convex subset which contains C is called the convex hull of C and is denoted by $co(C)$. It is easy to check that $co(C) = \bigcup_{n=1}^{\infty} C_n$, where $C_0 = C$ and $C_n = \{z \in \gamma_{x,y} : x,y \in C_{n-1}\}$.*

Definition 4. *([21]) Let $K \subset M$ be a nonempty closed geodesic convex set and $G : K \to 2^K$ be a set-valued mapping. We say that G is a KKM mapping if for any $\{x_1,...,x_m\} \subset K$, we have*

$$co(\{x_1,...,x_m\}) \subset \bigcup_{i=1}^{m} G(x_i).$$

Lemma 1. *([5]) Let K be a nonempty closed geodesic convex set and $G : K \to 2^K$ be a set-valued mapping such that for each $x \in K$, $G(x)$ is closed. Suppose that*

(i) there exists $x_0 \in K$ such that $G(x_0)$ is compact.

(ii) $\forall x_1,..,x_m \in K$, $co(\{x_1,..,x_m\}) \subset \bigcup_{i=1}^{m} G(x_i)$.

Then $\bigcap_{x\in K} G(x) \neq \emptyset$.

We recall that a geodesic triangle $\Delta(x_1x_2x_3)$ of a Riemannian manifold is the set consisting of three distinct points x_1, x_2, x_3 called the vertices and three minimizing geodesic segments γ_{i+1} joining x_{i+1} to x_{i+2} called the sides, where $i = 1,2,3(mod\ 3)$.

Theorem 1. *[15] Let M be a Hadamard manifold, $\Delta(x_1x_2x_3)$ a geodesic triangle and $\gamma_{i+1} : [0,l_{i+1}] \to M$ geodesic segments joining x_{i+1} to x_{i+2} and set $l_{i+1} = l(\gamma_{i+1})$, $\theta_{i+1} = \measuredangle(\gamma'_{i+1}(0), -\gamma'_i(l_i))$, for $i = 1,2,3(mod\ 3)$. Then*

$$\theta_1 + \theta_2 + \theta_3 \leq \pi,$$

$$l_{i+1}^2 + l_{i+2}^2 - 2l_{i+1}l_{i+2}\cos\theta_{i+2} \leq l_i^2,$$

$$d^2(x_{i+1},x_{i+2}) + d^2(x_{i+2},x_i) - 2\big\langle \exp^{-1}_{x_{i+2}} x_{i+1}, \exp^{-1}_{x_{i+2}} x_i\big\rangle \leq d^2(x_i,x_{i+1}). \tag{2.1}$$

By using the above inequality for any three points $x,y,z \in M$, we can get

$$d^2(x,y) \leq \big\langle \exp^{-1}_x z, \exp^{-1}_x y\big\rangle + \big\langle \exp^{-1}_y z, \exp^{-1}_y x\big\rangle. \tag{2.2}$$

Lemma 2. *([7]) Let $x_0 \in M$ and $\{x_n\} \in M$ such that $x_n \to x_0$. Then the following assertions hold.*

(i) For any $y \in M$

$$\exp^{-1}_{x_n} y \to \exp^{-1}_{x_0} y \text{ and } \exp^{-1}_y x_n \to \exp^{-1}_y x_0.$$

(ii) If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.

(iii) Given the sequence $\{u_n\}$ and $\{v_n\}$ with $u_n, v_n \in T_{x_n}M$, if $u_n \to u_0$ and $v_n \to v_0$ with $u_0, v_0 \in T_{x_0}M$, then $\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle$.

Proposition 1. *([1]) Let M be a Hadamard manifold, $x \in M$ and $u \in T_xM \setminus \{0\}$ nonzero. Then the function $g : M \to \mathbb{R}$ defined by*

$$g(y) = \big\langle u, \exp^{-1}_x y\big\rangle,$$

is a quasi-convex function.

Throughout the remaining part of the article we take M to be a finite dimensional Hadamard manifold and $K \subseteq M$ denote a nonempty closed geodesic convex set, unless explicitly stated otherwise.

3 Main Results

3.1 Equilibrium problems

An existence result for equilibrium problems on Hadamard manifolds was first established by [5] where the equilibrium problem was associated to a monotone bifunction.

Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the property $F(x,x) = 0$ for all $x \in K$. The equilibrium problem introduced by [5] is to find a point $\bar{x} \in K$, such that

$$\text{(EP)} \qquad F(\bar{x},y) \geq 0,\ \forall y \in K.$$

A point $\bar{x} \in K$ solving this problem (EP) is said to be an equilibrium point.

Definition 5. ([5]) We call a bifunction F to be monotone on K if for any $x,y \in K$, we have

$$F(x,y) + F(y,x) \leq 0.$$

Definition 6. We call a bifunction F to be pseudomonotone on K if for any

$x,y \in K$,

$$F(x,y) \geq 0 \Rightarrow F(y,x) \leq 0.$$

Remark 1. If a bifunction F is monotone on K then it is easy to see that it is also pseudomonotone. But the converse is not true which follows from the following counter example.

Example 1. Let $H^1 = \{x = (x_1,x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 = -1,\ x_2 > 0\}$ be the hyperbolic 1-space which forms a Hadamard manifold ([4]) endowed with the metric defined by

$$\langle x,y \rangle = x_1y_1 - x_2y_2,\ \forall x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2.$$

Let K be a subset of H^1 defined by $K = \{x = (x_1, x_2) \in H^1 : -1 \leq x_1 \leq 1\}$.

Now we define a bifunction $F : K \times K \to \mathbb{R}$ by

$$F(x,y) = x_2(x_1 - y_1).$$

To show that F is pseudomonotone on K, but not monotone.

$F(x,y) \geq 0$ on K when $x_1 \geq y_1$ (as $x_2 > 0$),

then $F(y,x) = y_2(y_1 - x_1) \leq 0$ (as $y_2 > 0$).

Therefore, F is pseudomonotone.

Particularly if we take $x = (1, \sqrt{2}) \in K$ and $y = (0,1) \in K$,

then $F(x,y) + F(y,x) = \sqrt{2} - 1 > 0$.

That is, F is not monotone.

3.2 Mixed equilibrium problems

Let $\psi : K \to \mathbb{R}$ be a mapping and $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the property $F(x,x) = 0$ for all $x \in K$. Then the problem is to find $\bar{x} \in K$ such that

$$\text{(MEP)} \qquad F(\bar{x},y) + \psi(y) - \psi(\bar{x}) \geq 0, \ \forall y \in K,$$

is called a mixed equilibrium problem (MEP) on K. We denote SOL(MEP), the solution set of (MEP). If $\psi \equiv 0$, then (MEP) reduces to the equilibrium problem (EP).

3.3 Existence of solutions of mixed equilibrium problems

In this section we study the existence of solutions of mixed equilibrium problems (MEP) under pseudomonotonicity assumptions.

Definition 7. A bifunction F is said to be pseudomonotone with respect to the function ψ if

$$F(x,y) + \psi(y) - \psi(x) \geq 0 \Rightarrow F(y,x) + \psi(x) - \psi(y) \leq 0.$$

Definition 8. Let K be a geodesic convex subset of M. A function $F : K \to \mathbb{R}$ is said to be hemicontinuous if for every geodesic $\gamma : [0,1] \to K$, whenever $t \to 0$, $F(\gamma(t)) \to F(\gamma(0))$.

Next we give the following lemma which will be needed in the sequel.

Lemma 3. Let $F : K \times K \to \mathbb{R}$ be a bifunction which is hemicontinuous in the first argument and for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ be geodesic convex. Also assume that the map $\psi : K \to \mathbb{R}$ is geodesic convex and F is pseudomonotone with respect to ψ. Then $x \in K$ solves (MEP), if and only if

$$F(y,x) + \psi(x) - \psi(y) \leq 0, \text{ for all } y \in K.$$

Proof. Suppose $x \in K$ solves (MEP), then

$$F(x,y) + \psi(y) - \psi(x) \geq 0, \text{ for all } y \in K. \tag{3.3}$$

Since F is pseudomonotone with respect to the function ψ, we have

$$F(y,x) + \psi(x) - \psi(y) \leq 0.$$

Conversely, let $x \in K$ be a solution of (3.3). Let $\gamma(t)$ be a geodesic joining x and y such that $\gamma(0) = x$. As the set K is geodesic convex, we have

$$F(\gamma(t),x) + \psi(x) - \psi(\gamma(t)) \leq 0, \text{ for } 0 \leq t \leq 1. \tag{3.4}$$

Since ψ is geodesic convex, we have

$$\psi(\gamma(t)) \leq t\psi(y) + (1-t)\psi(x)$$

$$\Rightarrow \psi(\gamma(t)) - \psi(x) \leq t[\psi(y) - \psi(x)].$$

As $z \mapsto F(x,z)$ is geodesic convex,

$0 = F(\gamma(t),\gamma(t)) \leq tF(\gamma(t),y) + (1-t)F(\gamma(t),x),$

$\Rightarrow \psi(\gamma(t)) - \psi(x) \leq tF(\gamma(t),y) + (1-t)F(\gamma(t),x) + \psi(\gamma(t)) - \psi(x)$

$$\leq tF(\gamma(t),y)+(1-t)F(\gamma(t),x)+t[\psi(y)-\psi(x)]$$

$\Rightarrow t[F(\gamma(t),y)-F(\gamma(t),x)+\psi(y)-\psi(x)] \geq -[F(\gamma(t),x)+\psi(x)-\psi(\gamma(t))] \geq 0$ (by (3.4)).

That is, $F(\gamma(t),y)-F(\gamma(t),x)+\psi(y)-\psi(x) \geq 0,$ as $t \geq 0$.

By the hemicontinuity of F, taking $t \to 0$, we have

$F(x,y)-F(x,x)+\psi(y)-\psi(x) \geq 0 \Rightarrow F(x,y)+\psi(y)-\psi(x) \geq 0, \ \forall\, y \in K.$

This completes the proof. □ □

Next we prove the main existence theorem. First we consider the case when the set K is bounded.

Theorem 2. Let K be a bounded subset of M and $F : K \times K \to \mathbb{R}$ be a bifunction which is hemicontinuous in the first argument. Suppose for fixed $x \in K$, the mappings $z \mapsto F(x,z)$ and $\psi : K \to \mathbb{R}$ are geodesic convex, lower semicontinuous. Also assume that the bifunction F is pseudomonotone with respect to ψ. Then (MEP) has a solution.

Proof. Consider the two set-valued mappings $G_1 : K \to 2^K$ and $G_2 : K \to 2^K$ such that

$$G_1(y) = \{x \in K : F(x,y)+\psi(y)-\psi(x) \geq 0\}, \ \ \forall\, y \in K,$$

$$G_2(y) = \{x \in K : F(y,x)+\psi(x)-\psi(y) \leq 0\}, \ \ \forall\, y \in K.$$

It is easy to see that $\bar{x} \in K$ solves (MEP) if and only if $\bar{x} \in \bigcap_{y\in K} G_1(y)$. Thus it suffices to prove that $\bigcap_{y\in K} G_1(y) \neq \emptyset$. First we show G_1 is a (KKM) mapping. So we have to prove that for any choice of $x_1,\ldots,x_m \in K$

$$co(\{x_1,\ldots,x_m\}) \subset \bigcup_{i=1}^{m} G_1(x_i).$$

On the contrary assume that there exists a point $x_0 \in K$, such that $x_0 \in co(\{x_1,\ldots,x_m\})$ but $x_0 \notin \bigcup_{i=1}^{m} G_1(x_i)$. That is

$$F(x_0,x_i)+\psi(x_i)-\psi(x_0) < 0, \ \forall i \in \{1,\ldots,m\}.$$

This implies that for any $i \in \{1,\ldots,m\}$, $x_i \in \{y \in K : F(x_0,y)+\psi(y)-\psi(x_0) < 0\}$. As $y \mapsto F(x_0,y)$ and ψ are geodesic convex functions their sum is a geodesic convex function. That is $y \mapsto F(x_0,y)+\psi(y)$ is geodesic convex. So the set $\{y \in K : F(x_0,y)+\psi(y)-\psi(x_0)<0\}$ is a geodesic convex set. Since

$$x_0 \in co(\{x_1,...,x_m\}) \subseteq \{y \in K : F(x_0,y)+\psi(y)-\psi(x_0)<0\}.$$

Therefore $F(x_0,x_0)+\psi(x_0)-\psi(x_0)<0$. But we have $F(x_0,x_0)=0$, a contradiction. Hence G_1 is a (KKM) mapping.

From Lemma 3, we have $G_1(y) \subset G_2(y),\ \forall y \in K$. That is,

$$co(\{x_1,...,x_m\}) \subset \bigcup_{i=1}^{m} G_2(x_i).$$

Hence G_2 is also a (KKM) mapping.

Since $F(y,.)$ and ψ are lower semicontinuous, $G_2(y)$ is closed for all $y \in K$.

Now $G_2(y)$ is a closed subset of a compact set K. So $G_2(y)$ is compact for all $y \in K$.

Hence by Lemma 1, there exists a point $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y\in K} G_2(y)$.

By Lemma 3, we have $\bigcap_{y\in K} G_1(y) = \bigcap_{y\in K} G_2(y)$. That is $\bar{x} \in \bigcap_{y\in K} G_1(y)$. So there exists a point $\bar{x} \in K$, such that

$$F(\bar{x},y)+\psi(y)-\psi(\bar{x}) \geq 0,\ \forall y \in K.$$

Therefore, $\bar{x} \in K$ solves (MEP). □ □

As a monotone bifunction is also pseudomonotone. The following corollary follows.

Corollary 1. Let K be bounded and $F : K \times K \to \mathbb{R}$ be a bifunction which is hemicontinuous in the first argument. Assume that for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ is geodesic convex and lower semicontinuous. Also assume that $\psi : K \to \mathbb{R}$ is geodesic convex and lower semicontinuous map. If F is monotone then (MEP) has a solution.

Suppose K is an unbounded subset of M. For a point $0 \in M$, let $\Sigma_R = \{x \in M : d(0,x) \leq R\}$ be the closed geodesic ball of radius R and center 0.

Theorem 3. Let K be an unbounded subset of M and $F : K \times K \to \mathbb{R}$ be a bifunction which is hemicontinuous in the first argument. Suppose for fixed $x \in K$, the mappings $z \mapsto F(x,z)$ and $\psi : K \to \mathbb{R}$ are geodesic convex, lower semicontinuous. Also assume that F is pseudomonotone with respect to ψ. If there exists a point $x_0 \in K$, such that

$$F(x,x_0) + \psi(x_0) - \psi(x) < 0, \text{ whenever } d(0,x) \to +\infty,\ x \in K, \tag{3.5}$$

holds, then (MEP) has a solution.

Proof. Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then by Theorem 2, there exists at least one $x_R \in K_R$ such that

$$F(x_R,y) + \psi(y) - \psi(x_R) \geq 0,\ \forall y \in K_R. \tag{3.6}$$

We now take a point $x_0 \in K$ satisfying (3.5) with $d(0,x_0) < R$, so $x_0 \in K_R$.
Hence by (3.6), we have

$$F(x_R,x_0) + \psi(x_0) - \psi(x_R) \geq 0. \tag{3.7}$$

If $d(0,x_R) = R$ for all R, we may choose R large enough so that $d(0,x_R) \to +\infty$.
Hence by (3.5), $F(x_R,x_0) + \psi(x_0) - \psi(x_R) < 0$ contradicts (3.7). So there exists an R such that $d(0,x_R) < R$.
Given $y \in K$, let $\gamma(t)$ be a geodesic joining x_R to y with $\gamma(0) = x_R$. Now since $d(0,x_R) < R$, we can choose $0 < t < 1$, sufficiently small so that $\gamma(t) \in K_R$.
Hence

$$\begin{aligned} 0 &\leq F(x_R,\gamma(t)) + \psi(\gamma(t)) - \psi(x_R) \\ &\leq tF(x_R,y) + (1-t)F(x_R,x_R) + t[\psi(y) - \psi(x_R)] \\ &= t[F(x_R,y) + \psi(y) - \psi(x_R)], \\ or, &F(x_R,y) + \psi(y) - \psi(x_R) \geq 0, \text{ for } y \in K. \end{aligned}$$

That is x_R is a solution of (MEP). □ □

The above theorem also holds good for monotone bifunctions.

Corollary 2. Let K be an unbounded subset of M and $F : K \times K \to \mathbb{R}$ be a bifunction which is hemicontinuous in the first argument and monotone. Suppose for fixed $x \in K$, the mappings $z \mapsto F(x,z)$ and $\psi : K \to \mathbb{R}$ are geodesic convex, lower semicontinuous. If there exists a point $x_0 \in K$, such that

$$F(x,x_0) + \psi(x_0) - \psi(x) < 0, \text{ whenever } d(0,x) \to +\infty, \ x \in K,$$

holds, then (MEP) has a solution.

3.4 Existence results for equilibrium problems

If $\psi \equiv 0$, then the mixed equilibrium problem (MEP) reduces to the equilibrium problem (EP). Therefore we can get the existence of solution of the equilibrium problems under both monotonicity and pseudomonotonicity assumptions on the associated bifunctions.

Lemma 4. Let $F : K \times K \to \mathbb{R}$ be a bifunction which is pseudomonotone and hemicontinuous in the first argument. Suppose for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ is geodesic convex. Then $x \in K$ is a solution of the equilibrium problem (EP) if and only if

$$F(y,x) \leq 0, \text{ for all } y \in K.$$

We now provide the main existence theorem. First we take K to be bounded.

Theorem 4. Let K be bounded and $F : K \times K \to \mathbb{R}$ be pseudomonotone and hemicontinuous in the first argument. Let for fixed $x \in K$, the mapping $z \mapsto F(x,z)$ be geodesic convex and lower semicontinuous. Then (EP) has a solution.

As a monotone bifunction is also pseudomonotone. The following corollary follows.

Corollary 3. Let K be bounded subset of M and $F : K \times K \to \mathbb{R}$ be a bifunction which is monotone and hemicontinuous in the first argument. Suppose for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ be geodesic convex and lower semicontinuous. Then (EP) has a solution.

Next we consider the case when K is unbounded.

Theorem 5. Let K be an unbounded subset of M and $F : K \times K \to \mathbb{R}$ be a bifunction which is pseudomonotone and hemicontinuous in the first argument. Suppose for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ is geodesic convex and lower semicontinuous. If there exists a point $x_0 \in K$, such that

$$F(x,x_0) < 0, \text{ whenever } d(0,x) \to +\infty, \ x \in K.$$

Then (EP) has a solution.

The following corollary is obvious.

Corollary 4. Let K be an unbounded subset of M and $F : K \times K \to \mathbb{R}$ be a bifunction which is monotone and hemicontinuous in the first argument. Suppose for fixed $x \in K$, the mapping $z \mapsto F(x,z)$ is geodesic convex, lower semicontinuous. If there exists a point $x_0 \in K$, such that

$$F(x,x_0) < 0, \text{ whenever } d(0,x) \to +\infty, \ x \in K.$$

Then (EP) has a solution.

The following example is constructed in support of Theorem 4.

Example 2. We now consider Example 1. Here K is a compact subset of the Hadamard manifold H^1. Moreover the normalized geodesic $\gamma : \mathbb{R} \to H^1$ starting from $x \in H^1$ is given by

$$\gamma(t) = (\cosh t)x + (\sinh t)v, \ \ \forall t \in \mathbb{R},$$

where $v \in T_x H^1$ is a unit vector. Clearly K is geodesic convex.

The bifunction F defined by

$$F(x,y) = x_2(x_1 - y_1)$$

is pseudomonotone. Also F is continuous in both the variables, it is hemicontinuous in the first variable and lower semicontinuous in the second variable. Clearly F is geodesic convex in the second variable.

Hence F satisfies all the conditions of Theorem 4. If we take $\bar{x} = (1, \sqrt{2}) \in K$ and any point $y = (y_1, y_2)$ in K. Then

$$F(\bar{x}, y) = \sqrt{2}(1 - y_1) \geq 0, \text{ as } y_1 \leq 1.$$

That is $F(\bar{x}, y) \geq 0$, for all $y \in K$. Therefore there exists a point $\bar{x} \in K$ such that $F(\bar{x}, y) \geq 0$, for all $y \in K$.

3.5 Existence results for variational inequality problems

Németh [11] generalized some basic existence and uniqueness theorems of the classical theory of variational inequality problems on the Hadamard manifolds. In this section we establish the existence of solutions of a variational inequality problem under weak pseudomonotonicity assumptions on Hadamard manifolds.

Definition 9. ([16]) A vector field V on K is said to be weak monotone if for any two points $x, y \in K$, there exists a real number $\mu > 0$, such that

$$\langle V_x, \exp_x^{-1} y \rangle + \langle V_y, \exp_y^{-1} x \rangle \leq \mu d^2(x, y). \tag{3.8}$$

Definition 10. A vector field V on K is said to be weak pseudomonotone if for any two points $x, y \in K$, there exists a real number $\mu > 0$, such that

$$\langle V_x, \exp_x^{-1} y \rangle \geq 0 \Rightarrow \langle V_y, \exp_y^{-1} x \rangle \leq \mu d^2(x, y).$$

Remark 2. If a vector field V is weak monotone on K then it is easy to see that it is also weak pseudomonotone. But the converse does not hold in general.

Let M and N be connected Riemannian manifolds and $\Phi : M \to N$ is an isometry, that is Φ is C^∞, and for all $x \in M$ and $u, v \in T_xM$, we have $\langle d\Phi_x u, d\Phi_x v \rangle = \langle u, v \rangle$. One can verify that Φ preserves geodesics, i.e., β is a geodesic in M if and only if $\gamma = \Phi \circ \beta$ is a geodesic in N, and that $d\Phi_{\gamma(t)} \gamma'(t) = \beta'(t)$. Furthermore, Φ preserves the distance function, i.e., $d(\Phi(x), \Phi(y)) = d(x, y)$, for all $x, y \in M$.

Proposition 2. Let M and N be connected Riemannian manifolds and $\chi(M)$ denotes the set of all vector fields in M. Suppose $V \in \chi(M)$ and $\Phi : M \to N$ is an isometry. Let $W \in \chi(N)$ be defined by $W = d\Phi \circ V \circ \Phi^{-1}$. Then

(i) V is monotone if and only if W is monotone;

(ii) V is weakly monotone if and only if W is weakly monotone;

(iii) V is pseudomonotone if and only if W is pseudomonotone;

(iv) V is weak pseudomonotone if and only if W is weak pseudomonotone.

Proof. The proof of the items (i), (ii) and (iii) are similar to those of Proposition 2.3 of [16], hence omitted. The proof of item (iv) follows directly from that of item (iii). □ □

We now construct an example on a Hadamard manifold which shows that weak pseudomonotonicity generalizes weak monotonicity.

Example 3. Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $V(x_1,x_2) = (x_1^2, x_2^2)$ and $K = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \leq 0, x_2 = 0\}$. It is easy to check that K is nonempty, closed and convex subset of $\mathbb{R}^2$ and V is weak pseudomonotone with $\mu = 1$ on K, but it is not weak monotone with $\mu = 1$ on K. Endowing $\mathbb{R}^2$ with the Riemannian metric g, given by

$$g(x_1,x_2) = \begin{pmatrix} 4x_1^2+1 & -2x_1 \\ -2x_1 & 1 \end{pmatrix},$$

we obtain the Riemannian manifold (M,g) that is complete and of constant curvature 0. Note that $\Phi(x_1,x_2) = (x_1, x_1^2 - x_2)$ is an isometry ([6]) between $\mathbb{R}^2$ and (M,g). We define $W = d\Phi \circ V \circ \Phi^{-1}$. From Proposition 2 we conclude that W is weak pseudomonotone with $\mu = 1$ on the closed and geodesic convex subset $\Phi(K)$ of (M,g) but it is not weak monotone with $\mu = 1$ on $\Phi(K)$.

Also Example 3.1 of [16] shows that weak monotonicity generalizes monotonicity. Therefore the implication relationship between monotonicity and some generalized monotonicity is shown as follows:

$$\text{monotonicity} \Rightarrow \text{weak monotonicity} \Rightarrow \text{weak pseudomonotonicity.}$$

To prove the main existence theorem, the following lemma is required.

Lemma 5. Suppose that a vector field V is hemicontinuous and weak pseudomonotone on K. Also assume that for a geodesic $\gamma: [0,1] \to K$, with $\gamma(0) = x$, we have $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then x satisfies

$$\left\langle V_x, \exp_x^{-1} y\right\rangle \geq 0, \ \forall y \in K, \tag{3.9}$$

if and only if it satisfies

$$\left\langle V_y, \exp_y^{-1} x\right\rangle \leq \mu d^2(x,y), \ \forall y \in K. \tag{3.10}$$

Proof. By the definition of weak psudomonotonicity of V it follows that (3.9) implies (3.10). Next to show (3.10) implies (3.9).

Let $z \in K$ and as K is geodesic convex $y = \exp_x(t\exp_x^{-1} z) \in K$ for $0 \leq t \leq 1$. Hence by (3.10) for $t > 0$,

$$\left\langle V_{\exp_x(t\exp_x^{-1} z)}, \exp^{-1}_{\exp_x(t\exp_x^{-1} z)} x\right\rangle \leq \mu d^2(x, \exp_x(t\exp_x^{-1} z)). \tag{3.11}$$

Let $P_{\gamma(t),\gamma(0)}$ be the parallel transport along the geodesic $\gamma: [0,1] \to K$, defined by $\gamma(t) = \exp_x(t\exp_x^{-1} z)$, from $\gamma(t)$ to $\gamma(0) = x$. Since the parallel transport along a curve is an isometry and the tangent vector of a geodesic is parallel along the geodesic, (3.11) implies

$$\left\langle P_{\gamma(t),\gamma(0)} V_{\exp_x(t\exp_x^{-1} z)}, -t\exp_x^{-1} z\right\rangle \leq \mu d^2(x, \exp_x(t\exp_x^{-1} z)),$$

or

$$\left\langle P_{\gamma(t),\gamma(0)} V_{\exp_x(t\exp_x^{-1} z)}, \exp_x^{-1} z\right\rangle \geq -\mu \frac{d^2(x, \exp_x(t\exp_x^{-1} z))}{t} \text{ for all } z \in K. \tag{3.12}$$

Passing $t \to 0$, in inequality (3.12), we get

$$\left\langle V_x, \exp_x^{-1} z\right\rangle \geq 0 \text{ for any } z \in K.$$

So x is a solution of (3.9). The proof completes. □ □

Next we prove the main existence theorem. First we consider the case when K is bounded.

Theorem 6. Let K be a bounded subset of M and for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, we have $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. If the vector field V is hemicontinuous, weak pseudomonotone, then (VIP) has a solution.

Proof. Consider the two set-valued mappings $F : K \to 2^K$ and $G : K \to 2^K$ such that

$$F(y) = \{x \in K : \langle V_x, \exp_x^{-1} y\rangle \geq 0\}, \text{ for all } y \in K;$$

$$G(y) = \{x \in K : \langle V_y, \exp_y^{-1} x\rangle \leq \mu d^2(x,y)\}, \text{ for all } y \in K.$$

It is easy to see that $\bar{x} \in K$ solves (VIP) if and only if $\bar{x} \in \bigcap_{y\in K} F(y)$. Thus it suffices to prove that $\bigcap_{y\in K} F(y) \neq \emptyset$. First we show that F is a (KKM) mapping. So we have to prove that for any choice of $x_1, ..., x_m \in K$,

$$co(\{x_1, ..., x_m\}) \subset \bigcup_{i=1}^{m} F(x_i).$$

Suppose on the contrary that there exists a point $x_0 \in K$, such that $x_0 \in co(\{x_1, ..., x_m\})$ but $x_0 \notin \bigcup_{i=1}^m F(x_i)$. That is

$$\langle V_{x_0}, \exp_{x_0}^{-1} x_i\rangle < 0, \ \forall i \in \{1, ..., m\}.$$

This implies that for any $i \in \{1, ..., m\}$, $x_i \in \{y \in K : \langle V_{x_0}, \exp_{x_0}^{-1} y\rangle < 0\}$. Since the function $y \mapsto \langle u, \exp_x^{-1} y\rangle$ is quasi-convex, the set $\{y \in K : \langle V_{x_0}, \exp_{x_0}^{-1} y\rangle < 0\}$ is a geodesic convex set ([1]). Then

$$x_0 \in co(\{x_1, ..., x_m\}) \subseteq \{y \in K : \langle V_{x_0}, \exp_{x_0}^{-1} y\rangle < 0\}.$$

Therefore

$$\langle V_{x_0}, \exp_{x_0}^{-1} x_0\rangle < 0.$$

This is a contradiction as $\langle V_{x_0}, \exp_{x_0}^{-1} x_0\rangle = 0$.

So F is a (KKM) mapping.

From weak pseudomonotonicity we have $F(y) \subset G(y),\ \forall y \in K$. Hence G is also a (KKM) mapping.
Let $x \in K$ and $\{x_n\} \in G(y)$, such that $x_n \to x$ as $n \to \infty$. Then $\exp_y^{-1} x_n \to \exp_y^{-1} x$. Hence

$$\left\langle V_y, \exp_y^{-1} x_n \right\rangle \to \left\langle V_y, \exp_y^{-1} x \right\rangle.$$

Also since d is continuous $d(x_n, y) \to d(x,y)$ as $n \to \infty$. This implies for all $y \in K$, $G(y)$ is closed subset of the compact set K. Hence $G(y)$ is compact.
Hence by Lemma 1, there exists a point $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y \in K} G(y)$.
By Lemma 5 we have $\bigcap_{y \in K} G(y) = \bigcap_{y \in K} F(y)$. That is $\bar{x} \in \bigcap_{y \in K} F(y)$.
So there exists a point $\bar{x} \in K$ such that

$$\left\langle V_{\bar{x}}, \exp_{\bar{x}}^{-1} y \right\rangle \geq 0,\ \forall y \in K.$$

Therefore, $\bar{x} \in K$ solves (VIP). □ □

As weak pseudomonotonicity generalizes weak monotonicity. The following corollary is obvious.

Corollary 5. Let K be a bounded subset of M and a vector field V be hemicontinuous, weak monotone. Also assume that for a geodesic $\gamma\colon [0,1] \to K$, with $\gamma(0) = x$, we have $\lim_{t \to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then (VIP) has a solution.

Next we consider the case when K is unbounded. Given a point $0 \in M$, we denote $\Sigma_R = \{x \in M : d(0,x) \leq R\}$ to be the closed geodesic ball of radius R and center 0.

Theorem 7. Suppose that K is an unbounded subset of M and for a geodesic $\gamma\colon [0,1] \to K$, with $\gamma(0) = x$, we have $\lim_{t \to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Also in addition we assume that the following condition (A) holds.
(A) There exists a point $x_0 \in K$ such that

$$\left\langle V_x, \exp_x^{-1} x_0 \right\rangle < 0, \text{ with } d(0,x) \to +\infty,\ x \in K. \tag{3.13}$$

If a vector field V is hemicontinuous and weak pseudomonotone, then (VIP) has a solution.

Proof. Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then there exists at least one $x_R \in K_R$

$$\left\langle V_{x_R}, \exp_{x_R}^{-1} y \right\rangle \geq 0,\ \forall y \in K_R, \tag{3.14}$$

by Theorem 6. We now take a point $x_0 \in K$ satisfying (3.13) with $d(0,x_0) < R$, so $x_0 \in K_R$.

Hence by (3.14), we have

$$\left\langle V_{x_R}, \exp_{x_R}^{-1} x_0 \right\rangle \geq 0. \tag{3.15}$$

If $d(0,x_R) = R$ for all R, we may choose R large enough so that $d(0,x_R) \to \infty$.

Hence by assumption (A), $\left\langle V_{x_R}, \exp_{x_R}^{-1} x_0 \right\rangle < 0$, which contradicts (3.15). So there exists an R such that $d(0,x_R) < R$.

Hence given $y \in K$, we can choose $t \geq 0$ sufficiently small so that $w = \exp_{x_R}(t \exp_{x_R}^{-1} y) \in K_R$. Consequently

$$0 \leq \left\langle V_{x_R}, \exp_{x_R}^{-1} w \right\rangle = t \left\langle V_{x_R}, \exp_{x_R}^{-1} y \right\rangle \text{ for } y \in K,$$

or

$$\left\langle V_{x_R}, \exp_{x_R}^{-1} y \right\rangle \geq 0 \text{ for } y \in K,$$

which means that x_R is a solution of (VIP). □ □

From the above theorem, the following corollary follows immediately.

Corollary 6. Let $K \subset M$ be an unbounded and a vector field V be hemicontinuous, weak monotone. Also assume that for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, we have $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. If there exists a point $x_0 \in K$, such that

$$\left\langle V_x, \exp_x^{-1} x_0 \right\rangle < 0, \text{ with } d(0,x) \to +\infty,\ x \in K.$$

Then the variational inequality problem (VIP) has a solution.

3.6 Existence results for mixed variational inequality problems

This section is devoted to the study of mixed variational inequality problems on Hadamard manifolds under weak monotonicity assumptions. Let V be a vector field on K and $\psi : K \to \mathbb{R}$ be a mapping. Then the problem is to find $\bar{x} \in K$, such that

$$\text{(MVIP)} \qquad \left\langle V_{\bar{x}}, \exp_{\bar{x}}^{-1} y \right\rangle + \psi(y) - \psi(\bar{x}) \geq 0, \ \forall y \in K,$$

is called a mixed variational inequality problem ([5]). We denote SOL(MVIP), the solution set of mixed variational inequality (MVIP).

Remark 3. If $\psi(\bar{x}) \equiv 0$, then (MVIP) reduces to the variational inequality problem (VIP).

Now we need the following lemma which will be used to prove our main results on mixed variational inequality problems.

Lemma 6. Suppose that a vector field V is hemicontinuous, weak monotone on K and $\psi : K \to \mathbb{R}$ is geodesic convex. Also for a geodesic $\gamma : [0,1] \to K$ with $\gamma(0) = x$, $\lim_{t \to 0} \frac{d^2(x, \gamma(t))}{t} = 0$. Then x satisfies

$$\left\langle V_x, \exp_x^{-1} y \right\rangle + \psi(y) - \psi(x) \geq 0, \ \forall y \in K, \tag{3.16}$$

if and only if it satisfies

$$\left\langle V_y, \exp_y^{-1} x \right\rangle + \psi(x) - \psi(y) \leq \mu d^2(x,y), \ \forall y \in K. \tag{3.17}$$

Proof. First we show that (3.16) implies (3.17). Since V is weak monotone we have

$$\left\langle V_x, \exp_x^{-1} y \right\rangle + \left\langle V_y, \exp_y^{-1} x \right\rangle \leq \mu d^2(x,y)$$

$$\Rightarrow \left\langle V_y, \exp_y^{-1} x \right\rangle + \psi(x) - \psi(y) \leq \mu d^2(x,y) - \left\langle V_x, \exp_x^{-1} y \right\rangle + \psi(x) - \psi(y)$$

$$= \mu d^2(x,y) - \{\left\langle V_x, \exp_x^{-1} y \right\rangle + \psi(y) - \psi(x)\}$$

$$\leq \mu d^2(x,y).$$

Next to show that (3.17) implies (3.16).

Let $\gamma(t) = \exp_x(t\exp_x^{-1} y), 0 \le t \le 1$ be the geodesic joining x and y in K with $\gamma(0) = x$.

Since K is geodesic convex, $\gamma(t) \in K$. Hence

$$\left\langle V_{\gamma(t)}, \exp_{\gamma(t)}^{-1} x\right\rangle + \psi(x) - \psi(\gamma(t)) \le \mu d^2(x, \gamma(t)). \tag{3.18}$$

As ψ is geodesic convex, we have

$$\psi(\gamma(t)) \le t\psi(y) + (1-t)\psi(x)$$

$$\Rightarrow \psi(\gamma(t)) - \psi(x) \le t[\psi(y) - \psi(x)].$$

From (3.18) we have,

$$\left\langle V_{\gamma(t)}, \exp_{\gamma(t)}^{-1} x\right\rangle \le \mu d^2(x, \gamma(t)) + \psi(\gamma(t)) - \psi(x)$$

or,

$$\left\langle V_{\gamma(t)}, \exp_{\gamma(t)}^{-1} x\right\rangle \le \mu d^2(x, \gamma(t)) + t[\psi(y) - \psi(x)]. \tag{3.19}$$

Let $P_{\gamma(t),\gamma(0)}$ be the parallel transport along the geodesic $\gamma : [0,1] \to K$, from $\gamma(t)$ to $\gamma(0) = x$. Since the parallel transport along a curve is an isometry and the tangent vector of a geodesic is parallel along the geodesic from (3.19), we have

$$\left\langle P_{\gamma(t),\gamma(0)} V_{\gamma(t)}, -t\exp_x^{-1} y\right\rangle \le \mu d^2(x, \gamma(t)) + t[\psi(y) - \psi(x)],$$

or,

$$\left\langle P_{\gamma(t),\gamma(0)} V_{\gamma(t)}, \exp_x^{-1} y\right\rangle \ge -\mu \frac{d^2(x, \gamma(t))}{t} - \psi(y) + \psi(x). \tag{3.20}$$

Taking $t \to 0$ in inequality (3.20) we get

$$\left\langle V_x, \exp_x^{-1} y\right\rangle \ge \psi(x) - \psi(y),$$

or,

$$\left\langle V_x, \exp_x^{-1} y\right\rangle + \psi(y) - \psi(x) \ge 0.$$

□ □

Next we prove the existence theorem for (MVIP) on Hadamard manifolds. First we take the set K to be bounded.

Theorem 8. Let K be a bounded subset of M. Suppose a vector field V is hemicontinuous and weak monotone. Assume that $\psi : K \to \mathbb{R}$ is a geodesic convex, lower semicontinuous map. Also assume that for a geodesic $\gamma : [0,1] \to K$ with $\gamma(0) = x$, we have $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then (MVIP) has a solution.

Proof. Consider the two set-valued mappings $F : K \to 2^K$ and $G : K \to 2^K$ such that

$$F(y) = \{x \in K : \langle V_x, \exp_x^{-1} y\rangle + \psi(y) - \psi(x) \geq 0\}, \text{ for all } y \in K.$$

$$G(y) = \{x \in K : \langle V_y, \exp_y^{-1} x\rangle + \psi(x) - \psi(y) \leq \mu d^2(x,y)\}, \text{ for all } y \in K.$$

It is easy to see that $\bar{x} \in K$ solves (MVIP) if and only if $\bar{x} \in \bigcap_{y\in K} F(y)$. Thus it suffices to prove that $\bigcap_{y\in K} F(y) \neq \emptyset$. First we show that F is a (KKM) mapping. So we have to prove that for any choice of $x_1,...,x_m \in K$

$$co(\{x_1,...,x_m\}) \subset \bigcup_{i=1}^{m} F(x_i).$$

On the contrary suppose that there exists a point $x_0 \in K$, such that $x_0 \in co(\{x_1,...,x_m\})$ but $x_0 \notin \bigcup_{i=1}^{m} F(x_i)$. That is

$$\langle V_{x_0}, \exp_{x_0}^{-1} x_i\rangle + \psi(x_i) - \psi(x_0) < 0, \ \forall i \in \{1,...,m\}.$$

This implies that for any $i \in \{1,...,m\}$, $x_i \in \{y \in K : \langle V_{x_0}, \exp_{x_0}^{-1} y\rangle + \psi(y) - \psi(x_0) < 0\}$. Since the function $y \mapsto \langle u, \exp_x^{-1} y\rangle$ is quasi-convex, the set $\{y \in K : \langle V_{x_0}, \exp_{x_0}^{-1} y\rangle + \psi(y) - \psi(x_0) < 0\}$ is a geodesic convex set ([1]). Therefore,

$$x_0 \in co(\{x_1,...,x_m\}) \subseteq \{y \in K : \langle V_{x_0}, \exp_{x_0}^{-1} y\rangle + \psi(y) - \psi(x_0) < 0\}.$$

Hence $0 = \langle V_{x_0}, \exp_{x_0}^{-1} x_0\rangle < 0$, a contradiction.

So F is a (KKM) mapping.

From Lemma 6, we have $F(y) \subset G(y)$ $\forall y \in K$. Hence G is also a (KKM) mapping.

Let $x \in K$ and $\{x_n\} \in G(y)$, such that $x_n \to x$ as $n \to \infty$.

Then we have, $\exp_y^{-1} x_n \to \exp_y^{-1} x$.

Hence

$$\langle V_y, \exp_y^{-1} x_n \rangle \to \langle V_y, \exp_y^{-1} x \rangle.$$

Also since d is continuous, $d(x_n, y) \to d(x, y)$ as $n \to \infty$ and ψ is lower semicontinuous, which implies $G(y)$ is closed for all $y \in K$.

Now $G(y)$ is a closed subset of a compact set K. So G(y) is compact for all $y \in K$.

Hence by Lemma 1 there exists a point $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y \in K} G(y)$.

By Lemma 6 we have $\bigcap_{y \in K} G(y) = \bigcap_{y \in K} F(y)$. That is $\bar{x} \in \bigcap_{y \in K} F(y)$.

So there exists a point $\bar{x} \in K$, such that

$$\langle V_{\bar{x}}, \exp_{\bar{x}}^{-1} y \rangle + \psi(y) - \psi(x) \geq 0, \ \forall y \in K.$$

Therefore, $\bar{x} \in K$ solves (MVIP). □ □

As weak monotonicity generalizes monotonicity, the following corollary is obvious.

Corollary 7. Let $K \subset M$ be bounded. Assume that a vector field V is hemicontinuous, monotone. And $\psi : K \to \mathbb{R}$ is a geodesic convex, lower semicontinuous function. Also suppose that for a geodesic $\gamma : [0,1] \to K$ with $\gamma(0) = x$, we have $\lim_{t \to 0} \frac{d^2(x, \gamma(t))}{t} = 0$. Then (MVIP) has a solution.

Next we consider the case where K is unbounded.

Theorem 9. Let K be an unbounded subset of M. Assume that a vector field V is hemicontinuous, weak monotone. And $\psi : K \to \mathbb{R}$ is a geodesic convex, lower semicontinuous map. Suppose that for a geodesic $\gamma : [0,1] \to K$ with $\gamma(0) = x$, $\lim_{t \to 0} \frac{d^2(x, \gamma(t))}{t} = 0$ holds. If

(B) there exists a point $x_0 \in K$ such that

$$\langle V_x, \exp_x^{-1} x_0 \rangle + \psi(x_0) - \psi(x) < 0, \text{ with } d(0, x) \to +\infty, \ x \in K; \tag{3.21}$$

then (MVIP) has a solution.

Proof. We denote $\Sigma_R = \{x \in M : d(0,x) \leq R\}$ the closed geodesic ball of radius R and center 0. Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then there exists a point $x_R \in K_R$ such that

$$\left\langle V_{x_R}, \exp_{x_R}^{-1} y \right\rangle + \psi(y) - \psi(x_R) \geq 0, \ \forall y \in K_R, \tag{3.22}$$

by Theorem 8. We now take a point $x_0 \in K$ satisfying (3.21) with $d(0,x_0) < R$, so $x_0 \in K_R$. Hence by (3.22), we have

$$\left\langle V_{x_R}, \exp_{x_R}^{-1} x_0 \right\rangle + \psi(x_0) - \psi(x_R) \geq 0. \tag{3.23}$$

If $d(0,x_R) = R$ for all R, we may choose R large enough so that $d(0,x_R) \to \infty$.
Hence by assumption (B), $\left\langle V_{x_R}, \exp_{x_R}^{-1} x_0 \right\rangle + \psi(x_0) - \psi(x_R) < 0$, contradicts (3.23). So there exists an R such that $d(0,x_R) < R$.
Hence given $y \in K$, we can choose $t \geq 0$ sufficiently small so that $\gamma(t) = \exp_{x_R}(t \exp_{x_R}^{-1} y) \in K_R$. Consequently

$$\left\langle V_{x_R}, \exp_{x_R}^{-1} \gamma(t) \right\rangle + \psi(\gamma(t)) - \psi(x_R) \geq 0,$$

or,

$$\left\langle V_{x_R}, t \exp_{x_R}^{-1} y \right\rangle + \psi(\gamma(t)) - \psi(x_R) \geq 0. \tag{3.24}$$

As ψ is geodesic convex

$$\psi(\gamma(t)) \leq t\psi(y) + (1-t)\psi(x_R)$$

$$\Rightarrow \psi(\gamma(t)) - \psi(x_R) \leq t[\psi(y) - \psi(x_R)].$$

Now from (3.24),

$$\left\langle V_{x_R}, t \exp_{x_R}^{-1} y \right\rangle \geq \psi(x_R) - \psi(\gamma(t)) \geq t(\psi(x_R) - \psi(y)),$$

as $t \geq 0$, we get

$$\left\langle V_{x_R}, \exp_{x_R}^{-1} y \right\rangle \geq \psi(x_R) - \psi(y)$$

or,

$$\left\langle V_{x_R}, \exp_{x_R}^{-1} y \right\rangle + \psi(y) - \psi(x_R) \geq 0,\ y \in K.$$

which means that x_R is a solution of (MVIP). □ □

The above theorem holds for monotone vector fields also.

Corollary 8. Let a vector field V be hemicontinuous, monotone. And $\psi : K \to \mathbb{R}$ is a geodesic convex, lower semicontinuous map. Also assume that for a geodesic $\gamma : [0,1] \to K$ with $\gamma(0) = x$, we have $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. If there exists a point $x_0 \in K$, such that

$$\left\langle V_x, \exp_x^{-1} x_0 \right\rangle + \psi(x_0) - \psi(x) < 0, \text{ with } d(0,x) \to +\infty,\ x \in K;$$

then (MVIP) has a solution.

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Stochastic Programming Problems Involving Some Continuous Random Variables

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Abstract

Stochastic programming(SP) is an active branch of mathematical programming dealing with optimization problems involving uncertain data. Chance constrained linear programming problems and two-stage stochastic programming problems are two very popular approaches to solve SP problems. In this paper, we consider some chance constrained programming problems and two-stage stochastic programming problems where the right hand side parameters of the constraints follow some non-normal continuous distributions namely two parameter exponential distribution and three parameter gamma distribution with known parameters. To find the solution of the stated problems, we first convert the problems in to equivalent deterministic models. Standard mathematical programming technique is applied to solve the problems. Some numerical examples are presented to illustrate the proposed methodology.

Keywords: Stochastic programming, Chance Constrained Programming, Two-stage stochastic Programming, Two parameter exponential distribution, Three parameter gamma distribution.

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1 Introduction

In Operations Research (OR), a decision maker formulate a real life decision making problem into a mathematical programming model. By solving such a mathematical programming model the decision maker can take his/her decision. A mathematical programming model can be represented as :

$$\max : \phi(X;p) \tag{1.1}$$

subject to

$$h_i(X;p) \leq b_i, \quad i = 1,2,...,m. \tag{1.2}$$

$$X \geq 0 \tag{1.3}$$

where $X \subset R^n$ is called a decision vector, p is a parameter which forms a parametric space, and the functions ϕ and h_i $(i = 1,2,\ldots,m)$ are defined as $\phi, h_i : R^n \to R$. The function ϕ is called an objective function, and (1.2)-(1.3) are known as constraints. If all the functions ϕ and h_i $(i = 1,2,\ldots,m)$ in the model are linear and X is a continuous vector then the mathematical programming model is called a linear programming (LP) problem otherwise it is called a non-linear programming (NLP) problem. There exists several type of mathematical programming problem depending upon their nature e.g., single objective and multi-objective optimization (depending on the number of objective functions), continuous, discrete, mixed integer optimization (depending on the nature of decision variables), deterministic, fuzzy, stochastic, possibilistic optimization (depending on the parameter space), multi-level optimization (depending on the multiple decision makers in a hierarchical organization).

Deterministic mathematical programming has been successfully used for modeling and analyzing a wide variety of systems requiring optimization. One of its assumptions is that the parameters in the models are known with certainty. In many practical applications, this assumption is quite restrictive and usually violated because of the inherent randomness in the system. For instance, in planning the capacity of a telecommunications network, the future demand pattern under which the system will operate is typically not known with certainty. In financial portfolio management, one chooses to invest in financial assets whose future returns are not yet realized. In these settings, an optimal network design or investment policy obtained via a deterministic mathematical program is not satisfying because it either completely disregards or does not capture well the underlying random effect. In other words, the deterministic model does not take possible future scenarios into account when specifying an optimal solution. Decision makers often have to make decisions in the presence of uncertainty. Decision making problems are often formulated as optimization problems, and thus in many situations decision makers wish to solve optimization problems which depend on parameters which are unknown. Typically it is quite difficult to formulate and solve such problems, both conceptually and numerically. The difficulty already starts at the conceptual stage of modeling. Usually there are a variety of ways in which the uncertainty can be formalized. In the formulation of optimization problems, one usually attempts to find a good trade-off between the realism of the optimization model, which usually affects the usefulness and quality of the obtained decisions, and the tractability of the problem, so that it could be solved analytically or numerically. As a result of these considerations there are a large number of different approaches for formulating and solving optimization problems under uncertainty. One of the most significant approaches of handling the uncertainty is known as stochastic programming (SP). Other approaches namely fuzzy programming and interval programming are also used for handling uncertainties. Our objective is to study some stochastic programming programming problems and their variants.

In most of the real-life decision-making problem, decision maker needs to take decision under some uncertain environment. The uncertainty can be found in parameter space as well as in the decision space of a decision making problem. These uncertainties are addressed by using probability distribution or fuzzy numbers or intervals. Stochastic programming (SP) is concerned with the decision making problems in which some or all parameters are treated as random variables in order to capture the uncertainty. SP is used in several real world decision making areas such as energy management, financial modeling, supply chain and scheduling, hydro thermal power production planning, transportation, agriculture, de-

fence, environmental and pollution control, production and control management, telecommunications, etc. Several models and methodologies have been developed in the field of stochastic programming. In the literature, there exist two very popular approaches to solve SP problems, namely,
(i) Chance constrained programming, and
(ii) Two-stage programming.

Chance constrained programming was developed as a means of describing constraints in mathematical programming models in the form of probability levels of attainment. The chance constrained programming (CCP) can be used to solve problems involving chance-constraints, i.e.constraints having violation up to a pre-specified probability level. The use of chance-constraints was initially introduced by Charnes and Cooper[**?**]. They established three different models for the objective functions with random cost coefficients:
(i) E-model which maximizes the expected value of the objective function,
(ii) V-model which minimizes the generalized mean square of the objective function, and
(iii) P-model which maximizes the probability of the aspiration level of the objective function.
These techniques at first transforms a SP problem into an equivalent deterministic model. Then it is solved by using the standard mathematical programming techniques.

Similarly, when the optimal decision is not specified to the realization of future events, a static stochastic programming model is formulated, although in many context the decision maker has to make a decision before observing random events which influence the system he/she wants to control. Further, the optimum solution can be obtained after observation of the random events. For this case, a special class of dynamic programming model has to be formulated known as two-stage stochastic programming problem(TSP). TSP is very effective for problems where an analysis of policy scenarios is desired and when the right-hand side goals of the constraints are random variables with known probability distributions. TSP deals with recourse, where corrective actions can be taken after a random event has taken place. In TSP, a decision is firstly made before values of random variables are known; after the random events occurred and their values are known, a second-stage decision can be made to minimize "penalties" that may appear due to any infeasibility. The formulation of two-stage stochastic programming problems was first introduced by Dantzig [9]. Further it was developed by Beale [6] and Dantzig and Madansky [10]. Unlike the chance constrained programming, the two-stage programming does not allow any constraint to be violated.

In the next Section, we have presented some literature available on stochastic programming. In rest of the paper, we have discussed about our models and methodologies.

2 Literature Survey

In the literature of the stochastic linear programming [15, 16, 13], various models have been suggested by several researchers. Bibliographical review is presented by Stancu and Wets [28], Infanger [13]. Most of the applications of the stochastic models assume normal distribution for model coefficients. Apart from the normal distribution, other distributions have been considered for the model coefficients. Goicoechea et al.[11] presented some probabilistic model involving uniform, exponential, normal and other random variables. Further, Goicoechea and Duckstein[12] presented some deterministic equivalent models for the probabilistic programming with non-normal distributions. Miller and Wagner [23] presented a method for solving chance constrained programming with joint constraints. Later, Jagannathan[14] has presented a single-objective joint chance constrained programming model by considering the coefficient matrices whose elements are normal random variables. Biswal et al.[7] presented some probabilistic linear programming problems by considering some parameters as exponential random variables. Later, Biswal et al.[8] proposed a solution scheme for solving probabilistic constrained programming problems involving log-normal random variables. Sahoo and Biswal [25] have also presented some stochastic programming problems with cauchy and extreme value distributions. Further, they presented some probabilistic linear programming problems by assuming the random parameters as normal and log-normal random variables with joint constraint [26]. Barik et al.[1] presented some stochastic programming problems involving pareto distributions. Recently, Pradhan and Biswal [24] presented a solution procedure based on chance constrained programming technique to solve a multi-choice probabilistic linear programming problem where alternative choices of any multi-choice parameter are considered as random variables.

The formulation of two-stage stochastic programming problems was first introduced by Dantzig[9]. This model was further developed by Beale [6] and Dantzig and Madansky [10]. Later, Wets [29] proposed an equivalent convex program of a two stage stochastic programming under uncertainty, while Maarten [21] presented an additional bibliographical study of stochastic programming based on the study of nearly 351 research papers, from 1996-2007. Quite successfully, Maqsood et al.[22] presented an interval-parameter fuzzy two-stage stochastic programming method for the planning of water-resources-management systems under uncertainty. This study was further developed by Li et al.[20] who proposed an interval-parameter two-stage stochastic mixed integer programming technique for waste management under uncertainty. Bashiri and Rezaei [5] proposed an extended relocation model for warehouses configuration in a supply chain network, in which uncertainty is associated to operational costs, production capacity and demands. Barik et al.[3] established a solution procedure for solving the two-stage stochastic linear programming problem considering both randomness and interval parameters in the problem formulation. Barik et al. [4]also developed a solution procedure for the multiobjective two-stage stochastic linear programming problem considering some parameters of the linear constraints as interval type discrete random variables with known probability distribution. Barik et al.[2] further presented solution procedures for a two-stage stochastic programming problem where the right hand side parameters follow some continuous distribution such as either uniform or exponential or normal or log-normal distribution with known mean and variance.

In the literature, there is no article on the stochastic programming problem where some parameters follow either two parameter exponential distribution or three parameter gamma distribution. So, in this study, we propose solution procedures of a chance constrained programming problem and a two-stage stochastic programming problem where the right hand side parameters follow either two parameter exponential distribution or three parameter gamma distribution with known parameters. To establish the solution procedures of the proposed problem, we transform the problem into an equivalent deterministic model. Then a standard mathematical programming technique is used to solve the transformed deterministic model.

3 Stochastic Programming Model

Mathematically, a stochastic programming problem can be stated as :

$$\max : z = \sum_{j=1}^{n} c_j x_j \tag{3.4}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, ..., m \tag{3.5}$$

$$x_j \ge 0, \qquad j = 1, 2, ..., n \tag{3.6}$$

where $x_j\,(j = 1, 2, ..., n)$ are n decision variables, $a_{ij}\,(i = 1, 2, ..., m; j = 1, 2, ..., n)$ are the coefficients of the technological matrix, $c_j\,(j = 1, 2, ..., n)$ are the coefficients associated with the decision variables in the objective function. Only the right hand side parameters $b_i\,(i = 1, 2, ..., m)$ of the constraints are considered as random variable which follow some continuous distribution with finite mean and variance.

3.1 Chance constrained programming model

Mathematical model of a chance constrained programming problem is given by:

$$\max : z = \sum_{j=1}^{n} c_j x_j \tag{3.7}$$

subject to

(3.8) $$Pr(\sum_{j=1}^{n} a_{ij}x_j \leq b_i) \geq (1-\gamma_i), \quad i=1,2,...,m$$

(3.9) $$0 < \gamma_i < 1, \qquad i = 1,2...,m$$

(3.10) $$x_j \geq 0, \qquad j = 1,2...,n$$

where $x_j\,(j = 1,2,...,n)$ are n decision variables, $a_{ij}\,(i = 1,2,...,m; j = 1,2,...,n)$ are the constraint coefficients, $c_j\,(j = 1,2,...,n)$ are the coefficients associated with the decision variables in the objective function, where Pr means probability, γ_i is the given probability of the extents to which the i-th constraint violations are admitted. The inequalities given by (3.8) are called chance constraints. Only the right hand side parameters $b_i\,(i = 1,2,...,m)$ are considered as random variables which follows some continuous distributions with finite mean and variance.

3.1.1 Chance constrained programming problem involving two parameter exponential distribution

It is assumed that $b_i\,(i = 1,2,...,m)$ are independent random variables those follows two parameters exponential distribution[19] with parameters θ_i, σ_i where

$$E(b_i) = \theta_i + \sigma_i, i = 1,2,..,m \text{ and} \tag{3.11}$$

$$Var(b_i) = \sigma_i^2, \;\; i = 1,2,...,m \tag{3.12}$$

The probability density function (pdf) of the i-th two parameter exponential variable b_i is given by

$$f(b_i) = \frac{1}{\sigma_i}\exp\left(\frac{-(b_i-\theta_i)}{\sigma_i}\right), i = 1,2,..,m \tag{3.13}$$

where $b_i \geq \theta_i$, $\sigma_i > 0$.

To solve the problem (3.7)-(3.10), we establish the deterministic form of the problem. In this case, right hand side parameters of the chance-constraints follows two parameter exponential distribution. Then from the chance-constraint (3.8), we have

$$Pr(\sum_{j=1}^{n} a_{ij}x_j \leq b_i) \geq (1-\gamma_i)$$
$$\Rightarrow Pr(b_i \geq \sum_{j=1}^{n} a_{ij}x_j) \geq (1-\gamma_i)$$
$$\Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} f(b_i)db_i \geq (1-\gamma_i)$$
$$\Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} \frac{1}{\sigma_i} \exp\Big(\frac{-(b_i-\theta_i)}{\sigma_i}\Big)db_i \geq (1-\gamma_i)$$

Integrating, we obtain

(3.14) $$\sum_{j=1}^{n} a_{ij}x_j \leq \theta_i - \sigma_i ln(1-\gamma_i)$$

Using this value in (3.8), we establish equivalent deterministic model of (3.7)-(3.10) as follows:

$$\max : z = \sum_{j=1}^{n} c_jx_j \tag{3.15}$$

subject to

(3.16) $$\sum_{j=1}^{n} a_{ij}x_j \leq \theta_i - \sigma_i ln(1-\gamma_i)$$

(3.17) $$0 < \gamma_i < 1, \qquad i = 1,2...,m$$

(3.18) $$x_j \geq 0, \qquad j = 1,2...,n$$

3.1.2 Chance constrained programming problem involving three parameter gamma distribution

Here, we assume that $b_i\,(i = 1,2,...,m)$ are independent random variables follow three parameter gamma distribution[18] with parameters α_i, β_i and θ_i where

$$E(b_i) = \alpha_i\beta_i + \theta_i, i = 1,2,..,m \; and \tag{3.19}$$

$$Var(b_i) = \alpha_i\beta_i^2, \quad i = 1,2,...,m \tag{3.20}$$

The probability density function (pdf) of the i-th three parameter gamma variable b_i is given by

$$f(b_i) = \frac{1}{\Gamma(\alpha_i)\beta_i^{\alpha_i}}(b_i-\theta_i)^{\alpha_i-1} \exp\Big(\frac{-(b_i-\theta_i)}{\beta_i}\Big), i = 1,2,....m \tag{3.21}$$

where $\alpha_i > 0, \beta_i > 0$ and $b_i \geq \theta_i$. It is further considered that α_i is a positive integer.

To solve the CCP problem (3.7)-(3.10), we establish the deterministic form of the problem. In this case, right hand side parameters of the chance-constraints follows three parameter gamma distribution. Then from the chance-constraint (3.8), we have

$$\begin{aligned}
& Pr(\sum_{j=1}^{n} a_{ij}x_j \leq b_i) \geq (1-\gamma_i) \\
& \Rightarrow Pr(b_i \geq \sum_{j=1}^{n} a_{ij}x_j) \geq (1-\gamma_i) \\
& \Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} f(b_i)db_i \geq (1-\gamma_i) \\
& \Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} \frac{1}{\Gamma(\alpha_i)\beta_i^{\alpha_i}} (b_i-\theta_i)^{\alpha_i-1} \exp\Big(\frac{-(b_i-\theta_i)}{\beta_i}\Big) db_i \geq (1-\gamma_i)
\end{aligned}$$

Integrating, we obtain

$$\exp\left(\frac{-(\sum_{j=1}^{n} a_{ij}x_j-\theta_i)}{\beta_i}\right)\left(\sum_{k=0}^{\alpha_i-1}\frac{1}{k!}\Big(\frac{(\sum_{j=1}^{n} a_{ij}x_j-\theta_i)}{\beta_i}\Big)^k\right) \geq (1-\gamma_i) \tag{3.22}$$

Using the above result in (3.8) we get equivalent deterministic model of (3.7)-(3.10) as follows:

$$\max : z = \sum_{j=1}^{n} c_j x_j \tag{3.23}$$

subject to

$$\exp\left(\frac{-(\sum_{j=1}^{n} a_{ij}x_j-\theta_i)}{\beta_i}\right)\left(\sum_{k=0}^{\alpha_i-1}\frac{1}{k!}\Big(\frac{(\sum_{j=1}^{n} a_{ij}x_j-\theta_i)}{\beta_i}\Big)^k\right) \geq (1-\gamma_i) \tag{3.24}$$

$$0 < \gamma_i < 1, \qquad i = 1,2...,m \tag{3.25}$$

$$x_j \geq 0, \qquad j = 1,2...,n \tag{3.26}$$

3.2 Two-stage stochastic programming model

Basically, two-stage stochastic programming problems are formulated to optimize the decisions which are made in two different stages.The first phase decisions are made before the realization of the random events and the second stage decisions are made after they have been realized.

Here, we shall consider a stochastic linear programming(SLP) problem of the form

$$\min : z = \sum_{j=1}^{n} c_j x_j \tag{3.27}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, ..., m \tag{3.28}$$

$$\sum_{j=1}^{n} r_{sj} x_j \geq h_s, s = 1, 2, ..., l \tag{3.29}$$

$$x_j \geq 0, \qquad j = 1, 2, ..., n \tag{3.30}$$

where $x_j\,(j = 1,2,...,n)$ are n decision variables, $a_{ij}\,(i = 1,2,...,m;\ j = 1,2,...,n)$, $r_{sj}\,(s = 1,2,...,l;\ j = 1,2,...,n)$ are the constraint coefficients, $c_j\,(j = 1,2,...,n)$ are the coefficient associated with the decision variables in the objective function. Only the right hand side parameters $b_i\,(i = 1,2,...,m)$ are considered as random variables which follows some continuous distributions with finite mean and variance.

Suppose, we have found a vector $(x_1, x_2, ..., x_n), x_j \geq 0\,(j = 1,2,..,n)$,which is feasible to above model (3.27)-(3.30) for an estimated or guessed value of $b_i\,(i = 1,2,...,m)$. Thus this decision vector$(x_1, x_2, ..., x_n)$,$x_j \geq 0\,(j = 1,2,..,n)$ is found here before the actual value of random vector$(b_1, b_2, ..., b_m)$ is known. This is called first stage technique.

Let $y_i = b_i - \sum_{j=1}^{n} a_{ij} x_j\,(i = 1,2,...,m)$. Suppose the discrepancy between $\sum_{j=1}^{n} a_{ij} x_j$ and b_i is $y_i\,(i = 1,2,...,m)$.

$$\text{We minimize } \sum_{i=1}^{m} p_i \mid y_i \mid$$

$$\text{subject to}$$

$$y_i = b_i - \sum_{j=1}^{n} a_{ij} x_j, i = 1, 2, ..., m$$

$$\sum_{j=1}^{n} r_{sj} x_j \geq h_s, s = 1, 2, ..., l$$

$$x_j \geq 0, \qquad j = 1, 2, ..., n$$

where, $p_i\,(i=1,2,...,m)$ are the penalty costs associated with the discrepancy between $\sum_{j=1}^{n} a_{ij}x_j$ and b_i. Let $b_i - \sum_{j=1}^{n} a_{ij}x_j = \eta_i - \rho_i \;\; (i=1,2,...,m)$, where $\eta_i \geq 0, \rho_i \geq 0\,(i=1,2,...,m)$ are called negative and positive deviational variables respectively. So, the problem (3.27)-(3.30) can be stated as

$$
\begin{aligned}
& min : \tilde{z} = \sum_{j=1}^{n} c_j x_j + E(\sum_{i=1}^{m} p_i(\eta_i + \rho_i)) \\
& subject \quad to \\
& \qquad \sum_{j=1}^{n} a_{ij}x_j + \eta_i - \rho_i = b_i, i = 1,2,...,m \\
& \qquad \sum_{j=1}^{n} r_{sj}x_j \geq h_s, s = 1,2,...,l \\
& \qquad x_j \geq 0, \qquad j = 1,2,...,n \\
& \qquad \eta_i, \rho_i \geq 0, \quad i = 1,2,...,m
\end{aligned}
$$

Further, it can be stated as

$$
\min : \tilde{z} = \sum_{j=1}^{n} c_j x_j + E(\sum_{i=1}^{m} p_i(\mid b_i - \sum_{j=1}^{n} a_{ij}x_j \mid)) \tag{3.31}
$$

subject to

$$
\sum_{j=1}^{n} r_{sj}x_j \geq h_s, s = 1,2,...,l \tag{3.32}
$$

$$
x_j \geq 0, \qquad j = 1,2,...,n \tag{3.33}
$$

where it is assumed that the first stage decision variable $x_j\,(j=1,2,...,n)$ are deterministic and the second stage decision variables $y_i\,(i=1,2,...,m)$ are random in the problem, E is used to represent the expected value associated with the random variables $b_i\,(i=1,2,...,m)$.

The deterministic model of the two-stage programming problem when the right hand side parameter of i-th constraint b_i follows either two parameter exponential distribution or three parameter gamma distribution can be obtained as follows.

3.2.1 When b_i follows two parameter exponential distribution

It is assumed that $b_i\,(i=1,2,...,m)$ are independent random variables which follows two parameter exponential distribution with parameters θ_i and σ_i, whose mean, variance and probability density function (pdf) are given by (3.11), (3.12) and (3.13) respectively.

First we compute $E(p_i \mid y_i \mid)) = p_i E(\mid b_i - g_i \mid)\ \ (i=1,2,...,m)$ using above pdf of the $i-th$ random variable b_i, where $g_i = \sum_{j=1}^{n} a_{ij}x_j\ \ (i=1,2,...,m)$ and $g_i \geq 0$.

$$E(\mid b_i - g_i \mid) = \int_{\theta_i}^{\infty} (\mid b_i - g_i \mid) f(b_i) db_i, \quad as\ b_i \geq \theta_i \tag{3.34}$$

Integrating (3.34), we obtain

$$E(\mid b_i - g_i \mid) = \begin{cases} -(\theta_i + \sigma_i) + g_i + 2\sigma_i \exp\left(\frac{\theta_i - g_i}{\sigma_i}\right), & \text{if} \quad g_i \geq \theta_i \\ (\theta_i + \sigma_i) - g_i, & \text{if} \quad g_i < \theta_i \end{cases} \tag{3.35}$$

where $g_i = \sum_{j=1}^{n} a_{ij}x_j,\ (i=1,2,...,m)$.

Using (3.35) in model (3.31)-(3.33), we can have two deterministic two-stage stochastic programming models as:

Model-I: If $g_i \geq \theta_i$, then the corresponding deterministic model is:

$$\min : \tilde{z} = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} p_i \Big[-(\theta_i + \sigma_i) + g_i + 2\sigma_i \exp\Big(\frac{\theta_i - g_i}{\sigma_i}\Big)\Big] \tag{3.36}$$

subject to

(3.37) $$\sum_{j=1}^{n} r_{ij}x_j \geq h_s, s = 1,2,...,l$$

(3.38) $$x_j \geq 0, \qquad j = 1,2,...,n$$

where $g_i = \sum_{j=1}^{n} a_{ij}x_j,\ (i=1,2,...,m)$.

Model-II: If $g_i < \theta_i$, then the corresponding deterministic model is:

$$\min : \tilde{z} = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} p_i \Big[(\theta_i + \sigma_i) - g_i\Big] \tag{3.39}$$

subject to

$$\sum_{j=1}^{n} r_{ij}x_j \geq h_s, s = 1,2,...,l \tag{3.40}$$

$$x_j \geq 0, \qquad j = 1,2,...,n \tag{3.41}$$

where $g_i = \sum_{j=1}^{n} a_{ij}x_j$, $(i = 1,2,...,m)$.

3.2.2 When b_i follows three parameter gamma distribution

Here, we assume that $b_i, i = 1,2,...,m$ are independent random variables which follows three parameter gamma distribution with parametersα_i, β_i and θ_i, whose mean, variance and probability density function (pdf) are given by (3.19), (3.20) and (3.21) respectively.

We compute $E(p_i \mid y_i \mid)) = p_i E(\mid b_i - g_i \mid)$,$(i = 1,2,...,m)$ using above pdf of the $i-th$ random variable b_i where $g_i = \sum_{j=1}^{n} a_{ij}x_j$, $(i = 1,2,...,m)$ and $g_i \geq 0$.

$$E(\mid b_i - g_i \mid) = \int_{\theta_i}^{\infty} (\mid b_i - g_i \mid) f(b_i) db_i, \quad as\ b_i \geq \theta_i \tag{3.42}$$

Integrating (3.42), we obtain

$$E(\mid b_i - g_i \mid) = \begin{cases} -(\alpha_i\beta_i + \theta_i) + g_i + \exp\left(\frac{-(g_i-\theta_i)}{\beta_i}\right) \\ \left[2\alpha_i\beta_i \sum_{k=0}^{\alpha_i} \frac{1}{k!}\left(\frac{(g_i-\theta_i)}{\beta_i}\right)^k + 2\theta_i \sum_{k=0}^{\alpha_i-1} \frac{1}{k!}\left(\frac{(g_i-\theta_i)}{\beta_i}\right)^k - 2g_i \sum_{k=0}^{\alpha_i-1} \frac{1}{k!}\left(\frac{(g_i-\theta_i)}{\beta_i}\right)^k\right], & \text{if} \quad g_i \geq \theta_i \\ (\alpha_i\beta_i + \theta_i) - g_i, & \text{if} \quad g_i < \theta_i \end{cases} \tag{3.43}$$

where $g_i = \sum_{j=1}^{n} a_{ij}x_j$, $i = 1,2,...,m$.

Using (3.43) in model (3.31)-(3.33), we can have two deterministic two-stage stochastic programming models as:

Model-(i): If $g_i \geq \theta_i$, then the corresponding deterministic model is:

$$\begin{aligned} \min : \tilde{z} = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} p_i \Bigg[& -(\alpha_i\beta_i + \theta_i) + g_i + \exp\left(\frac{-(g_i-\theta_i)}{\beta_i}\right)\Big[2\alpha_i\beta_i \sum_{k=0}^{\alpha_i} \frac{1}{k!}\left(\frac{(g_i-\theta_i)}{\beta_i}\right)^k \\ & + 2\theta_i \sum_{k=0}^{\alpha_i-1} \frac{1}{k!}\left(\frac{(g_i-\theta_i)}{\beta_i}\right)^k - 2g_i \sum_{k=0}^{\alpha_i-1} \frac{1}{k!}\left(\frac{(g_i-\theta_i)}{\beta_i}\right)^k\Big]\Bigg] \end{aligned} \tag{3.44}$$

subject to

$$\sum_{j=1}^{n} r_{ij}x_j \geq h_s, s = 1,2,...,l \tag{3.45}$$

$$x_j \geq 0, \qquad j = 1,2,...,n \tag{3.46}$$

where $g_i = \sum_{j=1}^{n} a_{ij}x_j$, $(i = 1,2,...,m)$.

Model-(ii) If $g_i < \theta_i$, then the corresponding deterministic model is:

$$\min : \tilde{z} = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} p_i \Big[(\alpha_i\beta_i + \theta_i) - g_i\Big] \tag{3.47}$$

subject to

$$\sum_{j=1}^{n} r_{ij}x_j \geq h_s, s = 1,2,...,l \tag{3.48}$$

$$x_j \geq 0, \qquad j = 1,2,...,n \tag{3.49}$$

where $g_i = \sum_{j=1}^{n} a_{ij}x_j$, $(i = 1,2,...,m)$.

3.3 Numerical Examples

In the following Section, we present some numerical examples to illustrate the models and methodology described in the previous sections.

3.3.1 Example -1: Chance constrained programming problems

Case-(I):Two parameter exponential distribution

Here, we consider a CCP problem where the right hand side parameter of the constraints follow two parameter exponential distribution.

$$max : z = 5x_1 + 8x_2 + 7x_3 \tag{3.50}$$

subject to

(3.51) $$Pr(2x_1+6x_2+5x_3 \leq b_1) \geq 0.99$$

(3.52) $$Pr(5x_1+11x_2+4x_3 \leq b_2) \geq 0.95$$

(3.53) $$Pr(4x_1+5x_2+x_3 \leq b_3) \geq 0.90$$

(3.54) $$x_j \geq 0, \quad j=1,2,3$$

Here, we assume that $b_i\,(i=1,2,3)$ are random variables following two parameter exponential distribution with following parameters:

$E(b_1)=161, E(b_2)=144, E(b_3)=106$ and

$Var(b_1)=25, Var(b_2)=36, Var(b_3)=64.$

Using (3.11) and (3.12), the parameters are calculated as follows:

$\theta_1=156, \sigma_1=5$, $\theta_2=138, \sigma_2=6$, $\theta_3=98$ and $\sigma_3=8$.

Now, using (3.15)-(3.18), the equivalent deterministic model of (3.50)-(3.54) can be formulated as follows:

$$max: z=5x_1+8x_2+7x_3 \tag{3.55}$$

subject to

(3.56) $$2x_1+6x_2+5x_3 \leq 156.05$$

(3.57) $$5x_1+11x_2+4x_3 \leq 138.308$$

(3.58) $$4x_1+5x_2+x_3 \leq 98.4103$$

(3.59) $$x_j \geq 0, \quad j=1,2,3$$

The above deterministic model is solved by LINGO(11.0) [27] and MAPLE software and the optimal solution are found as follows:

$x_1^*=0.0, x_2^*=10.07921, x_3^*=6.859181$ and the value of the objective function is $Z^*=128.6749$.

Case-(II): Three parameter gamma distribution

Here, we consider a CCP problem where the right hand side parameter of the constraints follow three parameter gamma distribution.

$$max : z = 5x_1 + 8x_2 + 7x_3 \tag{3.60}$$

subject to

$$Pr(3x_1 + 6x_2 + 5x_3 \leq b_1) \geq 0.99 \tag{3.61}$$

$$Pr(5x_1 + 3x_2 + 4x_3 \leq b_2) \geq 0.95 \tag{3.62}$$

$$Pr(4x_1 + 5x_2 + 7x_3 \leq b_3) \geq 0.90 \tag{3.63}$$

$$x_j \geq 0, \quad j = 1,2,3 \tag{3.64}$$

Here, we assume that $b_i\,(i = 1,2,3)$ are random variables following three parameter gamma distribution with following parameters:

$\alpha_1 = 3, \beta_1 = 20, \theta_1 = 15$, $\alpha_2 = 2, \beta_2 = 30, \theta_2 = 9$, $\alpha_3 = 3, \beta_3 = 25$ and $\theta_3 = 17$. The means and variances of random variable $b_i, i = 1,2,3$ are :

$E(b_1) = 75, E(b_2) = 69, E(b_3) = 92$ and

$Var(b_1) = 1200, Var(b_2) = 1800, Var(b_3) = 1875$.

Now, using (3.23)-(3.26), the equivalent deterministic model of (3.65)-(3.64) can be formulated as follows:

$$max : z = 5x_1 + 8x_2 + 7x_3 \tag{3.65}$$

subject to

$$\exp\left(\frac{-(g_1 - 15}{20}\right)\left(\sum_{k=0}^{2} \frac{1}{k!}\Big(\frac{g_1 - 15}{20}\Big)^k\right) \geq 0.99 \tag{3.66}$$

$$\exp\left(\frac{-(g_2 - 9}{30}\right)\left(\sum_{k=0}^{1} \frac{1}{k!}\Big(\frac{g_2 - 9}{30}\Big)^k\right) \geq 0.95 \tag{3.67}$$

$$\exp\left(\frac{-(g_3 - 17}{25}\right)\left(\sum_{k=0}^{1} \frac{1}{k!}\Big(\frac{g_3 - 17}{25}\Big)^k\right) \geq 0.90 \tag{3.68}$$

$$x_j \geq 0, \quad j = 1,2,3 \tag{3.69}$$

where $g_1 = 3x_1 + 6x_2 + 5x_3$, $g_2 = 5x_1 + 3x_2 + 4x_3$ and $g_3 = 4x_1 + 5x_2 + 7x_3$.

The above deterministic model is solved by LINGO(11.0) and MAPLE software and the optimal solutions are found as follows:

$x_1^* = 2.22868, x_2^* = 2.839142, x_3^* = 0.0$ and the value of the objective function is $Z^* = 33.85655$

3.3.2 Example -2: Two-stage stochastic programming problems

Case-(I):Two parameter exponential distribution

In this Section, we have considered two numerical examples to verify the solution procedure of the above two-stage stochastic programming (TSP) models by considering only the right hand side parameter as two parameter exponential distribution and three parameter gamma distribution. Consider the following TSP problem:

$$min : z = 5x_1 + 8x_2 + 7x_3 \tag{3.70}$$

subject to

$$7x_1 + 2x_2 + 5x_3 \leq b_1 \tag{3.71}$$

$$5x_1 + 3x_2 + 4x_3 \leq b_2 \tag{3.72}$$

$$3x_1 + 4x_2 + 2x_3 \leq b_3 \tag{3.73}$$

$$6x_1 + x_2 + 4x_3 \leq 40 \tag{3.74}$$

$$4x_1 + 2x_2 + 3x_3 \geq 35 \tag{3.75}$$

$$x_j \geq 0, \quad j = 1, 2, 3 \tag{3.76}$$

where, it is assumed that b_1, b_2 and b_3 are independent two parameter exponential random variables with given means and variances as:

$E(b_1) = 54, E(b_2) = 48, E(b_3) = 38$ and

$Var(b_1) = 2025, Var(b_2) = 1600, Var(b_3) = 1024.$

Using (3.11) and (3.12), the parameters are calculated as follows:

$\theta_1 = 9, \sigma_1 = 45$, $\theta_2 = 8, \sigma_2 = 40$, $\theta_3 = 6$ and $\sigma_3 = 32$.

Now, using (3.36)-(3.38), the equivalent deterministic model of (3.70)-(3.76) can be formulated as follows:

$$\begin{aligned} \min : \tilde{z} = 5x_1 + 8x_2 + 7x_3 - 54 + g_1 + 90\exp\left(\frac{9-g_1}{45}\right) \\ -48 + g_2 + 80\exp\left(\frac{8-g_2}{40}\right) \\ -38 + g_3 + 66\exp\left(\frac{6-g_3}{32}\right) \end{aligned} \tag{3.77}$$

subject to

$$6x_1 + x_2 + 4x_3 \leq 40 \tag{3.78}$$

$$4x_1 + 2x_2 + 3x_3 \geq 35 \tag{3.79}$$

$$x_j \geq 0, \quad j = 1,2,3 \tag{3.80}$$

where $g_1 = 7x_1 + 2x_2 + 5x_3$, $g_2 = 5x_1 + 3x_2 + 4x_3$ and $g_3 = 4x_1 + 4x_2 + x_3$. Here we have taken $p_1 = p_2 = p_3 = 1$.

The above deterministic model is solved by LINGO(11.0) and MAPLE software and the optimal solution are found as follows:

$x_1^* = 5.625, x_2^* = 6.25, x_3^* = 0.0$ and the value of the objective function is $Z^* = 164.5891$.

Case-(II):Three parameter gamma distribution

In the above two-stage stochastic programming model (3.70)-(3.76), the right hand side parametersb_1, b_2, b_3 are assumed to be random variables following three parameter gamma distributions with the following parameters:

$\alpha_1 = 3, \beta_1 = 15, \theta_1 = 9$ $\alpha_2 = 2, \beta_2 = 20, \theta_2 = 8$, $\alpha_3 = 2, \beta_2 = 16$ and $\theta_3 = 6$.

The means and variances of random variable $b_i, i = 1,2,3$ are :

$E(b_1) = 54, E(b_2) = 48, E(b_3) = 38$ and

$Var(b_1) = 675, Var(b_2) = 800, Var(b_3) = 512$.

Now using (3.44)-(3.46), the equivalent deterministic model of (3.70)-(3.76),can be formulated as follows:

$$\begin{aligned} \min : \tilde{z} = 5x_1 + 8x_2 + 7x_3 - 54 + g_1 + \exp\left(\frac{-(g_{1-9})}{15}\right)\Big[90\sum_{k=0}^{3}\frac{1}{k!}\left(\frac{g_1-9}{15}\right)^k \\ + 18\sum_{k=0}^{2}\frac{1}{k!}\left(\frac{g_1-9}{15}\right)^k - 2g_1\sum_{k=0}^{2}\frac{1}{k!}\left(\frac{g_1-9}{15}\right)^k\Big] \\ - 48 + g_2 + \exp\left(\frac{-(g_2-8)}{20}\right)\Big[80\sum_{k=0}^{2}\frac{1}{k!}\left(\frac{g_2-8}{20}\right)^k \\ + 16\sum_{k=0}^{1}\frac{1}{k!}\left(\frac{g_2-8}{20}\right)^k - 2g_2\sum_{k=0}^{1}\frac{1}{k!}\left(\frac{g_2-8}{20}\right)^k\Big] \\ - 38 + g_3 + \exp\left(\frac{-(g_3-6)}{16}\right)\Big[64\sum_{k=0}^{2}\frac{1}{k!}\left(\frac{g_3-6}{16}\right)^k \\ + 12\sum_{k=0}^{1}\frac{1}{k!}\left(\frac{g_3-6}{16}\right)^k - 2g_3\sum_{k=0}^{1}\frac{1}{k!}\left(\frac{g_3-6}{16}\right)^k\Big] \end{aligned} \tag{3.81}$$

subject to

$$6x_1 + x_2 + 4x_3 \leq 40 \tag{3.82}$$

$$4x_1 + 2x_2 + 3x_3 \geq 35 \tag{3.83}$$

$$x_j \geq 0, \quad j = 1,2,3 \tag{3.84}$$

where $g_1 = 7x_1 + 2x_2 + 5x_3$, $g_2 = 5x_1 + 3x_2 + 4x_3$ and $g_3 = 4x_1 + 4x_2 + x_3$. Here we have taken $p_1 = p_2 = p_3 = 1$.

The above deterministic model is solved by LINGO(11.0) and MAPLE software and the optimal solution are found as follows:

$x_1^* = 5.625, x_2^* = 6.25, x_3^* = 0.0$ and the value of the objective function is $Z^* = 101.0922$

.

4 Conclusions

In this study, we have considered a chance constrained programming problem and a two-stage programming problem, where right hand side parameters $b_i's$ are random variables following two parameter exponential distribution or three parameter gamma distribution. All other parameters of the model are assumed to be deterministic. Solution procedures for both the chance constrained programming problem and two-stage stochastic programming problem have been presented. Four different deterministic models, two each for different random variables have been established and examples are provided to illustrate the methodologies. The problem can be solved by using other distribution functions. One can consider c_j and a_{ij} as random variables in the problem, instead of b_i. Some heuristic techniques such as genetic algorithm (GA), neural network (NN), Tabu Search (TS) can be applied to solve CCP and TSP problems directly without transforming it to a deterministic model. The study can be extended for nonlinear chance constrained problem and in hierarchical decision making framework.

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Optimal Range for a General Interval Optimization Problem

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Abstract

Objective of the paper is to determine the lower and upper bound of the optimal value of a general optimization problem in which parameters of the objective function and constraints are closed intervals. Duality theory is used to construct the upper and lower level problems. The methodology is verified with an interval quadratic programming problem.

2010 AMS classification:91A06, 91A10, 91A35, 91A80.

Keywords: Interval optimization problem; Interval valued function; Quadratic programming; Efficient solution.

1 Introduction

Most of the optimization models in real life situations have ambiguities in the input data set, which may be considered as linguistic variables or random variables. Recently many researchers have accepted these uncertain parameters in the form of intervals, whose end points are estimated from the historical data. These type models are known as interval optimization problems. Objective function and constraints of an interval optimization problem are interval valued functions. Several methods have

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been developed to handle these interval parameters while solving the optimization models with intervals. Ishibuchi and Tanaka [7] studied linear interval optimization problem with a deterministic feasible set and obtained the Pareto optimal solutions by solving a bi-objective programming problem in terms of the mean and width of the objective of the problem, which is further extended by Inuiguchi and Sakawa [6] using minimum regrate method. Chanas and Kuchta [3] generalized the Ishibuchi and Tanaka [7]'s concepts using parametric representation of the intervals.
Sengupta et al. [13] determined the optimal solution by transforming the original model to an equivalent deterministic form using acceptability index of the intervals. Allahdadi and Nehi [1] obtained the set of all possible optimal solutions of this model.

An interval optimization problem, in which either objective function or at least on constraint is a nonlinear function, is called as nonlinear interval optimization problem. Liu and Wang [11] considered an interval quadratic programming problem in which quadratic part of the objective function is free from uncertainty and obtained the optimal bounds of the problem. Li and Tian [10] generalized the Liu and Wang [11]'s method for a general interval quadratic programming problem. Jiang et al. [9] solved a general nonlinear interval optimization model by solving a corresponding bi-objective programming problem in terms of the mean and width of the intervals. Hladik [5] obtained the bounds of the optimal value of the problem. Bhurjee and Panda [2] developed a methodology to discuss the existence of an efficient solution. Jana and Panda [8] discussed a methodology to obtain one preferable efficient solution using some preference function.

Since, an interval optimization problem is a family of infinitely many deterministic optimization problems, so one solution may not optimize all these deterministic problems. Hence, in this paper we develop a methodology to determine the optimal bounds of a general interval optimization problem in which the objective function and constraints may be linear or nonlinear function with all data represented as closed intervals. Lower and upper bounds of the optimal value of the problem are computed by solving two deterministic optimization problems. The developed methodology is applied in a general interval quadratic programming problem to find its optimal range.

2 Notations and Preliminaries on Interval Analysis

Throughout the paper the following notations are used:

Bold capital letters denote closed intervals, and small letters denote real numbers;

$\mathbb{I}$= The set of all closed intervals in $\mathbb{R}$; $\mathbb{I}^k$= The product space $\underbrace{\mathbb{I}\times\mathbb{I}\times\ldots\times\mathbb{I}}_{k \text{ times}}$;

$\mathbf{C}_v^k$= k dimensional column vector whose elements are intervals. That is,

$\mathbf{C}_v^k \in \mathbb{I}^k$, $\mathbf{C}_v^k = (\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_k)^T$, $\mathbf{C}_j = [c_j^L, c_j^U],\ j \in \Lambda_k,\ \Lambda_k = 1, 2, \ldots, k.$

Let $* \in \{+, -, \cdot, /\}$ be a binary operation on the set of real numbers. The binary operation $\circledast$ between two intervals $\mathbf{A} = [a^L, a^U]$ and $\mathbf{B} = [b^L, b^U]$ in $\mathbb{I}$, denoted by $\mathbf{A} \circledast \mathbf{B}$ is the set $\{a * b : a \in \mathbf{A}, b \in \mathbf{B}\}$. In the case of division, $\mathbf{A}/\mathbf{B}$, it is assumed that $0 \notin \mathbf{B}$. An interval can be expressed in terms of a parameter in several ways. Any point in $\mathbf{A}$ may be expressed as a_t, where $a_t = a^L + t(a^U - a^L)$, $t \in [0, 1]$. Throughout this paper we consider a specific parametric representation of an interval as

$$\mathbf{A} = [a^L, a^U] = \{a_t \mid\ t \in [0, 1]\}.$$

Using the parametric concepts of interval, the algebraic operations for $\mathbb{I}$ can be restated as follows.

$$\begin{aligned}\mathbf{A} \circledast \mathbf{B} &= \{a_{t_1} * b_{t_2} \mid t_1, t_2 \in [0, 1]\} \\ &\equiv \left[\min_{t_1, t_2}(a_{t_1} * b_{t_2}),\ \max_{t_1, t_2}(a_{t_1} * b_{t_2})\right]\end{aligned}$$

Hence we have

$$\begin{aligned}\mathbf{A} \oplus \mathbf{B} &= \{a_{t_1} + b_{t_2} \mid t_1, t_2 \in [0, 1]\}, \\ \mathbf{A} \ominus \mathbf{B} &= \{a_{t_1} - b_{t_2} \mid t_1, t_2 \in [0, 1]\}, \\ \mathbf{A} \odot \mathbf{B} &= \{a_{t_1} b_{t_2} \mid t_1, t_2 \in [0, 1]\}, \\ k\mathbf{A} &= \{k a_t \mid t \in [0, 1]\}, \\ \mathbf{A} \oslash \mathbf{B} &= \left\{a_{t_1} / b_{t_2} \mid t_1, t_2 \in [0, 1], a(t_2) \neq 0\right\}.\end{aligned}$$

$\mathbf{C}_v^k \in \mathbb{I}^k$ is the set

$$\left\{c_t \mid c_t = (c_{t_1}, c_{t_2}, \ldots, c_{t_k})^T,\ c_{t_j} = c_j^L + t_j(c_j^U - c_j^L),\ 0 \le t_j \le 1, j \in \Lambda_k\right\}. \tag{2.1}$$

2.1 Interval valued function

Many authors have been defined interval valued function in several ways in the literature (see [2, 4, 12, 14]). Moore [12] and Hansen [4] defined an interval valued function as an extension of a real valued function. However, Wu [14] considered the interval valued function, $\mathbf{F} : \mathbb{R}^n \to \mathbb{I}$ as,

$$\mathbf{F}(x) = [F^L(x), F^U(x)], \text{ where } F^L, F^U : \mathbb{R}^n \to \mathbb{R},\ F^L(x) \le F^U(x)\ \forall x \in \mathbb{R}^n.$$

Recently, Bhurjee and Panda [2] defined an interval valued function in a different way. For given $\mathbf{C}_v^k \in \mathbb{I}^k$, we define an interval valued function $\mathbf{F}_{\mathbf{C}_v^k} : \mathbb{R}^n \to \mathbb{I}$ by

$$\mathbf{F}_{\mathbf{C}_v^k}(x) = \left\{ f_{c_t}(x) \;\middle|\; f_{c_t} : \mathbb{R}^n \to \mathbb{R}, c_t \in \mathbf{C}_v^k \right\}.$$

Since for every fixed x, $f_{c_t}(x)$ is continuous in t so $\min_{t\in[0,1]^k} f_{c_t}(x)$ and $\max_{t\in[0,1]^k} f_{c_t}(x)$, exist. In that case

$$\mathbf{F}_{\mathbf{C}_v^k}(x) = \left[\min_{t\in[0,1]^k} f_{c_t}(x),\ \max_{t\in[0,1]^k} f_{c_t}(x) \right].$$

Now, we redefine the interval valued function in another way as follows.

Definition 2.1. For given $\mathbf{C}_v^k \in \mathbb{I}^k$, an interval valued function $\mathbf{F}(\mathbf{C}_v^k, x) : \mathbb{R}^n \to \mathbb{I}$ by

$$\mathbf{F}(\mathbf{C}_v^k, x) = \left\{ f(c_t, x) \;\middle|\; f(c_t, x) : [0,1]^k \times \mathbb{R}^n \to \mathbb{R}, c_t \in \mathbf{C}_v^k \right\}.$$

For every c_t, $f(c_t, x)$ is a function of x and it is continuous in t for every x. So $\min_{t\in[0,1]^k} f(c_t, x)$ and $\max_{t\in[0,1]^k} f(c_t, x)$ exist. In that case

$$\mathbf{F}(\mathbf{C}_v^k, x) = \left[\min_{t\in[0,1]^k} f(c_t, x),\ \max_{t\in[0,1]^k} f(c_t, x) \right].$$

If $f(c_t, x)$ is linear in t then

$$\min_{t\in[0,1]^k} f(c_t, x) = f(c_0, x) \text{ and } \max_{t\in[0,1]^k} f(c_t, x) = f(c_1, x)$$

exist in the set of vertices of $\mathbf{C}_v^k$. Then

$$\mathbf{F}(\mathbf{C}_v^k, x) = \left[f(c_0, x), f(c_1, x)\right].$$

3 Determination of Optimal Bounds

We consider a general interval nonlinear optimization problem $(GIOP)$ as follows,

$$(GIOP) \quad \min\ Z = \mathbf{F}(\mathbf{C}_v^k, x)$$

$$\text{subject to } \mathbf{G}_j(\mathbf{D}_v^{m_j}, x) \leq \mathbf{B}_j,\ j \in \Lambda_p,$$

$$x \geq 0, x \in \mathbb{D} \subseteq \mathbb{R}_+^n,$$

where $\mathbf{B}_j \in \mathbb{I}$ and $\mathbf{G}_j(\mathbf{D}_v^{m_j}, x) : \mathbb{D} \to \mathbb{I}$ is the set,

$$\mathbf{G}_j(\mathbf{D}_v^{m_j}, x) = \left\{g_j(d_{t'_j}, x) \mid g_j(d_{t'_j}, x) : [0,1]^{m_j} \times \mathbb{D} \to \mathbb{R}, d_{t'_j} \in \mathbf{D}_v^{m_j}\right\}.$$

The feasible region of $GIOP$ can be expressed as the set,

$$\mathbb{S} = \{x \in \mathbb{D} | \mathbf{G}_j(\mathbf{D}_v^{m_j}, x) \leq \mathbf{B}_j, j \in \Lambda_p, x \geq 0\}.$$

Since

$$\min_{x \in \mathbb{S}} \mathbf{F}(\mathbf{C}_v^k, x) \equiv \min_{x \in \mathbb{S}} \left\{f(c_t, x) |\ c_t \in \mathbf{C}_v^k, t \in [0,1]^k\right\},$$

so $GIOP$ can be treated as a multi-objective optimization problem in t for every $x \in \mathbb{S}$ over a continuous domain in which there are infinitely many objective functions. Also, conflict behavior of objectives is not necessary. Therefore, $GIOP$ can not be treated as a multi-objective optimization problem. Further, it is also seen that one solution may not be optimized all objectives simultaneously, so the solution concept for $GIOP$ is not sufficient to find the optimal solution for the problem. Therefore, we develop a procedure to obtain the optimal value bounds of the general interval nonlinear optimization problem.

The parametric form of $GIOP$ is denoted by $GIOP_t$ and define as follows.

$$(GIOP_t) \qquad \min_x \; Z_t = f(c_t, x)$$

$$\text{subject to } g_j(d_{t'_j}, x) \le b_{t''_j}, \; j \in \Lambda_p,$$

$$t \in [0,1]^k, t'_j \in [0,1]^{m_j}, t''_j \in [0,1],$$

$$x \ge 0, x \in \mathbb{D}.$$

The optimal value bounds of the $GIOP$ is calculated by solving the following two deterministic problems.

The lower bound of the optimal value of $GIOP$ can be calculated by solving the following optimization problem:

$$(GIOP_L) \qquad Z^L = \min_{t,\, t'_j,\, t''_j} \min_x f(c_t, x)$$

subject to

$$g_j(d_{t'_j}, x) \le b_{t''_j}, \; j \in \Lambda_p,$$

$$t \in [0,1]^k, t'_j \in [0,1]^{m_j}, t''_j \in [0,1],$$

$$x \ge 0, x \in \mathbb{D}.$$

The objective value Z^L is lower bound of the optimal value for $GIOP$.

The upper bound of the optimal value of $GIOP$ can be calculated by solving the following optimization problem:

$$(GIOP_U) \qquad Z^U = \max_{t,\, t'_j,\, t''_j} \min_x f(c_t, x)$$

subject to

$$g_j(d_{t'_j}, x) \le b_{t''_j}, \; j \in \Lambda_p,$$

$$t \in [0,1]^k, t'_j \in [0,1]^{m_j}, t''_j \in [0,1],$$

$$x \ge 0, x \in \mathbb{D}.$$

The objective value Z^U is upper bound of the optimal value for $GIOP$.

3.1 Lower bound

Since the inner model and outer model of $GIOP_L$ have the same minimization type, so both models can be combined into a deterministic optimization problem. Hence the lower bound of $GIOP$ can be calculated by the following problem:

$$
\begin{aligned}
Z^L = & \min_{x,\ t,\ t'_j,\ t''_j} f(c_t, x) \\
& \text{subject to} \\
& g_j(d_{t'_j}, x) \leq b_{t''_j}, \quad j \in \Lambda_p, \\
& t \in [0,1]^k, t'_j \in [0,1]^{m_j}, t''_j \in [0,1], \\
& x \geq 0, x \in \mathbb{D}.
\end{aligned}
$$

This is a nonlinear optimization problem with $x,\ t,\ t'_j,\ t''_j$ as decision variables, which can be easily solved by existing optimization problem solvers.

3.2 Upper bound

Since the inner model and outer model of $GIOP_U$ have the different optimization directions i.e. inner model is minimization type and outer model is maximization type, so both models can not be combined into a deterministic optimization problem. We convert the inner optimization problem into maximization type using duality theory for nonlinear optimization problem. The Lagrangian dual problem of the inner problem is defined as follows.

$$
\max_{\lambda \geq 0,\ \mu \geq 0} \ \inf_{x \in \mathbb{D}} \left\{ f(c_t, x) + \sum_{j=1}^{p} \lambda_j (g_j(d_{t'_j}, x) - b_{t''_j}) - \sum_{i=1}^{n} \mu_i x_i \right\},
$$

where the objective function is the Lagrange dual function. Dual of the inner problem is:

$$\begin{aligned}
&\max_{x,\,\lambda,\,\mu} f(c_t,x)+\sum_{j=1}^{p}\lambda_j(g_j(d_{t'_j},x)-b_{t''_j})-\sum_{i=1}^{n}\mu_i x_i\\
&\text{subject to}\\
&\frac{\partial f(c_t,x)}{\partial x_i}+\sum_{j=1}^{p}\lambda_j\frac{\partial g_j(d_{t'_j},x)}{\partial x_i}=\mu_i,\ i\in\Lambda_n,\\
&\lambda_j(g_j(d_{t'_j},x)-b_{t''_j})=0,\ \lambda_j\ge 0,\ j\in\Lambda_p,\\
&\mu_i x_i=0,\ \mu_i\ge 0,\ x_i\ge 0,\ i\in\Lambda_n,\\
&t\in[0,1]^k, t'_j\in[0,1]^{m_j}, t''_j\in[0,1].
\end{aligned}$$

$GIOP_U$ can be reformulated as

$$\begin{aligned}
Z^U=&\max_{t,\,t_j,\,t'_j}\ \max_{x,\,\lambda,\,\mu} f(c_t,x)+\sum_{j=1}^{p}\lambda_j(g_j(d_{t'_j},x)-b_{t''_j})-\sum_{i=1}^{n}\mu_i x_i\\
&\text{subject to}\\
&\frac{\partial f(c_t,x)}{\partial x_i}+\sum_{j=1}^{p}\lambda_j\frac{\partial g_j(d_{t'_j},x)}{\partial x_i}=\mu_i,\ i\in\Lambda_n,\\
&\lambda_j(g_j(d_{t'_j},x)-b_{t''_j})=0,\ \lambda_j\ge 0,\ j\in\Lambda_p,\\
&\mu_i x_i=0,\ \mu_i\ge 0,\ x_i\ge 0,\ i\in\Lambda_n,\\
&t\in[0,1]^k, t'_j\in[0,1]^{m_j}, t''_j\in[0,1].
\end{aligned}$$

At this stage, both the inner and outer problems are maximization type. Therefore, the upper bound of the optimal value of $GIOP$ can be calculated as

$$\begin{aligned}
Z^U=&\max_{t,\,t_j,\,t'_j,\,x,\,\lambda,\,\mu} f(c_t,x)+\sum_{j=1}^{p}\lambda_j(g_j(d_{t'_j},x)-b_{t''_j})-\sum_{i=1}^{n}\mu_i x_i\\
&\text{subject to}\\
&\frac{\partial f(c_t,x)}{\partial x_i}+\sum_{j=1}^{p}\lambda_j\frac{\partial g_j(d_{t'_j},x)}{\partial x_i}=\mu_i,\ i\in\Lambda_n,\\
&\lambda_j(g_j(d_{t'_j},x)-b_{t''_j})=0,\ \lambda_j\ge 0,\ j\in\Lambda_p,\\
&\mu_i x_i=0,\ \mu_i\ge 0,\ x_i\ge 0,\ i\in\Lambda_n,\\
&t\in[0,1]^k, t'_j\in[0,1]^{m_j}, t''_j\in[0,1].
\end{aligned}$$

This is a nonlinear optimization problem with $x, \lambda, \mu,\ t,\ t_j,\ t'_j$ as decision variables, which can be easily solved by existing optimization problem solver as LINGO, MATHEMATICA, MATLAB.

3.3 Numerical example

Consider the following interval quadratic programming problem,

$$
\begin{aligned}
(GIQPP)\ \ \min_x\ & [-10,-6]x_1 + [2,3]x_2 + [4,10]x_1^2 + [-1,1]x_1x_2 + [10,20]x_2^2 \\
& \text{subject to} \\
& [1,2]x_1 + 3x_2 \leq [1,10], \\
& [-2,8]x_1 + [4,6]x_2 \leq [4,6], \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

The parametric form of the given problem is define as follows.

$$
\begin{aligned}
(GIQPP_t)\ \ \min_x\ & f(c_t, x) \\
& \text{subject to} \\
& (1+t'_1)x_1 + 3x_2 \leq (1+9t''_1), \\
& (-2+10t'_2)x_1 + (4+2t'_3)x_2 \leq (4+2t''_2), \\
& x = (x_1, x_2)^T \geq 0, t = (t_1, t_2, t_3, t_4, t_5)^T \in [0,1]^5, \\
& t' = (t'_1, t'_2, t'_3)^T \in [0,1]^3, t'' = (t''_1, t''_2)^T \in [0,1]^2,
\end{aligned}
$$

where $f(c_t, x) = (-10+4t_1)x_1 + (2+t_2)x_2 + (4+6t_3)x_1^2 + (-1+2t_4)x_1x_2 + (10+10t_5)x_2^2$.

The lower bound of optimal value of $GIQPP$ is calculated by the solution of the following problem :

$$Z^L = \min_{x,\ t,\ t',\ t''} f(c_t, x)$$
subject to
$$(1+t_1')x_1 + 3x_2 \le (1+9t_1''),$$
$$(-2+10t_2')x_1 + (4+2t_3')x_2 \le (4+2t_2''),$$
$$x = (x_1, x_2)^T \ge 0,\ t = (t_1, t_2, t_3, t_4, t_5)^T \in [0,1]^5,$$
$$t' = (t_1', t_2', t_3')^T \in [0,1]^3,\ t'' = (t_1'', t_2'')^T \in [0,1]^2.$$

This is a nonlinear optimization problem which can be easily solve by existing optimization solver. Using LINGO software, the optimal solution of the above problem is

$$x_1^* = 1.25, x_2^* = 0;\ t = (0,0,0,0,0)^T;\ t' = (0,0,0)^T;\ t'' = (1,1)^T,$$

and the lower bound of the optimal value is $Z^L = -6.25$.

The upper bound of optimal value of $GIQPP$ is calculated by the solution of the following problem :

$$Z^U = \min_{x,\ \lambda,\ \mu,\ t,\ t',\ t''} f(c_t, x) + \lambda_1\Big((1+t_1')x_1 + 3x_2 - (1+9t_1'')\Big)$$
$$+\lambda_2\Big((-2+10t_2')x_1 + (4+2t_3')x_2 - (4+2t_2'')\Big) - \mu_1 x_1 - \mu_2 x_2$$
subject to
$$(-10+4t_1) + (8+12t_3)x_1 + (-1+2t_4)x_2 + \lambda_1(1+t_1') + \lambda_2(-2+10t_2') = \mu_1,$$
$$(2+t_2) + (-1+2t_4)x_1 + (20+20t_5)x_2 + 3\lambda_1 + \lambda_2(4+2t_3') = \mu_2,$$
$$\lambda_1\Big((1+t_1')x_1 + 3x_2 - (1+9t_1'')\Big) = 0,$$
$$\lambda_2\Big((-2+10t_2')x_1 + (4+2t_3')x_2 - (4+2t_2'')\Big) = 0,$$
$$\mu_1 x_1 = 0,\ \mu_2 x_2 = 0, x = (x_1, x_2)^T \ge 0, \mu = (\mu_1, \mu_2)^T \ge 0,$$
$$t = (t_1, t_2, t_3, t_4, t_5)^T \in [0,1]^5, t' = (t_1', t_2', t_3')^T \in [0,1]^3,$$
$$t'' = (t_1'', t_2'')^T \in [0,1]^2, \lambda = (\lambda_1, \lambda_2)^T \ge 0.$$

This is also a nonlinear optimization problem. Using LINGO software, the optimal solution of the above problem is

$$x_1^* = 0.3,\ x_2^* = 0;\ t^* = (1, 0.18179, 1, 0.13075, 0)^T;$$

$$t' = (1, 0.28585, 0.1817)^T; t'' = (0, 0.2917)^T; \lambda = (0, 0)^L; \mu = (0, 1.96)^T,$$

and the upper bound of the optimal value is $Z^U = -0.90$. Hence the bounds in which optimal value of the problem lies, is $[-6.25, -0.90]$. □

4 Conclusion

In this paper a methodology is developed to determined the optimal bounds of a general interval optimization problem. This concept can be further developed to multi-objective case, which is the future scope of the present work.

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