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Discontinuous Fractal Interpolation

M.A. Navascués*

Abstract

The fractal interpolation functions provide curves whose graph has generally a non-integer dimension. They own other characteristics as the interpolation of a set of data and the continuity. In this paper, the latter conditions are omitted, defining discontinuous fractal functions passing close to (but not necessarily through) the given data.

In a second part of the article we define affine fractal functions not linked to two-dimensional data. To do this we use the methodology of iterated functions systems. They are composed of a finite set of contractive affinities whose attractor is related to the graph of a bounded function. In this way the paper introduces a very large class of affine fractal functions which are generally discontinuous (though they contain the classical continuous case as a particular case) and whose relevance is not based only on the approximation.

Keywords: Fractals, Discontinuous functions, Interpolation, Approximation

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1 Introduction

The most popular methods of data interpolation are polynomial, linear and spline. These procedures cannot describe the fine microscopic and irregular structure of some experimental, natural and social variables. In this regard (and in many others), the creation of the Fractal Theory by B. Mandelbrot has been truly revolutionary.

Fractal Interpolation Functions (denoted in the text by FIFs), defined by Iterated Function Systems ([8]), provide new methods to model measurable natural phenomena, and new ways of data processing and visualization. This kind of systems generates a class of functions, whose graphs agree with the attractor associated to the system. The difference with the standard mappings is the creation of very complicated geometries with a few elements. The most well-known and used is definitely the affine case, called in some texts linear fractal interpolation (see for instance [2]). Classical references on this matter are [2], [7], [5], [17], etc. that analyze the properties of differentiability and smoothness of the affine FIFs. The article [3] proposes a generalization of the model to higher dimensions. In the reference [6], the authors study the range of values of some particular cases, in terms of the elements of the system, and many papers deal with the stability of the coefficients. A more general scenario is developed in [4].

An important open question is the resolution of the "inverse problem", that is to say, how to compute the coefficients of the system in order to mimic a set of data. A suitable reference is that of Mazel & Hayes ([10]), that use an analytical approach to compute contraction factors (special coefficients of the system). Zhou et al. ([18]) propose a method of signal forecast using self-affine linear mappings where the contraction factors follow an auto-regressive process. The procedure is applied to simulate Weierstrass-Mandelbrot maps and real radar sea clutter data (see also [9] for this topic). The number of experiments related to the affine fractal model is very large and difficult to summarize. For instance, the applications to data compression are numerous (one example is given in [1]). Our team has used this kind of procedures for electroencephalographic processing ([13]).

The fractal functions may bridge the gap between deterministic and random variables. This kind of maps provide a nice framework for natural and social phenomena far from smoothness and periodicity (see for instance [12], [11]). For many stochastic processes, a model with continuity hypothesis may not be acceptable. In terms of Mandelbrot ([12]): "If the requirement of continuity is abandoned, many other interesting self-similar processes suggest themselves. One may for example replace $B(t)$ (Brownian motion) by a non-Gaussian process whose increments are stable in the sense of Paul Lévy". Thus discontinuity adds "degrees of freedom" to the modelling of experimental variables.

The graph of the affine fractal interpolation functions has a self-similar structure that can be quantified by means of a fractal dimension. Besides this feature, these maps possess two characteristics:

- Interpolation of a set of data and
- Continuity.

In this paper, we omit both conditions, defining discontinuous functions by means of affinities of the plane, passing close to (but not necessarily through) the given data.

The curve obtained approximates the set as far as desired, and this fact suggests the definition of affine functions not linked to two-dimensional data. The paper introduces a very large class of affine fractal functions which are generally discontinuous (though they contain the classical continuous case as a particular case) and whose importance is not based only on approximation.

The nonlinear case has been treated in the reference [16].

2 Fractal Approximation Functions

The affinities in the plane are classical in fractal theory because they helped to define the first fractal sets. For instance, the Koch's curve $K \subset \mathbb{R}^2$ can be decomposed as $K = \cup_{n=1}^4 K_n$, where every K_n is similar to the total curve K. Accordingly there exist contractivities w_n $(n = 1, \ldots, 4)$ such that $w_n(K) = K_n$ and

$$K = \cup_{n=1}^4 w_n(K).$$

The transformations w_n are affinities defined as

$$w_1(t, x) = (\frac{t}{3}, \frac{x}{3}),$$

$$w_2(t, x) = (\frac{t\cos(60) - x\sin(60) + 1}{3}, \frac{t\sin(60) + x\cos(60)}{3}),$$

$$w_3(t, x) = (\frac{t\cos(60) + x\sin(60)}{3} + \frac{1}{2}, \frac{-t\sin(60) + x\cos(60)}{3} + \frac{\sqrt{3}}{6}),$$

$$w_4(t, x) = (\frac{t+2}{3}, \frac{x}{3}).$$

Something similar happens in the Sierpinski's triangle, etc.

In the systems studied by Barnsley, and others. ([2]) the first component of the affinities depends only on t:

$$w_n(t, x) = (L_n(t), F_n(t, x)).$$

These maps are associated to a real compact interval $I = [a, b]$, a partition $\Delta : a = t_0 < t_1 < \ldots < t_N = b$ and a set of data $D = \{(t_n, x_n)\}_{n=0}^N$. The functions L_n transform I in $I_n = [t_{n-1}, t_n]$. Any initial point (in fact any set) is taken to the subinterval I_n for the application of w_n (regarding the first coordinate). The iterated images by a sequence of $w_m's$ tend to attracting points which depend only on the path (choice of the successive affinities), and take part of a curve of the plane which can be univocally represented as $x = f(t)$ for any $t \in I$. The function f is called a Fractal Interpolation Function (FIF). The join-up conditions:

$$L_n(t_0) = t_{n-1}, \quad L_n(t_N) = t_n, \tag{2.1}$$

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \tag{2.2}$$

make f continuous.

If $F_n(t,x) = \alpha_n x + q_n(t)$ where $q_n(t) = c_n^0 t + d_n^0$, the function f is called an Affine Fractal Interpolation Function (AFIF). The parameter $\alpha = (\alpha_n)_{n=1}^N$ is the scale vector of the system, and represents the vertical contractivity factors of the transformations. The conditions (2.2) determine the values of the coefficients c_n^0 and d_n^0 in terms of the data and the scale factors:

$$c_n^0 = \frac{x_n - x_{n-1}}{t_N - t_0} - \alpha_n \frac{x_N - x_0}{t_N - t_0}, \tag{2.3}$$
$$d_n^0 = \frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0} - \alpha_n \frac{t_N x_0 - t_0 x_N}{t_N - t_0}. \tag{2.4}$$

Figure 1 represents an Affine Fractal Interpolation Function in the interval $I = [0,1]$ with scale vector $\alpha = (0.2, -0.2, 0.3, 0.1, -0.1)$. The data are $\{(0,0.5), (0.2,3), (0.4,1), (0.6,1.4), (0.8,2), (1,0)\}$.

In this article the conditions (2.2) on F_n are weakened, considering a more general iterated function system.

As said before, let us consider a compact interval $I = [a,b]$, a partition $\Delta : a = t_0 < t_1 < ... < t_N = b$ and a set of real data $D = \{(t_n, x_n)\}_{n=0}^N$.

Let $\epsilon > 0$ be fixed. The purpose of this Section is to construct a fractal function $\widetilde{f}$ (not necessarily continuous) such that

$$|\widetilde{f}(t_n) - x_n| \leq \epsilon, \qquad \forall n = 0, 1, \dots N. \tag{2.5}$$

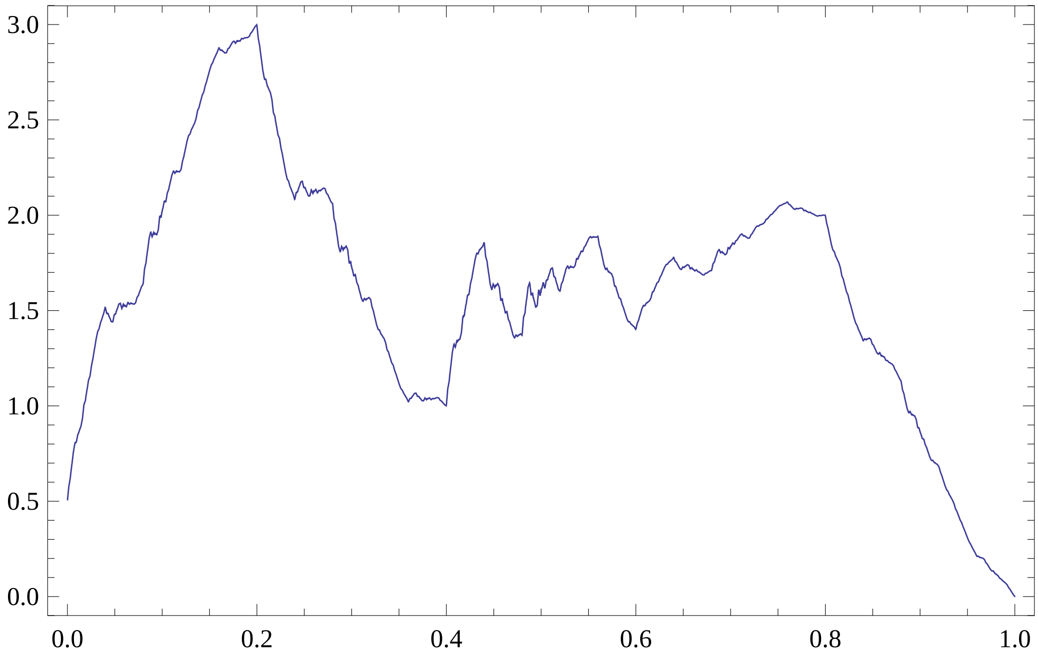

Figure 1: Graph of an affine fractal interpolation function of a set of data in the interval I=[0, 1].

For it let us consider the metric space:

$$\mathcal{G}_{D,\epsilon} = \{f \in \mathcal{B}(I) : |f(t_i) - x_i| \leq \epsilon \quad \text{for} \quad i = 0, N\}, \tag{2.6}$$

where $\mathcal{B}(I)$ is the Banach algebra of bounded functions on I, with the supremum (or uniform) norm:

$$\|f\|_\infty = \sup\{|f(t)| : t \in I\}.$$

Proposition 2.1. *$\mathcal{G}_{D,\epsilon}$ is a complete metric space with respect to the uniform norm.*

Proof. The subspace $\mathcal{G}_{D,\epsilon}$ is complete because is closed. To prove this, let us consider a sequence $g_m \in \mathcal{G}_{D,\epsilon}$ converging to $g \in \mathcal{B}(I)$. In this case, $g_m(t_i)$ converges to $g(t_i)$, for $i = 0, N$. Since $g_m(t_i)$ belongs to the closed ball $\overline{B}(x_i, \epsilon)$ in the real line for all m, the limit $g(t_i)$ belongs to it too, and consequently $g \in \mathcal{G}_{D,\epsilon}$. □

Let $L_n,\ n \in \{1, 2, ..., N\}$, be contractive affinities such that:

$$L_n(t_0) = t_{n-1},\ L_n(t_N) = t_n,$$

and $F_n(t, x) = \alpha(t)x + q_n(t)$ where $\alpha \in \mathcal{B}(I)$, $\|\alpha\|_\infty < 1$ and $q_n(t) = c_n t + d_n$ satisfying

$$|F_n(t_N, x_N) - x_n| = |\alpha(t_N)x_N + q_n(t_N) - x_n| \le \frac{\epsilon}{2}, \tag{2.7}$$

for $n = 1, 2, \ldots, N$,

$$|F_1(t_0, x_0) - x_0| = |\alpha(t_0)x_0 + q_1(t_0) - x_0| \le \frac{\epsilon}{2}, \tag{2.8}$$

and

$$|\alpha(t_i)| \le 1/2, \tag{2.9}$$

for $i = 0, N$.

The inequalities (2.7) and (2.8) generalize the join-up conditions (2.2) of the affine continuous case, where $\epsilon = 0$. The scale constants α_n of the classical case are replaced here by a function $\alpha(t)$.

Let us consider now the operator $T : \mathcal{G}_{D,\epsilon} \to \mathcal{G}_{D,\epsilon}$ defined as

$$Tg(t) = F_n(L_n^{-1}(t), g \circ L_n^{-1}(t)), \tag{2.10}$$

that is to say

$$Tg(t) = \alpha(L_n^{-1}(t))g \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t), \tag{2.11}$$

for $t \in I_n$. The intervals I_n are defined as $I_n = (t_{n-1}, t_n]$ for $n = 2, \ldots, N$, and $I_1 = [t_0, t_1]$.

Let us see that T is well defined. If $g \in \mathcal{G}_{D,\epsilon}$, since $t_0 \in I_1$,

$$Tg(t_0) = \alpha(L_1^{-1}(t_0))g \circ L_1^{-1}(t_0) + q_1 \circ L_1^{-1}(t_0),$$

and thus

$$\begin{aligned}|Tg(t_0) - x_0| &= |\alpha(t_0)g(t_0) + q_1(t_0) - x_0| \\ &\leq |\alpha(t_0)||g(t_0) - x_0| + |\alpha(t_0)x_0 + q_1(t_0) - x_0| \\ &\leq \epsilon,\end{aligned}$$

due to (2.9), the definition of $\mathcal{G}_{D,\epsilon}$ and the condition (2.8). Similar arguments provide the inequality

$$|Tg(t_N) - x_N| \leq \epsilon,$$

and consequently $Tg \in \mathcal{G}_{D,\epsilon}$.

Let us see that T is a contraction in $\mathcal{G}_{D,\epsilon}$. If $t \in I_n$,

$$|Tg(t) - Tg'(t)| = |\alpha(L_n^{-1}(t))||(g - g') \circ L_n^{-1}(t)|,$$

then

$$|Tg(t) - Tg'(t)| \leq \|\alpha\|_\infty \|g - g'\|_\infty,$$

and thus

$$\|Tg - Tg'\|_\infty \leq \|\alpha\|_\infty \|g - g'\|_\infty.$$

Since T is contractive on the complete space $\mathcal{G}_{D,\epsilon}$, it admits a unique fixed point, denoted by $\widetilde{f}_{D,\epsilon}$. The equation satisfied by this map is

$$\widetilde{f}_{D,\epsilon}(t) = \alpha(L_n^{-1}(t))\widetilde{f}_{D,\epsilon} \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t), \tag{2.12}$$

for $t \in I_n$.

Definition 2.2. *The map $\widetilde{f}_{D,\epsilon}$ is a fractal approximation function (FAF) of D with respect to the mapping α, the tolerance ϵ and the operator T defined through the maps L_n, F_n.*

For simplicity the function $\widetilde{f}_{D,\epsilon}$ will be denoted by $\widetilde{f}$. This map need not be continuous but only bounded on the interval I. If we consider $\epsilon = 0$, α continuous (or piecewise constant on the subintervals I_n) and the space $\mathcal{G}_{D,0}$ contained in $\mathcal{C}[a,b]$, the fractal approximation function is continuous.

The fractal approximation functions generalize the fractal interpolation functions addressed, for instance, in the reference [14], considering a much more general approach (weaker join-up conditions (2.7), (2.8) and a wider space of functions).

Proposition 2.3. *The FAF $\widetilde{f}$ satisfies the following approximation inequalities on the nodes of the partition:*

$$|\widetilde{f}(t_n) - x_n| \leq \epsilon,$$

for all $n = 0, 1, 2, \ldots, N$.

Proof. The fixed point equation (2.12) is equivalent to:

$$\widetilde{f}(L_n(t)) = F_n(t, \widetilde{f}(t)), \tag{2.13}$$

for any $t \neq t_0$ ($L_n(t_0) = t_{n-1} \in I_{n-1}$). Taking $t = t_N$,

$$\widetilde{f}(t_n) = \widetilde{f}(L_n(t_N)) = F_n(t_N, \widetilde{f}(t_N)),$$

for $n = 1, 2, \ldots, N$. Therefore,

$$|\widetilde{f}(t_n) - x_n| = |F_n(t_N, \widetilde{f}(t_N)) - x_n| = |\alpha(t_N)\widetilde{f}(t_N) + q_n(t_N) - x_n|,$$

Then

$$|\widetilde{f}(t_n) - x_n| \leq |\alpha(t_N)||\widetilde{f}(t_N) - x_N| + |\alpha(t_N)x_N + q_n(t_N) - x_n|$$

$$|\widetilde{f}(t_n) - x_n| \leq |\widetilde{f}(t_N) - x_N|/2 + |F_n(t_N, x_N) - x_n|,$$

and

$$|\widetilde{f}(t_n) - x_n| \leq \epsilon$$

for $n = 1, 2, \ldots N$, due to the definition of $\mathcal{G}_{D,\epsilon}$ (2.6), and the conditions (2.7) and (2.9). Since $\widetilde{f} \in \mathcal{G}_{D,\epsilon}$,

$$|\widetilde{f}(t_0) - x_0| \leq \epsilon.$$

□

If we consider $\epsilon = 0$ in the set (2.6) and in conditions (2.7) and (2.8), we have the interpolatory (not necessarily continuous) case, and the hypothesis (2.9) is not necessary.

2.1 Discrete Case in the Scale Function

Let us consider now the standard case where the map α is replaced by a vector composed of N constant scale factors, i.e., $\alpha(t) = \alpha_n \in \mathbb{R}$ in the map F_n, and let us define

$$|\alpha|_\infty = \max\{|\alpha_n| : n = 1, 2, \ldots, N\}.$$

Definition of the coefficients of q_n:

If we consider the maps $q_n^0(t) = c_n^0 t + d_n^0$ with the join-up conditions of continuous fractal interpolation:

$$F_n^0(t_N, x_N) = x_n,$$

$$F_n^0(t_0, x_0) = x_{n-1},$$

the coefficients c_n^0, d_n^0 are given by the expressions (2.3) and (2.4) (see for instance [14]).

Let us assume the same scale factors α_n in both systems (continuous (F_n^0) and discontinuous (F_n)) and let us see how to choose c_n, d_n to satisfy (2.7) and (2.8) :

$$|q_n(t_N) - q_n^0(t_N)| = |(F_n(t_N, x_N) - x_n) - (F_n^0(t_N, x_N) - x_n)| \leq \frac{\epsilon}{2},$$

due to (2.7). It suffices to choose

$$|q_n(t_N) - q_n^0(t_N)| = |(c_n - c_n^0)t_N + (d_n - d_n^0)| \leq \frac{\epsilon}{2}.$$

If $t_N = 1$, one can take $|c_n - c_n^0| \leq \epsilon/4$, $|d_n - d_n^0| \leq \epsilon/4$.

Figure 2 represents a discontinuous fractal approximation of the data

$$\{(0, 3.3), (0.1, 2.9), (0.2, 3.5), (0.3, 3.6), (0.4, 2.3), (0.5, 3.8),$$

$$(0.6, 3.9), (0.7, 3.7), (0.8, 3.4), (0.9, 3.3), (1, 3)\}$$

with scale vector

$$\alpha = (-0.3, 0.3, 0.2, 0.3, -0.2, 0.3, -0.3, 0.2, 0.2, -0.3),$$

along with the polygonal joining the data. The coefficients c_n and d_n were defined as $c_n = c_n^0 + \psi_n$, $d_n = d_n^0 + \eta_n$, where ψ_n and η_n were randomly chosen satisfying $|\psi_n| \leq 0.5$ and $|\eta_n| \leq 0.5$.

We deduce now upper limits of the distance between $\widetilde{f}$ and the polygonal g_0 whose vertices are the data (t_n, x_n), in order to bound the range of $\widetilde{f}$.

Proposition 2.4. *For all* $t \in I_n$*,*

$$Tg(t) = g_0(t) + \alpha_n(g - r) \circ L_n^{-1}(t) + (c_n - c_n^0)L_n^{-1}(t) + (d_n - d_n^0),$$

where $g_0(t)$ *is the polygonal whose vertices are the data* (t_n, x_n)*,* r *is the line joining* (t_0, x_0) *and* (t_N, x_N) *and* c_n^0*,* d_n^0 *are defined by the expressions (2.3) and (2.4).*

Proof. The following function is the line passing through (t_{n-1}, x_{n-1}) and (t_n, x_n) :

$$\frac{x_n - x_{n-1}}{t_N - t_0} L_n^{-1}(t) + \frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0}.$$

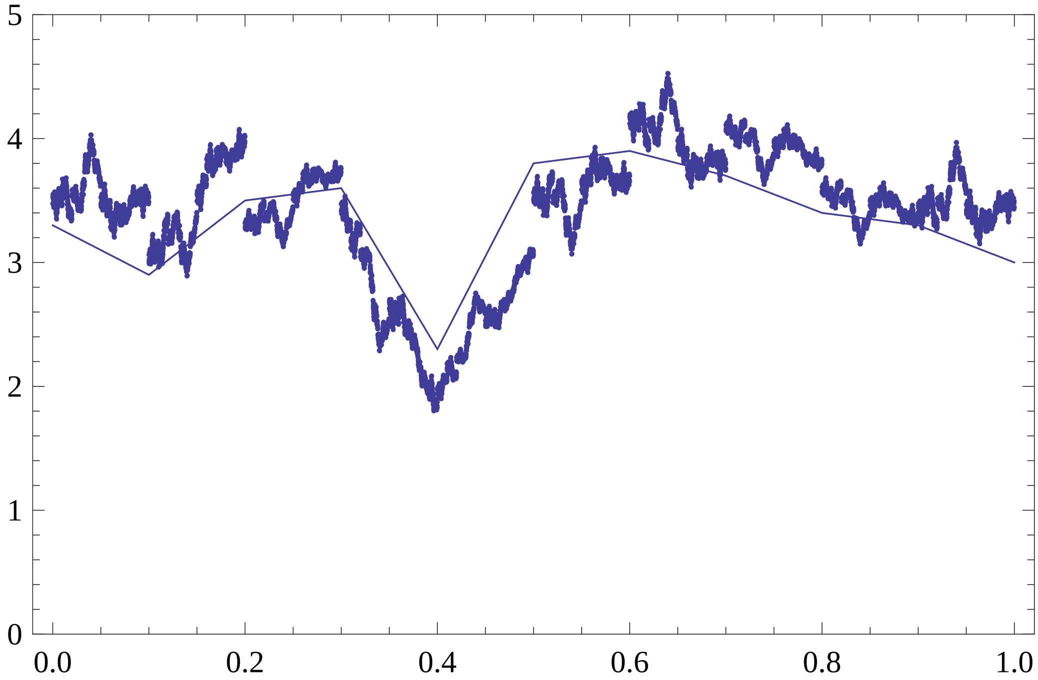

Figure 2: Graph of a fractal approximation function of a set of data, along with the polygonal joining them in the interval I=$[0,1]$

The line through the extreme data has the expression

$$r(t)=\frac{x_N-x_0}{t_N-t_0}t+\frac{t_Nx_0-t_0x_N}{t_N-t_0}.$$

Then, for $t\in I_n$ (2.11),

$$Tg(t)-g_0(t)=\alpha_n g\circ L_n^{-1}(t)+\big(c_n-\frac{x_n-x_{n-1}}{t_N-t_0}\big)L_n^{-1}(t)+\big(d_n-\frac{t_Nx_{n-1}-t_0x_n}{t_N-t_0}\big).$$

The equalities (2.3) and (2.4) imply

$$Tg(t)-g_0(t)=\alpha_n g\circ L_n^{-1}(t)+(c_n-c_n^0)L_n^{-1}(t)+(d_n-d_n^0)-\alpha_n\big(\frac{x_N-x_0}{t_N-t_0}L_n^{-1}(t)+\frac{t_Nx_0-t_0x_N}{t_N-t_0}\big).$$

and thus

$$Tg(t)=g_0(t)+(c_n-c_n^0)L_n^{-1}(t)+(d_n-d_n^0)+\alpha_n(g-r)\circ L_n^{-1}(t).$$

$\square$

Corollary 2.5. *The following inequality holds*

$$\|\widetilde{f} - g_0\|_\infty \leq |\alpha|_\infty \|\widetilde{f} - r\|_\infty + \max_n |c_n - c_n^0||t_N| + \max_n |d_n - d_n^0|.$$

Proof. According to the previous Proposition, the fixed point equation of the fractal function $\widetilde{f}$ can be written for $t \in I_n$ as

$$\widetilde{f}(t) = g_0(t) + \alpha_n(\widetilde{f} - r) \circ L_n^{-1}(t) + (c_n - c_n^0)L_n^{-1}(t) + (d_n - d_n^0),$$

then

$$|\widetilde{f}(t) - g_0(t)| \leq |\alpha_n|\|\widetilde{f} - r\|_\infty + |(c_n - c_n^0)L_n^{-1}(t) + (d_n - d_n^0)|$$

and we obtain the result proposed. □

Proposition 2.6. *The uniform distance between the fractal function $\widetilde{f}$ and the polygonal g_0 joining the data is bounded as*

$$\|\widetilde{f} - g_0\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} 2X_{max} + \frac{\gamma|t_N| + \delta}{1 - |\alpha|_\infty},$$

where $X_{max} = \max_n\{|x_n|\}$, $\gamma = \max_n |c_n - c_n^0|$ *and* $\delta = \max_n |d_n - d_n^0|$.

Proof. According to the previous Corollary

$$\|\widetilde{f} - g_0\|_\infty \leq |\alpha|_\infty(\|\widetilde{f} - g_0\|_\infty + \|g_0 - r\|_\infty) + \gamma|t_N| + \delta,$$

and thus

$$\|\widetilde{f} - g_0\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|g_0 - r\|_\infty + \frac{\gamma|t_N| + \delta}{1 - |\alpha|_\infty},$$

obtaining the result. □

3 Discontinuous Affine Fractal Functions

We consider in this Section the definition of iterated function systems $w_n(t,x) = (L_n(t), F_n(t,x))$ related to a partition $\Delta : a = t_0 < t_1 < ... < t_N = b$, where $N > 1$, of an interval $I = [a,b]$, but not related to a set of two-dimensional data.
That is to say, L_n is still an affine map $L_n(t) = a_n t + b_n$ preserving the join-up conditions $L_n(t_0) = t_{n-1}$ and $L_n(t_N) = t_n$, but we omit the join-up conditions on F_n (2.7) and (2.8).

We consider then a compact $I \times C$, where C is a real closed bounded interval and a partition of I, $\Delta : a = t_0 < t_1 < \ldots < t_N = b$, where $N > 1$. The maps $F_n : I \times C \to \mathbb{R}$ are defined as

$$F_n(t,x) = \alpha(t)x + q_n(t),$$

and $q_n(t) = c_n t + d_n$ does not satisfy any special condition (that is to say c_n and d_n are arbitrary real numbers).

Three questions arose:

1. Does the new system own an attractor?
2. If yes, is it still the graph of a function defined on the interval I?
3. In the positive case, is it a continuous mapping?

To answer the first question let us prove that, with some hypotheses on α, the system (w_n) is contractive with respect to a given metric.

Proposition 3.1. *If $\alpha : I \to \mathbb{R}$ satisfies a Lipschitz condition:*

$$|\alpha(t) - \alpha(t')| \leq p|t - t'|, \tag{3.14}$$

for $t, t' \in I$ (this restriction is trivially fulfilled in the discrete case where $\alpha(t) = \alpha_n \in \mathbb{R}$ in the definition of F_n), and $\|\alpha\|_\infty < 1$, the system is hyperbolic (w_n is contractive with respect to a given metric).

Proof. Let us define the distance in $\mathbb{R}^2$

$$d_\lambda((t,x),(t',x')) = |t-t'| + \lambda|x-x'|,$$

for $\lambda > 0$. Then

$$d_\lambda(w_n(t,x), w_n(t',x')) = |L_n(t) - L_n(t')| + \lambda|\alpha(t)x + q_n(t) - \alpha(t')x' - q_n(t')|.$$

$$d_\lambda(w_n(t,x), w_n(t',x')) \leq |a_n||t-t'| + \lambda|c_n||t-t'| + \lambda|\alpha(t)||x-x'| + \lambda|x'||\alpha(t) - \alpha(t')|.$$

Let us denote $h = \max_n |a_n|$. The constant h is the maximum of the quantities

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0},$$

according to the join-up conditions of L_n (2.1), and is lower than 1 because $N > 1$.

Let us define $\gamma = \max_n |c_n|$, and $k > |x|$ for all $x \in C$. Then, bearing in mind the inequality (3.14),

$$d_\lambda(w_n(t,x), w_n(t',x')) \leq (h + \lambda\gamma + \lambda kp)|t-t'| + \lambda\|\alpha\|_\infty|x-x'|.$$

Let us take now

$$\lambda = \frac{1-h}{2(\gamma + kp)} > 0.$$

We have

$$d_\lambda(w_n(t,x), w_n(t',x')) \leq (\frac{h+1}{2})|t-t'| + \lambda\|\alpha\|_\infty|x-x'|,$$

$$d_\lambda(w_n(t,x), w_n(t',x')) \leq \max\{(\frac{h+1}{2}), \|\alpha\|_\infty\} d_\lambda((t,x),(t',x')).$$

Consequently if α satisfies a Lipschitz condition and $\|\alpha\|_\infty < 1$ the system is hyperbolic. □

The hyperbolicity of (w_n) implies the existence of an attractor $A \subset \mathbb{R}^2$, and the first question is answered. The existence of the set A is independent of any data $\{(t_n, x_n)\}$. This is the difference with respect to the previous setting (Section 2).

Our second concern is the following: Is still A the graph of a function $\widehat{f}: I \to \mathbb{R}$?
Let us consider the set valued contractive map $W : \mathcal{K} \to \mathcal{K}$, where $\mathcal{K}$ is the family of compact subsets of $I \times C$ defined as

$$W(K) = \bigcup_{n=1}^{N} w_n(K),$$

for $K \in \mathcal{K}$.

Let us define, as in previous Sections, an operator $T : \mathcal{B}(I) \to \mathcal{B}(I)$, expressed as

$$Tg(t) = F_n(L_n^{-1}(t), g \circ L_n^{-1}(t)) = \alpha(L_n^{-1}(t))g \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t),$$

for $t \in I_n$. The subintervals I_n are defined as in Sec. 2. We assume the standard condition $\|\alpha\|_\infty < 1$ on the scale function.
Even for g continuous, Tg can be discontinuous since F_n does not verify any join-up condition. As in former Sections, T is contractive and, since $\mathcal{B}(I)$ is complete, there is a fixed point $\widehat{f} \in \mathcal{B}(I)$, satisfying

$$\widehat{f}(t) = F_n(L_n^{-1}(t), \widehat{f} \circ L_n^{-1}(t)), \tag{3.15}$$

for $t \in I_n$.
The function $\widehat{f}$ need not be generally continuous, but it is bounded.

Let us extend the map W to w, defined for any subset B of $I \times C$:

$$w(B) = \bigcup_{n=1}^{N} w_n(B).$$

Let us see that the graph G of $\widehat{f}$ is invariant by w, that is to say,

$$G = w(G).$$

For it, let us consider a point $P = (t, \widehat{f}(t)) \in G$. The value t belongs to a subinterval I_n and thus $t = L_n(t')$.

Then, by (3.15),

$$P = (L_n(t'), \widehat{f} \circ L_n(t')) = (L_n(t'), F_n(t', \widehat{f}(t')) = w_n(t', \widehat{f}(t'))$$

and thus $P \in w_n(G) \subset w(G)$.

Let us prove now $w(G) \subset G$. If $(t, x) \in w(G)$, then $(t, x) = (L_n(t'), F_n(t', x'))$ for some $(t', x') \in G$ and $n \in \{1, 2, \ldots, N\}$. Therefore

$$(t, x) = (L_n(t'), F_n(t', \widehat{f}(t')) = (L_n(t'), \widehat{f}(L_n(t'))) \in G,$$

due to the fixed point condition of $\widehat{f}$. As a consequence $G = w(G)$.

The closure of G, $\overline{G}$, belongs to $\mathcal{K}$ because G is bounded.

Proposition 3.2.

$$\overline{G} = W(\overline{G}).$$

Proof. In order to prove the relation $\overline{G} \subset W(\overline{G})$, let us consider that

$$G = w(G) \subseteq w(\overline{G}) = W(\overline{G}).$$

Since $W(\overline{G})$ is closed $\overline{G} \subseteq W(\overline{G})$.

Let us prove now $W(\overline{G}) \subseteq \overline{w(G)} = \overline{G}$.

If $P \in \overline{G}$ then there exists a sequence of points $P_m \in G$ such that $\lim_m P_m = P$. In this case $W(P_m) \to W(P)$, $W(P) \in \overline{w(G)} = \overline{G}$. $\square$

Since $\overline{G}$ is an invariant set of W, it must be the attractor.

Corollary 3.3. *The attractor A of the iterated function system agrees with the closure of the graph of a bounded function $\widehat{f} : I \to \mathbb{R}$.*

The function $\widehat{f}$ is different from the map $\tilde{f}$ defined in Section 2 because the first one is not related to a set of two-dimensional data.

Figure 3 represents the graph of a discontinuous affine fractal function in the interval $I = [0, 1]$, for $t_n = n/10$, for $n = 0, 1, 2, \ldots, 10$ and scale $\alpha(t) = t/2$. The coefficients c_n were randomly chosen in the interval $[0, 1]$, and d_n random in the interval $[-0.5, 0.5]$.

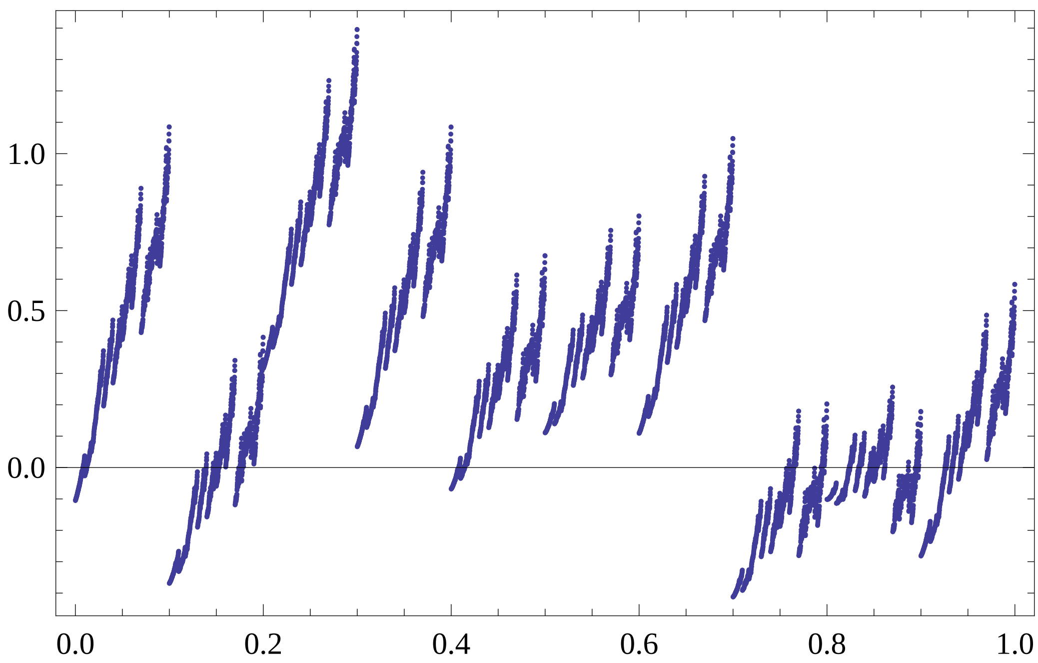

Figure 3: Graph of a discontinuous affine fractal function with coefficients c_n and d_n randomly chosen.

Our purpose now is to obtain the values of the function $\widehat{f}$ on the nodes of the partition $(x_n = \widehat{f}(t_n))$ in terms of the coefficients of the system.

Since $t_0 = L_1(t_0)$, using the fixed point equation,

$$\widehat{f}(t_0) = \alpha(L_1^{-1}(t_0))\widehat{f} \circ L_1^{-1}(t_0) + c_1 L_1^{-1}(t_0) + d_1,$$

then

$$x_0 = \widehat{f}(t_0) = \frac{c_1 t_0 + d_1}{1 - \alpha(t_0)}.$$

For $t_N \in I_N$,

$$\widehat{f}(t_N) = \alpha(L_N^{-1}(t_N))\widehat{f} \circ L_N^{-1}(t_N) + c_N L_N^{-1}(t_N) + d_N,$$

and

$$x_N = \widehat{f}(t_N) = \frac{c_N t_N + d_N}{1 - \alpha(t_N)}.$$

For the rest of the nodes $n = 1, 2, \dots N - 1$

$$\widehat{f}(t_n) = \alpha(L_n^{-1}(t_n))\widehat{f} \circ L_n^{-1}(t_n) + c_n L_n^{-1}(t_n) + d_n,$$

and

$$x_n = \widehat{f}(t_n) = \alpha(t_N)x_N + c_n t_N + d_n.$$

In the case $\alpha(t) = 0$, the graph of $\widehat{f}$ is composed of the lines $q_n \circ L_n^{-1}$ on the subintervals $I_n = (t_{n-1}, t_n]$ for $n = 2, 3, \dots N$ and the graph of $q_1 \circ L_1^{-1}$ on the interval $I_1 = [t_0, t_1]$.

The fractal function passes through the data $(t_n, x_n = \widehat{f}(t_n))$, the iterated function system satisfies the conditions (2.7) and (2.8) for $\epsilon = 0$, and the results of Sec. 2 are applicable here.

3.1 Affine Fractal Functions Close to a Bounded Mapping

In this Subsection we consider a given function $f \in \mathcal{B}(I)$ and we look for an affine fractal function $\widehat{f}$ (not necessarily continuous) close to it.

We consider the discrete case $F_n(t,x) = \alpha_n x + q_n(t)$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is such that

$$|\alpha|_\infty = \max\{|\alpha_n| : n = 1, 2, \ldots, N\} < 1,$$

$c = (c_1, c_2, \ldots, c_N)$, $d = (d_1, d_2, \ldots, d_N)$ and $q_n(t) = c_n t + d_n$. The subintervals I_n are defined as in previous Sections.

If $\widehat{f}$ is the fractal function associated with the operator T (fixed point of it), the Collage Theorem states that, if r is the contractivity factor of T,

$$\|f - \widehat{f}\|_\infty \leq \frac{1}{1-r}\|Tf - f\|_\infty.$$

Based on this result, we can minimize the quantity $\|Tf - f\|_\infty$ in order to find $\widehat{f}$ close to f.

Proposition 3.4. *Let us consider $f \in \mathcal{B}(I)$, a constant δ such that $0 < \delta < 1$, $\overline{B}_\delta = \{\alpha \in \mathbb{R}^N : |\alpha|_\infty \leq \delta\}$ and J, K closed and bounded intervals of $\mathbb{R}^N$. Let us define the mapping $H = H_f : \overline{B}_\delta \times J \times K \to \mathcal{B}(I)$ defined as $H(\alpha, c, d) = T_{\alpha,c,d}(f) : I \to \mathbb{R}$ where*

$$T_{\alpha,c,d}f(t) = \alpha_n f \circ L_n^{-1}(t) + c_n L_n^{-1}(t) + d_n,$$

for $t \in I_n$. The map H is Lipschitz with constant $M = \max\{\|f\|_\infty, |t_N|, 1)$, and thus continuous.

Proof. For $\alpha, \beta \in \overline{B}_\delta$, $c, c' \in J$, $d, d' \in K$ and $t \in I_n$,

$$(H(\alpha, c, d) - H(\beta, c', d'))(t) = (\alpha_n - \beta_n) f \circ L_n^{-1}(t) + (c_n - c'_n) L_n^{-1}(t) + (d_n - d'_n).$$

Therefore

$$\|H(\alpha, c, d) - H(\beta, c', d')\|_\infty \leq |\alpha - \beta|_\infty \|f\|_\infty + |c - c'|_\infty |t_N| + |d - d'|_\infty$$

and

$$\|H(\alpha, c, d) - H(\beta, c', d')\|_\infty \leq M(|\alpha - \beta|_\infty + |c - c'|_\infty + |d - d'|_\infty),$$

where $M = \max\{\|f\|_\infty, |t_N|, 1)$, obtaining the result. □

Corollary 3.5. *Let $P = P_f : \overline{B}_\delta \times J \times K \to \mathbb{R}$ be defined as $P(\alpha, c, d) = \|H(\alpha, c, d) - f\|_\infty$. The function $P(\alpha, c, d)$ reaches a minimum at some $\alpha^* \in \overline{B}_\delta$, $c^* \in J$ and $d^* \in K$.*

Proof. The function $P(\alpha, c, d)$ is continuous and defined on the compact $\overline{B}_\delta \times J \times K$, consequently it reaches a global minimum. □

Proposition 3.6. *The map $P(\alpha, c, d)$ is convex.*

Proof. Let $\lambda \in \mathbb{R}$ be such that $0 \leq \lambda \leq 1$, and α, β scale vectors, $c, c' \in J$, $d, d' \in K$.

$$P(\lambda\alpha + (1-\lambda)\beta, \lambda c + (1-\lambda)c', \lambda d + (1-\lambda)d') =$$

$$\sup\{|T_{\lambda\alpha+(1-\lambda)\beta, \lambda c+(1-\lambda)c', \lambda d+(1-\lambda)d'} f(t) - f(t)|;\ t \in I\} =$$

$$\sup_{1\leq n\leq N}\{|(\lambda\alpha_n + (1-\lambda)\beta_n) f \circ L_n^{-1}(t) + (\lambda c_n + (1-\lambda)c'_n) L_n^{-1}(t) + (\lambda d_n + (1-\lambda)d'_n) - f|;\ t \in I_n\} =$$

$$\sup_{1\leq n\leq N}\{|\lambda(\alpha_n f \circ L_n^{-1}(t) + c_n L_n^{-1}(t) + d_n) + (1-\lambda)(\beta_n f \circ L_n^{-1}(t) + c'_n L_n^{-1}(t) + d'_n) - f(t)|;\ t \in I_n\}.$$

Considering that f can be expressed as $f = \lambda f + (1-\lambda) f$:

$$P(\lambda\alpha + (1-\lambda)\beta, \lambda c + (1-\lambda)c', \lambda d + (1-\lambda)d') \leq \lambda P(\alpha, c, d) + (1-\lambda) P(\beta, c', d').$$

□

Corollary 3.7. *The problem of finding a minimum of $P(\alpha, c, d) = \|T_{\alpha,c,d}(f) - f\|_\infty$ on the feasible region $\overline{B}_\delta \times J \times K$ is a non-smooth convex optimization problem with solution.*

If (α^*, c^*, d^*) is one of the optimum values, then the fractal function associated to the operator T_{α^*,c^*,d^*}, $\widetilde{f}_{\alpha^*,c^*,d^*}$, is the sought function. The Collage Theorem states that

$$\|f - \widetilde{f}_{\alpha^*,c^*,d^*}\|_\infty \leq \frac{\|T_{\alpha^*,c^*,d^*}(f) - f\|_\infty}{1 - |\alpha^*|_\infty} = \frac{P(\alpha^*, c^*, d^*)}{1 - |\alpha^*|_\infty}, \tag{3.16}$$

since $|\alpha^*|_\infty$ is the contractivity factor of the operator.

Proposition 3.8. *If f is an interpolant or an approximant of a set of data $\{(t_n, x_n)\}_{n=0}^N$, and g is an original function providing them ($g(t_n) = x_n$ for $n = 0, 1, \dots N$), then*

$$\|g - \widetilde{f}_{\alpha^*,c^*,d^*}\|_\infty \leq E(f) + \frac{P(\alpha^*, c^*, d^*)}{1 - |\alpha^*|_\infty},$$

where $E(f)$ is the error of the approximation f.

Proof. It is a consequence of the triangular inequality

$$\|g - \widetilde{f}_{\alpha^*,c^*,d^*}\|_\infty \leq \|g - f\|_\infty + \|f - \widetilde{f}_{\alpha^*,c^*,d^*}\|_\infty,$$

and the expression (3.16). □

The following result can be found in [15].

Lemma 3.9. *Let us consider a continuous function $f : I \to \mathbb{R}$ passing through the data $D = \{(t_n, x_n)\}_{n=0}^N$, such that $t_n - t_{n-1} = h$, and let $\omega(\delta)$ be the modulus of continuity of f, defined as*

$$\omega(\delta) = \sup\{|f(t) - f(t')| : |t - t'| \leq \delta\}.$$

If g_0 is the polygonal joining the data D,

$$\|f - g_0\|_\infty \leq \omega(h).$$

Proposition 3.10. *If f interpolates the set of data $D = \{(t_n, x_n = \widetilde{f}_{\alpha^*,c^*,d^*}(t_n))\}_{n=0}^N$, such that $t_n - t_{n-1} = h$, and it is continuous*

$$\|f - \widetilde{f}_{\alpha^*,c^*,d^*}\|_\infty \leq \omega(h) + \frac{|\alpha^*|_\infty}{1 - |\alpha^*|_\infty} 2X_{max} + \frac{\gamma|t_N| + \delta}{1 - |\alpha^*|_\infty},$$

where $\omega(\delta)$ is the modulus of continuity of f, $X_{max} = \max_n\{|x_n|\}$, $\gamma = \max_n |c_n - c_n^0|$ and $\delta = \max_n |d_n - d_n^0|$.

Proof. Let us consider that

$$\|f - \widetilde{f}_{\alpha^*,c^*,d^*}\|_\infty \leq \|f - g_0\|_\infty + \|g_0 - \widetilde{f}_{\alpha^*,c^*,d^*}\|_\infty,$$

where g_0 is the polygonal whose vertices are the points (t_n, x_n). Applying the previous Lemma and Proposition 2.6 the result is obtained. □

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Popoviciu Type Inequalities Via Green Function and Hermite's Polynomial

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Abstract

The Hermite polynomial and Green function are used to construct the identities related to Popoviciu type inequalities for higher order convex functions. We investigate the bounds for the identities related to the generalization of the Popoviciu inequality using inequalities for the Čebyšev functional. Some results relating to the Grüss and Ostrowski type inequalities are constructed. Further, we also construct new families of exponentially convex functions and Cauchy-type means by looking at linear functionals associated with the obtained inequalities.

Keywords: Convex Function, Divided Difference, Generalized Montgomery Identity, ČEbyŠEv Functional, GrÜSs Inequality, Ostrowski Inequality, Exponential Convexity.

1 Introduction and Preliminary Results

A characterization of convex function established by T. Popoviciu [18] is studied by many people (see [19, 17] and references with in). For recent work, we refer [7, 10, 11, 14, 15]. The following form of

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Popoviciu's inequality is by Vasić and Stanković in [19] (see also page 173 [17]):

Theorem 1.1. *Let $z,w \in \mathbb{N}$, $z \geq 3$, $2 \leq w \leq z-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,...,x_z) \in [\alpha,\beta]^z$, $\mathbf{p} = (p_1,...,p_z)$ be a positive z-tuple such that $\sum_{u=1}^{z} p_u = 1$. Also let $f : [\alpha,\beta] \to \mathbb{R}$ be a convex function. Then*

$$p_{w,z}(\mathbf{x},\mathbf{p};f) \leq \frac{z-w}{z-1} p_{1,z}(\mathbf{x},\mathbf{p};f) + \frac{w-1}{z-1} p_{z,z}(\mathbf{x},\mathbf{p};f), \tag{1.1}$$

where

$$p_{w,z}(\mathbf{x},\mathbf{p};f) = p_{w,z}(\mathbf{x},\mathbf{p};f(x)) := \frac{1}{\binom{z-1}{w-1}} \sum_{1 \leq u_1 < ... < u_w \leq z} \left(\sum_{v=1}^{w} p_{u_v} \right) f \left(\frac{\sum\limits_{v=1}^{w} p_{u_v} x_{u_v}}{\sum\limits_{v=1}^{w} p_{u_v}} \right)$$

is the linear functional with respect to f.

By inequality (1.1), we write

$$\mathbf{P}(\mathbf{x},\mathbf{p};f) := \frac{z-w}{z-1} p_{1,z}(\mathbf{x},\mathbf{p};f) + \frac{w-1}{z-1} p_{z,z}(\mathbf{x},\mathbf{p};f) - p_{w,z}(\mathbf{x},\mathbf{p};f). \tag{1.2}$$

Remark 1.2. *It is important to note that under the assumptions of Theorem 1.1, if the function f is convex then $\mathbf{P}(\mathbf{x},\mathbf{p};f) \geq 0$ and $\mathbf{P}(\mathbf{x},\mathbf{p};f) = 0$ for $f(x) = x$ or f is a constant function.*

The mean value theorems and exponential convexity of the linear functional $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ are given in [10] for a positive m-tuple $\mathbf{p}$. Some special classes of convex functions are considered to construct the exponential convexity of $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ in [10].

Consider the Green function $G : [\alpha,\beta] \times [\alpha,\beta] \to$ defined as

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \leq s \leq t; \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \leq s \leq \beta. \end{cases} \tag{1.3}$$

The function G is convex and continuous w.r.t s and due to symmetry also w.r.t t.

For any function $\psi : [\alpha,\beta] \to \mathbb{R}$, $\psi \in C^2([\alpha,\beta])$, we have

$$\psi(x) = \frac{\beta-x}{\beta-\alpha} \psi(\alpha) + \frac{x-\alpha}{\beta-\alpha} \psi(\beta) + \int_{\alpha}^{\beta} G(x,s) \psi''(s) ds, \tag{1.4}$$

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Discontinuous Fractal Interpolation

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Abstract

The fractal interpolation functions provide curves whose graph has generally a non-integer dimension. They own other characteristics as the interpolation of a set of data and the continuity. In this paper, the latter conditions are omitted, defining discontinuous fractal functions passing close to (but not necessarily through) the given data.

In a second part of the article we define affine fractal functions not linked to two-dimensional data. To do this we use the methodology of iterated functions systems. They are composed of a finite set of contractive affinities whose attractor is related to the graph of a bounded function. In this way the paper introduces a very large class of affine fractal functions which are generally discontinuous (though they contain the classical continuous case as a particular case) and whose relevance is not based only on the approximation.

Keywords: Fractals, Discontinuous functions, Interpolation, Approximation

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where the function G is defined in (1.3) (see [20]).

Let $-\infty < \alpha < \beta < \infty$ and $\alpha = a_1 < a_2 \cdots < a_r = \beta$, $(r \geq 2)$ be the given points. For $\psi \in C^n[\alpha, \beta]$ a unique polynomial $\rho_H(s)$ of degree $(n-1)$ exists satisfying any of the following conditions:

Hermite conditions:

$$\rho_H^{(i)}(a_j) = \psi^{(i)}(a_j);\ 0 \leq i \leq k_j,\ 1 \leq j \leq r,\ \sum_{j=1}^{r} k_j + r = n. \tag{H}$$

It is of great interest to note that Hermite conditions include the following particular cases:

Type $(m, n-m)$ **conditions:** $(r = 2, 1 \leq m \leq n-1,\ k_1 = m-1,\ k_2 = n-m-1)$

$$\rho_{(m,n)}^{(i)}(\alpha) = \psi^{(i)}(\alpha), 0 \leq i \leq m-1,$$

$$\rho_{(m,n)}^{(i)}(\beta) = \psi^{(i)}(\beta), 0 \leq i \leq n-m-1,$$

Two-point Taylor conditions: $(n = 2m,\ r = 2,\ k_1 = k_2 = m-1)$

$$\rho_{2T}^{(i)}(\alpha) = \psi^{(i)}(\alpha), \rho_{2T}^{(i)}(\beta) = \psi^{(i)}(\beta), 0 \leq i \leq m-1.$$

We have the following result from [1].

Theorem 1.3. *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ be the given points, and $\psi \in C^n([\alpha, \beta])$. Then we have*

$$\psi(t) = \rho_H(t) + R_{H,n}(\psi, t) \tag{1.5}$$

where $\rho_H(t)$ is the Hermite interpolating polynomial, i.e.

$$\rho_H(t) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \psi^{(i)}(a_j);$$

the H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}}\sum_{k=0}^{k_j-i}\frac{1}{k!}\frac{d^k}{dt^k}\left(\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right)\Bigg|_{t=a_j}(t-a_j)^k, \tag{1.6}$$

with

$$\omega(t) = \prod_{j=1}^{r}(t-a_j)^{k_j+1},$$

and the remainder is given by

$$R_{H,n}(\psi,t) = \int_{\alpha}^{\beta} G_{H,n}(t,s)\psi^{(n)}(s)ds$$

where $G_{H,n}(t,s)$ is defined by

$$G_{H,n}(t,s) = \begin{cases} \sum_{j=1}^{l}\sum_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t);\ s \le t, \\ -\sum_{j=l+1}^{r}\sum_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t);\ s \ge t, \end{cases} \tag{1.7}$$

for all $a_l \le s \le a_{l+1}$; $l = 0,\dots,r$ with $a_0 = \alpha$ and $a_{r+1} = \beta$.

Remark 1.4. *In particular cases,*

for type $(m,n-m)$ conditions, from Theorem 1.3 we have

$$\psi(t) = \rho_{(m,n)}(t) + R_{(m,n)}(\psi,t) \tag{1.8}$$

where $\rho_{(m,n)}(t)$ is $(m,n-m)$ interpolating polynomial, i.e

$$\rho_{(m,n)}(t) = \sum_{i=0}^{m-1}\tau_i(t)\psi^{(i)}(\alpha) + \sum_{i=0}^{n-m-1}\eta_i(t)\psi^{(i)}(\beta),$$

with

$$\tau_i(t) = \frac{1}{i!}(t-\alpha)^i\left(\frac{t-\beta}{\alpha-\beta}\right)^{n-m}\sum_{k=0}^{m-1-i}\binom{n-m+k-1}{k}\left(\frac{t-\alpha}{\beta-\alpha}\right)^k \tag{1.9}$$

and

$$\eta_i(t) = \frac{1}{i!}(t-\beta)^i\left(\frac{t-\alpha}{\beta-\alpha}\right)^{m}\sum_{k=0}^{n-m-1-i}\binom{m+k-1}{k}\left(\frac{t-\beta}{\alpha-\beta}\right)^k. \tag{1.10}$$

and the remainder $R_{(m,n)}(\psi,t)$ *is given by*

$$R_{(m,n)}(\psi,t)=\int_{\alpha}^{\beta}G_{(m,n)}(t,s)\psi^{(n)}(s)ds$$

with

$$G_{(m,n)}(t,s)=\begin{cases}\sum\limits_{j=0}^{m-1}\Big[\sum\limits_{p=0}^{m-1-j}\binom{n-m+p-1}{p}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^{p}\Big]\times & \\ \frac{(t-\alpha)^{j}(\alpha-s)^{n-j-1}}{j!(n-j-1)!}\Big(\frac{\beta-t}{\beta-\alpha}\Big)^{n-m}, & \alpha\le s\le t\le\beta \\ -\sum\limits_{i=0}^{n-m-1}\Big[\sum\limits_{q=0}^{n-m-i-1}\binom{m+q-1}{q}\Big(\frac{\beta-t}{\beta-\alpha}\Big)^{q}\Big]\times & \\ \frac{(t-\beta)^{i}(\beta-s)^{n-i-1}}{i!(n-i-1)!}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^{m}, & \alpha\le t\le s\le\beta.\end{cases} \tag{1.11}$$

For Type Two-point Taylor conditions, from Theorem 1.3 we have

$$\psi(t)=\rho_{2T}(t)+R_{2T}(\psi,t) \tag{1.12}$$

where $\rho_{2T}(t)$*is the two-point Taylor interpolating polynomial i.e,*

$$\begin{aligned}\rho_{2T}(t)=\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}\Big[&\frac{(t-\alpha)^{i}}{i!}\Big(\frac{t-\beta}{\alpha-\beta}\Big)^{m}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^{k}\psi^{(i)}(\alpha)\\ &+\frac{(t-\beta)^{i}}{i!}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^{m}\Big(\frac{t-\beta}{\alpha-\beta}\Big)^{k}\psi^{(i)}(\beta)\Big]\end{aligned} \tag{1.13}$$

and the remainder $R_{2T}(\psi,t)$ *is given by*

$$R_{2T}(\psi,t)=\int_{\alpha}^{\beta}G_{2T}(t,s)\psi^{(n)}(s)ds$$

with

$$G_{2T}(t,s)=\begin{cases}\frac{(-1)^{m}}{(2m-1)!}p^{m}(t,s)\sum\limits_{j=0}^{m-1}\binom{m-1+j}{j}(t-s)^{m-1-j}q^{j}(t,s), & s\le t;\\ \frac{(-1)^{m}}{(2m-1)!}q^{m}(t,s)\sum\limits_{j=0}^{m-1}\binom{m-1+j}{j}(s-t)^{m-1-j}p^{j}(t,s), & s\ge t;\end{cases} \tag{1.14}$$

where $p(t,s)=\frac{(s-\alpha)(\beta-t)}{\beta-\alpha}$, $q(t,s)=p(t,s),\forall t,s\in[\alpha,\beta]$.

The following Lemma describes the positivity of Green's function (1.7) see (Beesack [2] and Levin [12]).

Lemma 1.5. *The Green's function $G_{H,n}(t,s)$ has the following properties:*

(i) $\frac{G_{H,n}(t,s)}{w(t)} > 0, a_1 \leq t \leq a_r, a_1 \leq s \leq a_r$;

(ii) $G_{H,n}(t,s) \leq \frac{1}{(n-1)!(\beta-\alpha)}|w(t)|$;

(iii) $\int_\alpha^\beta G_{H,n}(t,s)ds = \frac{w(t)}{n!}$.

The organization of the paper is as follows: In Section 2, we use Green's function, Hermite interpolating polynomial and the $n-$convexity of the function ψ to establish a generalization of Popoviciu's inequality. We discuss the results for particular cases namely, $(m, n-m)$ interpolating polynomial and two-point Taylor interpolating polynomial. In Section 3, we present some interesting results by employing Čebyšev functional and Grüss-type inequalities, also results relating to the Ostrowski-type inequality. In Section 4, we study the functional defined as the difference between the R.H.S. and the L.H.S. of the generalized inequality. Here our objective is to investigate the properties of the functional, such as n-exponential and logarithmic convexity. Further, we prove monotonicity property of the generalized Cauchy means obtained via this functional. Finally, in Section 5 we give several examples of the families of functions for which the obtained results can be applied.

2 Popoviciu's Type Inequalities by Green's Function and Hermite Interpolating Polynomial

We begin this section with the proof of our main identity related to generalizations of Popoviciu's type inequality.

Theorem 2.1. *Let $z, w \in \mathbb{N}$, $z \geq 3$, $2 \leq w \leq z-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_z) \in [\alpha,\beta]^z$, $\mathbf{p} = (p_1, ..., p_z)$ be a real z-tuple such that $\sum_{v=1}^{w} p_{u_v} \neq 0$ for any $1 \leq u_1 < ... < u_w \leq z$ and $\sum_{u=1}^{z} p_u = 1$. Also let $\frac{\sum\limits_{v=1}^{w} p_{u_v} x_{u_v}}{\sum\limits_{v=1}^{w} p_{u_v}} \in$*

$[\alpha,\beta]$ *for any* $1 \leq u_1 < ... < u_w \leq z$ *with* $\alpha = a_1 < a_2 \cdots < a_r = \beta$, $(r \geq 2)$ *be the given points, and* $\psi \in C^n([\alpha,\beta])$. *Furthermore,* H_{ij}, $G_{H,n}$ *and* G *be as defined in* (1.6), (1.7) *and* (1.3) *respectively. Then we have the following identity:*

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) = \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t)) \sum_{j=1}^{r}\sum_{i=0}^{k_j} \psi^{(i+2)}(a_j)H_{ij}(t)dt$$
$$+ \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t))G_{H,n-2}(t,s))\psi^{(n)}(s)dsdt. \quad (2.15)$$

Proof. Using (1.4) in (1.2) and following the linearity of $\mathbf{P}(\mathbf{x},\mathbf{p};\cdot)$, we have

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) = \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t))\psi''(t)dt. \quad (2.16)$$

By Theorem 1.3,$\psi''(t)$ can be expressed as

$$\psi''(t) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\psi^{(i+2)}(a_j) + \int_{\alpha}^{\beta} G_{H,n-2}(t,s)\psi^{(n)}(s)ds. \quad (2.17)$$

Using (2.17) in (2.16) we get (2.15). □

For $n-$convex functions, we can give the following form of new identity (2.15).

Theorem 2.2. *Let all the assumptions of Theorem 2.1 be satisfied. If* $\psi : [\alpha,\beta] \to$ *is* $n-$*convex function and*

$$\int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t))G_{H,n-2}(t,s)dt \geq 0, \ t \in [\alpha,\beta]. \quad (2.18)$$

Then

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \geq \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t)) \sum_{j=1}^{r}\sum_{i=0}^{k_j} \psi^{(i+2)}(a_j)H_{ij}(t)dt \quad (2.19)$$

Proof. Since the function ψ is $n-$convex, therefore without loss of generality we can assume that ψ is n-times differentiable and $\psi^{(n)}(x) \geq 0$ for all $x \in [\alpha,\beta]$ (see [17], p. 16). Hence we can apply Theorem 2.1 to obtain (2.19). □

Now we obtain a generalization of Popoviciu's type inequality for z-tuples.

Theorem 2.3. *Let in addition to the assumptions of Theorem 2.1,* $\mathbf{p} = (p_1, ..., p_z)$ *be a positive z-tuple such that* $\sum_{u=1}^{z} p_u = 1$*, and* $\psi : [\alpha, \beta] \to$ *be an* $n-$*convex function.*

(i) If k_j is odd for each $j = 2, .., r$, then (2.19) holds.

(ii) Let the inequality (2.19) be satisfied and

$$F(.) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \psi^{(i+2)}(a_j) H_{ij}(.) \tag{2.20}$$

is non negative. Then we have

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; \psi(x)) \geq 0. \tag{2.21}$$

Proof. (i) Since Green's function $G(x,t)$ is convex and the weights are positive. So $\mathbf{P}(\mathbf{x}, \mathbf{p}; G(x,t)) \geq 0$ by virtue of Remark 1.2. Also as it is given that k_j is odd for each $j = 2, .., r$, therefore we have $\omega(t) \geq 0$ and by using Lemma 1.5(i) we have $G_{H,n-2}(t,s) \geq 0$, so (2.18) holds. Now following Theorem 2.2, we can obtain (2.19).

(ii) Using (2.20) in (2.19), we get (2.27).

□

As a particular cases of Hermite conditions, we can give the following corollaries to above Theorem 2.3. By using type $(m, n-m)$ conditions we can give the following result:

Corollary 2.4. *Let* τ_i, η_i *be as defined in* (1.9) *and* (1.10) *respectively. Also* $\mathbf{p} = (p_1, ..., p_z)$ *be a positive z-tuple such that* $\sum_{u=1}^{z} p_u = 1$*, and* $\psi : [\alpha, \beta] \to$ *be an* $n-$*convex function.*

(i) If $n-m$ *is even, then the following inequality holds*

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; \psi(x)) \geq \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x}, \mathbf{p}; G(x,t)) \left(\sum_{i=0}^{m-1} \tau_i(t) \phi^{(i+2)}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(t) \phi^{(i+2)}(\beta) \right) dt. \tag{2.22}$$

(ii) Let the inequality (2.22) *be satisfied and*

$$F(.) = \left(\sum_{i=0}^{m-1} \tau_i(.)\phi^{(i+2)}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(.)\phi^{(i+2)}(\beta) \right) \tag{2.23}$$

is non negative. Then we have

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \geq 0. \tag{2.24}$$

By using Two-point Taylor conditions we can give the following result.

Corollary 2.5. *Let $\mathbf{p} = (p_1, ..., p_z)$ be a positive z-tuple such that $\sum_{u=1}^{z} p_u = 1$, and $\psi : [\alpha,\beta] \to$ be an $n-$convex function.*

(i) If m is even, then the following inequality holds

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \geq \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t)) \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \times \left[\frac{(t-\alpha)^i}{i!} \left(\frac{t-\beta}{\alpha-\beta} \right)^m \left(\frac{t-\alpha}{\beta-\alpha} \right)^k \phi^{(i+2)}(\alpha) + \frac{(t-\beta)^i}{i!} \left(\frac{t-\alpha}{\beta-\alpha} \right)^m \left(\frac{t-\beta}{\alpha-\beta} \right)^k \phi^{(i+2)}(\beta) \right] dt. \tag{2.25}$$

(ii) Let the inequality (2.25) *be satisfied and*

$$F(t) = \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \times \left[\frac{(t-\alpha)^i}{i!} \left(\frac{t-\beta}{\alpha-\beta} \right)^m \left(\frac{t-\alpha}{\beta-\alpha} \right)^k \phi^{(i+2)}(\alpha) + \frac{(t-\beta)^i}{i!} \left(\frac{t-\alpha}{\beta-\alpha} \right)^m \left(\frac{t-\beta}{\alpha-\beta} \right)^k \phi^{(i+2)}(\beta) \right] \tag{2.26}$$

is non negative. Then we have

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \geq 0. \tag{2.27}$$

3 Bounds for Identities Related to Generalization of Popoviciu's Inequality

In this section we present some interesting results by using Čebyšev functional and Gr*ü*ss type inequalities. For two Lebesgue integrable functions $f, h : [\alpha,\beta] \to$, we consider the Čebyšev functional

$$\Delta(f,h) = \frac{1}{\beta-\alpha} \int_\alpha^\beta f(t)h(t)dt - \frac{1}{\beta-\alpha} \int_\alpha^\beta f(t)dt . \frac{1}{\beta-\alpha} \int_\alpha^\beta h(t)dt.$$

The following Grüss type inequalities are given in [6].

Theorem 3.1. *Let $f : [\alpha,\beta] \to$ be a Lebesgue integrable function and $h : [\alpha,\beta] \to$ be an absolutely continuous function with $(.-\alpha)(\beta-.)[h']^2 \in L[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leq \frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_{\alpha}^{\beta} (x-\alpha)(\beta-x)[h'(x)]^2 dx \right)^{\frac{1}{2}}. \tag{3.28}$$

The constant $\frac{1}{\sqrt{2}}$ in (3.28) is the best possible.

Theorem 3.2. *Assume that $h : [\alpha,\beta] \to$ is monotonic nondecreasing on $[\alpha,\beta]$ and $f : [\alpha,\beta] \to$ be an absolutely continuous with $f' \in L_\infty[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leq \frac{1}{2(\beta-\alpha)}||f'||_\infty \int_{\alpha}^{\beta} (x-\alpha)(\beta-x)dh(x). \tag{3.29}$$

The constant $\frac{1}{2}$ in (3.29) is the best possible.

In the sequel, we consider above theorems to derive generalizations of the results proved in the previous section. In order to avoid many notions let us denote

$$\tilde{\mathfrak{H}}(s) = \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t))G_{H,n-2}(t,s)dt, \quad s \in [\alpha,\beta], \tag{3.30}$$

Consider the Čebyšev functional $\Delta(\tilde{\mathfrak{H}},\tilde{\mathfrak{H}})$ given as:

$$\Delta(\tilde{\mathfrak{H}},\tilde{\mathfrak{H}}) = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \tilde{\mathfrak{H}}^2(s)ds - \left(\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \tilde{\mathfrak{H}}(s)ds\right)^2.$$

Theorem 3.3. *Let all the assumptions of Theorem 2.1 be valid with $-\infty < \alpha < \beta < \infty$ and $\alpha = a_1 < a_2 \cdots < a_r = \beta$, $(r \geq 2)$ be the given points. Moreover, $\psi \in C^n([\alpha,\beta])$ such that $\psi^{(n)}$ is absolutely continuous with $(.-\alpha)(\beta-.)[\psi^{(n+1)}]^2 \in L[\alpha,\beta]$. Also let H_{ij} be the fundamental polynomials of the Hermite basis and the functions G and $\tilde{\mathfrak{H}}$ be defined by (1.3) and (3.30) respectively. Then*

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) = \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t)) \sum_{j=1}^{r}\sum_{i=0}^{k_j} \psi^{(i+2)}(a_j)H_{ij}(t)dt + \frac{\psi^{(n-1)}(\beta)-\psi^{(n-1)}(\alpha)}{(\beta-\alpha)} \int_{\alpha}^{\beta} \tilde{\mathfrak{H}}(s)ds + \tilde{\mathfrak{K}}_n(\alpha,\beta;\psi). \tag{3.31}$$

where the remainder $\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)$ satisfy the bound

$$|\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)| \leq [\Delta(\tilde{\mathfrak{H}},\tilde{\mathfrak{H}})]^{\frac{1}{2}}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (s-\alpha)(\beta-s)[\psi^{(n+1)}(s)]^2 ds\right|^{\frac{1}{2}}. \tag{3.32}$$

Proof. The proof is similar to the Theorem [9] in [5]. □

The following Gr*ü*ss type inequalities can be obtained by using Theorem 3.2

Theorem 3.4. *Let all the assumptions of Theorem 2.1 be valid with $-\infty<\alpha<\beta<\infty$ and $\alpha=a_1<a_2\cdots<a_r=\beta$, $(r\geq 2)$ be the given points. Moreover, $\psi\in C^n([\alpha,\beta])$ such that $\psi^{(n)}$ is absolutely continuous and let $\psi^{(n+1)}\geq 0$ on $[\alpha,\beta]$ with $\tilde{\mathfrak{H}}$ defined in (3.30). Then the representation (3.31) and the remainder $\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)$ satisfies the estimation*

$$|\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)| \leq (\beta-\alpha)||\tilde{\mathfrak{H}}'||_\infty\left[\frac{\psi^{(n-1)}(\beta)+\psi^{(n-1)}(\alpha)}{2}-\frac{\psi^{(n-2)}(\beta)-\psi^{(n-2)}(\alpha)}{(\beta-\alpha)}\right]. \tag{3.33}$$

Proof. The proof is similar to the Theorem [10] in [5]. □

Now we intend to give the Ostrowski type inequalities related to generalizations of Popoviciu's inequality.

Theorem 3.5. *Suppose all the assumptions of Theorem 2.1 be satisfied. Moreover, assume (p,q) is a pair of conjugate exponents, that is $p,q\in[1,\infty]$ such that $1/p+1/q=1$. Let $|\psi^{(n)}|^p:[\alpha,\beta]\to$ be a R-integrable function for some $n\geq 2$. Then, we have*

$$\begin{aligned}&\left|\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x))-\textstyle\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t))\sum_{j=1}^{r}\sum_{i=0}^{k_j}\psi^{(i+2)}(a_j)H_{ij}(t)dt\right|\\ &\leq ||\psi^{(n)}||_p\left(\int_\alpha^\beta\left|\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t))G_{H,n-2}(t,s)dt\right|^q ds\right)^{1/q}.\end{aligned} \tag{3.34}$$

The constant on the R.H.S. of (3.34) is sharp for $1<p\leq\infty$ and the best possible for $p=1$, respectively.

Proof. The proof is similar to the Theorem [11] in [5]. □

Remark 3.6. *We can give all the above results of this sections for Type $(m,n-m)$ conditions and Two-point Taylor conditions.*

4 Mean Value Theorems and $n-$exponential Convexity

We recall some definitions and basic results from [3], [8] and [16] which are required in sequel.

Definition 1. A function $\psi : I \to \mathbb{R}$ is $n-$exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

hold for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$. A function $\psi : I \to \mathbb{R}$ is $n-$exponentially convex if it is $n-$exponentially convex in the Jensen sense and continuous on I.

Definition 2. A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is $n-$exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 4.1. *If $\psi : I \to \mathbb{R}$ is an $n-$exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \leq n$. Particularly,*

$$\det\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, ..., n$.

Remark 4.2. It is known that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta \, \psi\left(\frac{x+y}{2}\right) + \beta^2 \psi(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is $2-$exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is $2-$exponentially convex.

Remark 4.3. *By the virtue of Theorem 2.2, we define the positive linear functional with respect to $n-$convex function ψ as follows*

$$\Lambda(\psi) := \Upsilon(\mathbf{x},\mathbf{p};\psi(x)) - \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,t)) \sum_{j=1}^{r}\sum_{i=0}^{k_j} \psi^{(i+2)}(a_j)H_{ij}(t)dt \geq 0. \tag{4.35}$$

Lagrange and Cauchy type mean value theorems related to defined functional is given in the following theorems.

Theorem 4.4. *Let $\psi : [\alpha,\beta] \to$ be such that $\psi \in C^n[\alpha,\beta]$. If the inequality in (2.18) holds, then there exist $\xi \in [\alpha,\beta]$ such that*

$$\Lambda(\psi) = \psi^{(n)}(\xi)\Lambda(\varphi), \tag{4.36}$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\Lambda(\cdot)$ is defined by (4.35).

Proof. Similar to the proof of Theorem 4.1 in [9] (see also [4]). □

Theorem 4.5. *Let $\psi,\phi : [\alpha,\beta] \to$ be such that $\psi,\phi \in C^n[\alpha,\beta]$. If the inequality in (2.18) holds, then there exist $\xi \in [\alpha,\beta]$ such that*

$$\frac{\Lambda(\psi)}{\Lambda(\phi)} = \frac{\psi^{(n)}(\xi)}{\phi^{(n)}(\xi)}, \tag{4.37}$$

provided that the denominators are non-zero and $\Lambda(\cdot)$ is defined by (4.35).

Proof. Similar to the proof of Corollary 4.2 in [9] (see also [4]). □

Theorem 4.5 enables us to define Cauchy means, because if

$$\xi = \left(\frac{\psi^{(n)}}{\phi^{(n)}}\right)^{-1}\left(\frac{\Lambda(\psi)}{\Lambda(\phi)}\right),$$

which means that ξ is mean of α, β for given functions ψ and ψ.

Next we construct the non trivial examples of $n-$exponentially and exponentially convex functions from positive linear functional $\Lambda(\cdot)$. We use the idea given in [16]. In the sequel I and J are intervals in .

Theorem 4.6. *Let $\Gamma = \{\psi_t : t \in J\}$, where J is an interval in , be a family of functions defined on an interval I in such that the function $t \mapsto [x_0,\ldots,x_n;\psi_t]$ is $n-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n \in I$. Then for the linear functional $\Lambda(\psi_t)$ as defined by* (4.35)*, the following statements are valid:*

(i) The function $t \to \Lambda(\psi_t)$ is $n-$exponentially convex in the Jensen sense on J and the matrix $[\Lambda(\psi_{\frac{t_j+t_l}{2}})]_{j,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \le n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\Lambda(\psi_{\frac{t_j+t_l}{2}})]_{j,l=1}^{m} \ge 0 \textit{ for all } m \in \mathbb{N},\ m = 1,2,...,n.$$

(ii) If the function $t \to \Lambda(\psi_t)$ is continuous on J, then it is $n-$exponentially convex on J.

Proof. (i) For $\xi_j \in$ and $t_j \in J$, $j = 1,\ldots,n$, we define the function

$$h(x) = \sum_{j,l=1}^{n} \xi_j \xi_l \psi_{\frac{t_j+t_l}{2}}(x).$$

Using the assumption that the function $t \mapsto [x_0,\ldots,x_n;\psi_t]$ is $n-$exponentially convex in the Jensen sense, we have

$$[x_0,\ldots,x_n,h] = \sum_{j,l=1}^{n} \xi_j \xi_l \left[x_0,\ldots,x_n;\psi_{\frac{t_j+t_l}{2}}\right] \ge 0,$$

which in turn implies that h is a $n-$convex function on J, therefore from Remark 4.3 we have $\Lambda(h) \ge 0$. The linearity of $\Lambda(\cdot)$ gives

$$\sum_{j,l=1}^{n} \xi_j \xi_l \Lambda\left(\psi_{\frac{t_j+t_l}{2}}\right) \ge 0.$$

We conclude that the function $t \mapsto \Lambda(\psi_t)$ is $n-$exponentially convex on J in the Jensen sense.

The remaining part follows from Proposition 4.1.

(ii) If the function $t \to \Lambda(\psi_t)$ is continuous on J, then it is $n-$exponentially convex on J by definition.

□

The following corollary is an immediate consequence of the above theorem

Corollary 4.7. *Let $\Gamma = \{\psi_t : t \in J\}$, where J is an interval in , be a family of functions defined on an interval I in , such that the function $t \mapsto [x_0, \ldots, x_n; \psi_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Then for the linear functional $\Lambda(\psi_t)$ as defined by* (4.35)*, the following statements hold:*

(i) The function $t \to \Lambda(\psi_t)$ is exponentially convex in the Jensen sense on J and the matrix $\left[\Lambda\left(\psi_{\frac{t_j+t_l}{2}}\right)\right]_{j,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, .., t_m \in J$. Particularly,

$$\det\left[\Lambda\left(\psi_{\frac{t_j+t_l}{2}}\right)\right]_{j,l=1}^{m} \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, ..., n.$$

(ii) If the function $t \to \Lambda(\psi_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 4.8. *Let $\Gamma = \{\psi_t : t \in J\}$, where J is an interval in , be a family of functions defined on an interval I in , such that the function $t \mapsto [x_0, \ldots, x_n; \psi_t]$ is $2-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Let $\Lambda(\cdot)$ be linear functional defined by* (4.35)*. Then the following statements hold:*

(i) If the function $t \mapsto \Lambda(\psi_t)$ is continuous on J, then it is $2-$exponentially convex function on J. If $t \mapsto \Lambda(\psi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[\Lambda(\psi_s)]^{t-r} \leq [\Lambda(\psi_r)]^{t-s} [\Lambda(\psi_t)]^{s-r},$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto \Lambda(\psi_t)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(\Lambda, \Gamma) \leq \mu_{u,v}(\Lambda, \Gamma), \tag{4.38}$$

where

$$\mu_{p,q}(\Lambda, \Gamma) = \begin{cases} \left(\frac{\Lambda(\psi_p)}{\Lambda(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\Lambda(\psi_p)}{\Lambda(\psi_p)}\right), & p = q, \end{cases} \tag{4.39}$$

for $\psi_p, \psi_q \in \Gamma$.

Proof. (i) This is an immediate consequence of Theorem 4.6 and Remark 4.2.

(ii) Since $p \mapsto \Lambda(\psi_t)$ is positive and continuous, by (i) we have that $t \mapsto \Lambda(\psi_t)$ is log-convex on J, that is, the function $t \mapsto \log \Lambda(\psi_t)$ is convex on J. Hence we get

$$\frac{\log \Lambda(\psi_p) - \log \Lambda(\psi_q)}{p-q} \leq \frac{\log \Lambda(\psi_u) - \log \Lambda(\psi_v)}{u-v}, \tag{4.40}$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. So, we conclude that

$$\mu_{p,q}(\Lambda, \Gamma) \leq \mu_{u,v}(\Lambda, \Gamma).$$

Cases $p = q$ and $u = v$ follow from (4.40) as limit cases.

□

5 Applications to Cauchy Means

In this section, we present some families of functions which fulfil the conditions of Theorem 4.6, Corollary 4.7 and Corollary 4.8. This enables us to construct a large families of functions which are exponentially convex. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

Example 5.1. Let us consider a family of functions

$$\Gamma_1 = \{\psi_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\psi_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & t \neq 0, \\ \frac{x^n}{n!}, & t = 0. \end{cases}$$

Since $\frac{d^n \psi_t}{dx^n}(x) = e^{tx} > 0$, the function ψ_t is $n-$convex on for every $t \in$ and $t \mapsto \frac{d^n \psi_t}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.6 we also have that $t \mapsto$

$[x_0,\ldots,x_n;\psi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 4.7 we conclude that $t\mapsto\Lambda(\psi_t)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $t\mapsto\psi_t$ is not continuous for $t=0$), so it is exponentially convex. For this family of functions, $\mu_{t,q}(\Lambda,\Gamma_1)$, from (4.39), becomes

$$\mu_{t,q}(\Lambda,\Gamma_1)=\begin{cases}\left(\frac{\Lambda(\psi_t)}{\Lambda(\psi_q)}\right)^{\frac{1}{t-q}}, & t\neq q,\\ \exp\left(\frac{\Lambda(id\cdot\psi_t)}{\Lambda(\psi_t)}-\frac{n}{t}\right), & t=q\neq 0,\\ \exp\left(\frac{1}{n+1}\frac{\Lambda(id\cdot\psi_0)}{\Lambda(\psi_0)}\right), & t=q=0,\end{cases}$$

where "*id*" is the identity function. By Corollary 4.8 $\mu_{t,q}(\Lambda,\Gamma_1)$ is a monotone function in parameters t and q.

Since

$$\left(\frac{\frac{d^nf_t}{dx^n}}{\frac{d^nf_q}{dx^n}}\right)^{\frac{1}{t-q}}(\log x)=x,$$

using Theorem 4.5 it follows that:

$$M_{t,q}(\Lambda,\Gamma_1)=\log\mu_{t,q}(\Lambda,\Gamma_1),$$

satisfies

$$\alpha\leq M_{t,q}(\Lambda,\Gamma_1)\leq\beta.$$

Hence $M_{t,q}(\Lambda,\Gamma_1)$ is a monotonic mean.

Example 5.2. *Let us consider a family of functions*

$$\Gamma_2=\{g_t:(0,\infty)\to:t\in\}$$

defined by

$$g_t(x)=\begin{cases}\frac{x^t}{t(t-1)\cdots(t-n+1)}, & t\notin\{0,1,\ldots,n-1\},\\ \frac{x^j\log x}{(-1)^{n-1-j}j!(n-1-j)!}, & t=j\in\{0,1,\ldots,n-1\}.\end{cases}$$

Since $\frac{d^ng_t}{dx^n}(x)=x^{t-n}>0$*, the function* g_t *is* $n-$*convex for* $x>0$ *and* $t\mapsto\frac{d^ng_t}{dx^n}(x)$ *is exponentially convex by definition. Arguing as in Example 5.1 we get that the mappings* $t\mapsto\Lambda(g_t)$ *is exponentially convex.*

Hence, for this family of functions $\mu_{p,q}(\Lambda,\Gamma_2)$ *, from* (4.39)*, are equal to*

$$\mu_{t,q}(\Lambda,\Gamma_2)=\begin{cases}\left(\frac{\Lambda(g_t)}{\Lambda(g_q)}\right)^{\frac{1}{t-q}}, & t\neq q,\\ \exp\left((-1)^{n-1}(n-1)!\frac{\Lambda(g_0g_t)}{\Lambda(g_t)}+\sum\limits_{k=0}^{n-1}\frac{1}{k-t}\right), & t=q\notin\{0,1,\ldots,n-1\},\\ \exp\left((-1)^{n-1}(n-1)!\frac{\Lambda(g_0g_t)}{2\Lambda(g_t)}+\sum\limits_{\substack{k=0\\k\neq t}}^{n-1}\frac{1}{k-t}\right), & t=q\in\{0,1,\ldots,n-1\}.\end{cases}$$

Again, using Theorem 4.5 we conclude that

$$\alpha\leq\left(\frac{\Lambda(g_t)}{\Lambda(g_q)}\right)^{\frac{1}{t-q}}\leq\beta. \tag{5.41}$$

Hence $\mu_{t,q}(\Lambda,\Gamma_2)$ *is a mean and its monotonicity is followed by (4.38).*

Example 5.3. *Let*

$$\Gamma_3=\{\zeta_t:(0,\infty)\rightarrow:t\in(0,\infty)\}$$

be a family of functions defined by

$$\zeta_t(x)=\begin{cases}\frac{t^{-x}}{(-\log t)^n}, & t\neq 1;\\ \frac{x^n}{(n)!}, & t=1.\end{cases}$$

Since $\frac{d^n\zeta_t}{dx^n}(x)=t^{-x}$ *is the Laplace transform of a non-negative function (see [20]) it is exponentially convex. Obviously* ζ_t *are* $n-$*convex functions for every* $t>0$.

For this family of functions, $\mu_{t,q}(\Lambda,\Gamma_3)$ *, in this case for* $[\alpha,\beta]\subset\mathbb{R}^+$*, from (4.39) becomes*

$$\mu_{t,q}(\Lambda,\Gamma_3)\;=\;\begin{cases}\left(\frac{\Lambda(\zeta_t)}{\Lambda(\zeta_q)}\right)^{\frac{1}{t-q}}, & t\neq q;\\ exp\left(-\frac{\Lambda(id.\zeta_t)}{t\Lambda(\zeta_t)}-\frac{n}{t\log t}\right), & t=q\neq 1;\\ exp\left(-\frac{1}{n+1}\frac{\Lambda(id.\zeta_1)}{\Lambda(\zeta_1)}\right), & t=q=1,\end{cases}$$

where id is the identity function. By Corollary 4.8 $\mu_{p,q}(\Lambda,\Gamma_3)$ *is a monotone function in parameters t and q.*

Using Theorem 4.5 it follows that

$$M_{t,q}(\Lambda,\Gamma_3)=-L(t,q)log\mu_{t,q}(\Lambda,\Gamma_3),$$

satisfy

$$\alpha \leq M_{t,q}(\Lambda, \Gamma_3) \leq \beta.$$

This shows that $M_{t,q}(\Lambda, \Gamma_3)$ is a mean. Because of the inequality (4.38), this mean is monotonic. Furthermore, $L(t,q)$ is logarithmic mean defined by

$$L(t,q) = \begin{cases} \frac{t-q}{\log t - \log q}, & t \neq q; \\ \\ t, & t = q. \end{cases}$$

Example 5.4. *Let*

$$\Gamma_4 = \{\Lambda_t : (0,\infty) \rightarrow : t \in (0,\infty)\}$$

be a family of functions defined by

$$\Lambda_t(x) = \frac{e^{-x\sqrt{t}}}{(-\sqrt{t})^n}.$$

Since $\frac{d^n \Lambda_t}{dx^n}(x) = e^{-x\sqrt{t}}$ is the Laplace transform of a non-negative function (see [20]) it is exponentially convex. Obviously Λ_t are $n-$convex function for every $t > 0$.

For this family of functions, $\mu_{t,q}(\Lambda, \Gamma_4)$, in this case for $[\alpha, \beta] \subset \mathbb{R}^+$, from (4.39) becomes

$$\mu_{t,q}(\Lambda, \Gamma_4) = \begin{cases} \left(\frac{\Lambda(\Lambda_t)}{\Lambda(\Lambda_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Lambda(id.\Lambda_t)}{2\sqrt{t}\Lambda(\Lambda_t)} - \frac{n}{2t}\right), & t = q; \end{cases} \quad i = 1, 2.$$

By Corollary 4.8, it is a monotone function in parameters t and q.

Using Theorem 4.5 it follows that

$$M_{t,q}(\Lambda, \Gamma_4) = -\left(\sqrt{t} + \sqrt{q}\right) ln\mu_{t,q}(\Lambda, \Gamma_4),$$

satisfy

$$\alpha \leq M_{t,q}(\Lambda, \Gamma_4) \leq \beta.$$

This shows that $M_{t,q}(\Lambda, \Gamma_4)$ is a mean. Because of the above inequality (4.38), this mean is monotonic.

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Applications of Exponential Convexity

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Abstract

In this paper, we apply n-exponential convexity and log-convexity on a positive linear functional defined as the difference of the left hand side and right hand side of the inequalities from [3]. We obtain interesting inequalities and improvements of Hardy type inequality given in [3].

Keywords: Convex Function, Divided Difference, Generalized Montgomery Identity, ČEbyŠEv Functional, GrÜSs Inequality, Ostrowski Inequality, Exponential Convexity.

1 Introduction and Preliminaries

Steffensen [12] proved the following inequality: if $f,h:[\alpha,\beta]\to\mathbb{R}$, $0\leq h\leq 1$ and f is decreasing, then

$$\int_{\alpha}^{\beta} f(t)h(t)\,dt \leq \int_{\alpha}^{\alpha+\gamma} f(t)\,dt, \qquad \text{where } \gamma=\int_{\alpha}^{\beta} h(t)\,dt. \tag{1.1}$$

Several papers are devoted to studying generalizations of Steffensen's inequality (1.1). Convex functions are used in some generalization of Steffensen inequality. One recent generalization is given by Rabier [11].

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Theorem 1.1. *Let $\phi : [0,\infty) \to \mathbb{R}$ be convex and continuous with $\phi(0) = 0$. If $b > 0$ and $h \in L^\infty(0,b), h \geq 0$ and $\|h\|_\infty \leq 1$, then $h\phi' \in L^1(0,b)$ and*

$$\phi\Big(\int_0^b h(t)\,dt\Big) \leq \int_0^b h(t)\phi'(t)\,dt \tag{1.2}$$

In fact, Rabier's result, given in Theorem 1.1, is closely related to another generalization of Steffensen's inequality given by Pečarić [8].

Theorem 1.2. *Let $g : [a,b] \to \mathbb{R}$ be a nondecreasing and differentiable function and $f : I \to \mathbb{R}$ be a nondecreasing function (I is an interval in $\mathbb{R}$ such that $a, b, g(a), g(b) \in I$).*

(a) *If $g(x) \leq x$, then*
$$\int_a^b f(t)g'(t)\,dt \geq \int_{g(a)}^{g(b)} f(t)\,dt. \tag{1.3}$$

(b) *If $g(x) \geq x$, then the reverse of the above inequality holds.*

The assumptions of Theorem 1.2 can be weakened and differentiability of g can be replaced with absolute continuity. Indeed, for a nondecreasing function f, the function $F(x) = \int_a^x f(t)\,dt$ is well defined and satisfies $F^{(1)} = f$ at all except at most countably many points. For absolutely continuous nondecrasing function g the substitution $z = g(t)$ in the integral is justified (see [4, Corollary 20.5]), so

$$F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(z)\,dz = \int_a^b f(g(t))g'(t)\,dt \leq \int_a^b f(t)g'(t)\,dt, \tag{1.4}$$

where the last inequality holds when $g(t) \leq t$.

Steffensen's inequality (1.1) follows from Theorem 1.2 by making substitutions $g(x) \mapsto \int_a^x h(t+\alpha-a)\,dt + a$ and $f(x) \mapsto -f(x+\alpha-a)$ and taking $b = \beta - \alpha + a$.

Moreover, a convex function ϕ from Theorem 1.1 has a nondecreasing right-sided derivative $\phi_+^{(1)}$ such that $\phi(x) = \int_0^x \phi_+^{(1)}(t)\,dt$. Furthermore, for a function $h : [0,b] \to [0,1]$, the function $g(x) = \int_0^x h(t)dt$ is absolutely continuous and satisfies $g(x) \leq x$ and $g^{(1)} = h$. Therefore, by taking $a = 0$, $f = \phi_+^{(1)}$ and

$g(x) = \int_0^x h(t)\,dt$ in Theorem 1.2 (under the weaker assumptions) we get Theorem 1.1.

By replacing the equality

$$F(g(x)) = F(g(a)) + \int_{g(a)}^{g(x)} f(t)\,dt$$

with the n-th order Taylor expansion of the composition $F \circ g$, Fahad, Pečarić and Praljak [3] obtained the following generalization of Theorem 1.2.

Theorem 1.3. *Let $n \in \mathbb{N}$. Let $g : [a,b] \to \mathbb{R}$ and $F : I \to \mathbb{R}$ (where I is an interval in $\mathbb{R}$ such that $a,b,g(a),g(b) \in I$) are n times differentiable functions such that $g, g^{(1)}, \ldots g^{(n-1)}$, $F^{(1)}, F^{(2)}, \ldots, F^{(n)}$ are nondecreasing functions.*

(a) *If $g(x) \le x$, then*

$$F(g(b)) \le F(g(a)) + \sum_{k=1}^{n-1} F^{(k)}(g(a)) \sum_{i=k}^{n-1} B_{i,k}(g^{(1)}(a), \ldots, g^{(i-k+1)}(a)) \frac{(b-a)^i}{i!} + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(t) B_{n,k}(g^{(1)}(t), \ldots, g^{(n-k+1)}(t))\,dt.$$

(b) *If $g(x) \ge x$, then the reverse of the above inequality holds.*

We obtain the following corollary:

Corollary 1.4. *Let $n \in \mathbb{N}$, let $F : [0,b] \to \mathbb{R}$ be n times differentiable function such that $F^{(1)}, F^{(2)}, \ldots, F^{(n)}$ are nondecreasing functions and let $h : [0,b] \to [0,+\infty)$ be $n-1$ times differentiable function such that $h, h^{(1)}, \ldots, h^{(n-1)}$ are nonnegative.*

(a) *If $\int_0^x h(t)\,dt \le x$ for every $x \in [0,b]$, then*

$$F\left(\int_0^b h(t)\,dt\right) \le F(0) + \sum_{k=1}^{n-1} F^{(k)}(0) \sum_{i=k}^{n-1} B_{i,k}(h(0), h^{(1)}(0), \ldots, h^{(i-k)}(0)) \frac{b^i}{i!} + \int_0^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} F^{(k)}(t) B_{n,k}(h(t), h^{(1)}(t), \ldots, h^{(n-k)}(t))\,dt.$$

(b) If $x \leq \int_0^x h(t)\,dt$ for every $x \in [0,b]$, then the reverse of the above inequality holds.

Theorem 1.5. *Let $n \in \mathbb{N}$, h and F be as in Corollary 1.4, $k : [0,b] \to [0,+\infty)$ and denote $K_i(t) = \int_t^b \frac{(x-t)^{i-1}}{(i-1)!} k(x)\,dx$ for $i \in \mathbb{N}$.*

(a) If $\int_0^x h(t)\,dt \leq x$ for every $x \in [0,b]$, then

$$\int_0^b k(x)F\left(\int_0^x h(t)\,dt\right)dx \leq F(0)K_1(0) + \\ + \sum_{k=1}^{n-1} F^{(k)}(0) \sum_{i=k}^{n-1} B_{i,k}(h(0),h^{(1)}(0),\ldots,h^{(i-k)}(0))K_{i+1}(0) + \\ + \int_0^b K_n(t) \sum_{k=1}^{n} F^{(k)}(t)B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt.$$

(b) If $x \leq \int_0^x h(t)\,dt$ for every $x \in [0,b]$, then the reverse of the above inequality holds.

In 1929, the notion of exponential convexity was introduced by S. N. Bernstein in [1] and later on D. V. Widder [13] introduced these functions as a special of convex functions in a given interval (a,b). J. Pečarić and J. Perić [9] introduced the notion of n-exponentially convex functions. Much more about this class of functions has been given in [7].

We begin the section with the notions and results which we intend to use frequently. For details see [7] and [9].

Definition 1. A real valued function $\phi : I \to \mathbb{R}$ on an open interval I, $I \subset \mathbb{R}$, is called n-exponentially convex in the Jensen sense if

$$\sum_{j,k=1}^{n} \xi_j \xi_k \phi\left(\frac{x_j + x_k}{2}\right) \geq 0$$

is true for all $\xi_i \in \mathbb{R}$ and all $x_i \in I$, $i = 1,\ldots,n$.

A real valued function $\phi : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.6.

(i) *From above definition it is obvious that set of all n-exponentially convex functions on I form convex cone.*

(ii) *Less obvious is that a product of any two n-exponentially convex functions on I is again of the same type (see [7]) .*

(iii) *n-exponentially convex functions are invariant on taking admissible shifts and translations inside argument of the function i.e. if $x \mapsto f(x)$ is n-exponentially convex, then $x \mapsto f(x-c)$ and $x \mapsto f(x/\lambda)$ are also n-exponentially convex functions.*

Definition 2. A real valued function $\phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense, if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A real valued function $\phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 1.7. *It can be noted that a positive real valued function $\phi : I \to \mathbb{R}$ is* log*-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense, that is the following holds*

$$\xi_1^2 \phi(x) + 2\xi_1\xi_2\phi\left(\frac{x+y}{2}\right) + \xi_2^2\phi(y) \geq 0$$

for all $\xi_1, \xi_2 \in \mathbb{R}$ and $x, y \in I$.

If ϕ is $2-$exponentially convex, then ϕ is log*-convex. Converse is true if provided that ϕ is continuous also. $n-$exponentially convex functions are not, in general, exponentially convex. For example see [7].*

Proposition 1.8. *If $\Phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense then the matrix*

$$\left[\Phi\left(\frac{x_j + x_k}{2}\right)\right]_{j,k=1}^{m}$$

is positive semi definite matrix for all $m \in \mathbb{N}$, $m \leq n$.

In particular,

$$det\left[\Phi\left(\frac{x_j + x_k}{2}\right)\right]_{j,k=1}^{m} \geq 0,$$

for all $m \in \mathbb{N}$, $m = 1, 2, ..., n$.

Definition 3. The divided difference of order n, of real valued function ϕ defined on $[a,b]$, at distinct points in $[a,b]$ is defined recursively as:

$$[x_i;\phi] = \phi(x), (i = 0, 1, \ldots, n)$$

and

$$[x_0, x_1, \ldots, x_n;\phi] = \frac{[x_1, \ldots, x_n;\phi] - [x_0, \ldots, x_{n-1};\phi]}{x_n - x_0}$$

The value of $[x_0, x_1, \ldots, x_n;\phi]$ is independent of order of points.

Definition 4. A function $\phi : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$ if and only if for all choices of $(n+1)$ distinct points $x_0, x_1, \ldots, x_n \in [a,b]$, the nth order divided difference is non negative that is

$$[x_0, x_1, \ldots, x_n;\phi] \geq 0$$

for every $x_0, x_1, \ldots, x_n$ in $[a,b]$.

The properties related to n-convex function can be found in [10]. Let us also recall the following important theorem from [10].

Theorem 1.9. *Let ϕ be real valued function on $[a,b]$. If $\phi^{(n)}$ exists, then ϕ is n-convex if and only if $\phi^{(n)} \geq 0$.*

In [7], exponentially convexity method was given for n-convex functions. The next simple Lemma will be useful for extensions of this method.

Lemma 1.10. *Let $\phi : [0,\infty) \to \mathbb{R}$ be an n-convex function such that $\phi^{(n-1)}$ exists and $\phi^{(k)}(0) = 0$ for each $k = 2, 3, ..., n-1$. Then $\phi^{(k)}$ is non-decreasing for every $k = 1, 2, ...n-1$.*

Proof. By Proposition 1.9 we have that $\phi^{(n-1)}$ is non-decreasing. Since $\phi^{(n-1)}(0) = 0$, it follows that $\phi^{(n-1)}(x) \geq 0$ for every $x \in [0, \infty)$. By induction we conclude that $\phi^{(k)}$ is non-decreasing for every $k = 1, 2, ..., n-1$.

$\square$

2 $n-$exponential Convexity and Mean Value Theorems

We use an idea from [7] to give an elegant method for producing n-exponentially convex functions. Different variations of these method can be also found in [2], [5] and [6]. Let $\phi : I \to \mathbb{R}$ be n-times differentiable functions such that $\psi, \psi^{(1)}, \ldots \psi^{(n-1)}$ and $\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(n)}$ are nondecreasing functions. Let us define linear functional

$$\begin{aligned} \phi \mapsto \digamma(\phi) = \phi(\psi(a)) + \sum_{k=1}^{n-1} \phi^{(k)}(\psi(a)) \sum_{i=k}^{n-1} B_{i,k}(\psi^{(1)}(a), \ldots, \psi^{(i-k+1)}(a)) \frac{(b-a)^i}{i!} \\ + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum_{k=1}^{n} \phi^{(k)}(t) B_{n,k}(\psi^{(1)}(t), \ldots, \psi^{(n-k+1)}(t))\, dt - \phi(\psi(b)) \end{aligned} \tag{2.5}$$

From Theorem 1.3 it follows that $\digamma(\Phi) \geq 0$ if $\psi(x) \leq x$.

Theorem 2.1. *Let $\digamma$ be linear functional as defined in* (2.5). *Let $\phi \in C^{n+1}([0,a])$ with $\phi^{(k)}(0) = 0$ for $k = 2, 3, \ldots, n$. Then there exists $\xi \in [0,a]$ such that*

$$\digamma(\phi) = \phi^{(n+1)}(\xi) \digamma(\varphi_0)$$

where $\varphi_0(x) = \frac{x^{n+1}}{(n+1)!}$.

Proof. Let $m = \inf_{x \in [0,a]} \phi^{(n+1)}(x)$, $M = \sup_{x \in [0,a]} \phi^{(n+1)}(x)$. Let us define the functions $\phi_1, \phi_2 : I \to \mathbb{R}^+$ defined by

$$\begin{aligned} \phi_1(x) &= \frac{M}{(n+1)!} x^{n+1} - \phi(x) = M\varphi_0(x) - \phi(x) \\ \phi_2(x) &= \phi(x) - \frac{m}{(n+1)!} x^{n+1} = \phi(x) - m\varphi_0(x). \end{aligned} \tag{2.6}$$

Clearly, $\phi_1^{(k)}(0) = \phi_2^{(k)}(0) = 0$ for every $k = 2,3,\ldots,n$. Also $\phi_1^{(n+1)}(x) \geq 0$ and $\phi_2^{(n+1)}(x) \geq 0$ for every $x \in [0,a]$, so ϕ_1, ϕ_2 are $(n+1)$-convex. Therefore, Lemma 1.10 implies that $\phi_1^{(k)}, \phi_2^{(k)}$ are non-decreasing for each $k = 1,2,\ldots,n$. Hence, from Theorem1.3 we have $\digamma(\phi_1) \geq 0$ and $\digamma(\phi_2) \geq 0$ and

$$m\digamma(\varphi_0) \leq \digamma(\phi) \leq M\digamma(\varphi_0)$$

holds. So, Bolzano intermediate theorem ensures that there exist $\xi \in [0,a]$ with $\digamma(\phi) = \phi^{(n+1)}(\xi)\digamma(\varphi_0)$.

□

The following mean value theorem will enable us to generate various means.

Theorem 2.2. *Let $\digamma$ be linear functional as defined in (2.5). Let $\phi_1, \phi_2 \in C^{n+1}([0,a])$ with ${\phi_1}^{(k)}(0) = {\phi_2}^{(k)}(0) = 0$ for $k = 2,3,\ldots,n$. Then there exists $\xi \in [0,a]$ such that*

$$\frac{\digamma(\phi_1)}{\digamma(\phi_2)} = \frac{{\phi_1}^{(n+1)}(\xi)}{{\phi_2}^{(n+1)}(\xi)}, \tag{2.7}$$

assuming that denominators are not equal to zero.

Proof. Let us define $h : [0,a] \to \mathbb{R}$ in the following way:

$$h(x) = \digamma(\phi_2)\phi_1(x) - \digamma(\phi_1)\phi_2(x).$$

Clearly, $h \in C^{n+1}([0,a])$ and $h^{(k)}(0) = 0$ for every $k = 2,3,\ldots,n$. Therefore, from above theorem, it follows that there exists $\xi \in [0,a]$ such that $\digamma(h) = h^{(n+1)}(\xi)\digamma(\varphi_0)$, where $\varphi_0(x) = \frac{x^{n+1}}{(n+1)!}$.

Since

$$\digamma(h) = \digamma(\phi_2)\digamma(\phi_1) - \digamma(\phi_1)\digamma(\phi_2) = 0$$

and

$$h^{(n+1)}(\xi)\digamma(\varphi_0) = [\digamma(\phi_2){\phi_1}^{(n+1)}(\xi) - \digamma(\phi_1){\phi_2}^{(n+1)}(\xi)]\digamma(\varphi_0)$$

it follows,

$$\digamma(\phi_2){\phi_1}^{(n+1)}(\xi) - \digamma(\phi_1){\phi_2}^{(n+1)}(\xi) = 0$$

and hence we get,

$$\frac{\digamma(\phi_1)}{\digamma(\phi_2)} = \frac{{\phi_1}^{(n+1)}(\xi)}{{\phi_2}^{(n+1)}(\xi)}.$$

□

Remark 2.3. *If the inverse of* $\frac{{\phi_1}^{(n+1)}}{{\phi_2}^{(n+1)}}$ *exists then various types of means can be defined by using* (2.7). *That is,*

$$\xi = \left(\frac{{\phi_1}^{(n+1)}}{{\phi_2}^{(n+1)}}\right)^{-1}\left(\frac{\digamma(\phi)}{\digamma(\psi)}\right)$$

Now we will produce n-exponentially and exponentially convex functions by using the defined functional. We use an idea from [9].

Theorem 2.4. *Let* $I \subset \mathbb{R}$ *be an open interval, and* $\Gamma = \{\eta_t | t \in I\}$ *is a family of* $n+1$ *time continuously differentiable functions defined on* $J = [0,\infty)$ *with* $\eta^{(k)}(0) = 0$ *for every* $k = 2,3,\ldots,n$, *such that for every choice of* $n+2$ *mutually different points* $x_0, x_1, \ldots, x_{n+1}$ *in* J *the function* $t \mapsto [x_0, x_1, \ldots, x_{n+1}; \eta_t]$ *is* l*-exponentially convex on* I.

Consider the functional $\phi \mapsto \digamma(\phi)$ *as given in* (2.5). *Then,*

(i) $t \mapsto \digamma(\eta_t)$ *is an* l*-exponentially convex function in the Jensen sense on* I *and the matrix* $\left[\digamma\left(\eta_{\frac{p_j+p_k}{2}}\right)\right]_{j,k=1}^{m}$ *is positive semi definite matrix for all* $m \in \mathbb{N}$, $m \leq l$, $p_1, p_2, \ldots, p_m \in I$.

In particular,

$$det\left[\digamma\left(\eta_{\frac{p_j+p_k}{2}}\right)\right]_{j,k=1}^{m} \geq 0$$

for all $m \in \mathbb{N}$, $m = 1,2,\ldots,l$.

(ii) If $t \mapsto \digamma(\eta_t)$ *is continuous on* I *then it is* l*-exponentially convex on* I.

Proof. (i) Let $\xi_j \in \mathbb{R},\ p_j \in I,\ (j = 1,\ldots,l)$ be arbitrary and define auxiliary function $h : [a,b] \to \mathbb{R}$ by

$$h(x) = \sum_{j,k=1}^{l} \xi_j \xi_k \eta_{\frac{p_j+p_k}{2}}(x).$$

Clearly, $h^{(k)}(0) = 0$ for every $k = 2,3,\ldots,n$. Now, since $t \mapsto [x_0,x_1,\ldots,x_{n+1};\eta_t]$ is l-exponentially convex in the Jensen's sense. So, we have:

$$[x_0,x_1,...x_{n+1};h] = \sum_{j,k=1}^{l} \xi_j\xi_k[x_0,x_1,...x_{n+1};\eta_{\frac{p_j+p_k}{2}}] \geq 0,$$

so h is $(n+1)$-convex. Therefore, Lemma 1.10 gives that $h^{(1)},...,h^{(n)}$ are non-decreasing. Hence, Theorem 1.3 implies $\digamma(h) \geq 0$. Therefore, $t \mapsto \digamma(\eta_t)$ is l-exponentially convex in the Jensen sense and the rest follows from Proposition 1.8.

(ii) Follows from part (i) and definition of l-exponentially convex functions.

□

Corollary 2.5. *Let $I \subset \mathbb{R}$ be an open interval, and $\Gamma = \{\eta_t | t \in I\}$ is a family of $n+1$-time continuously differentiable functions defined on $J = [0,\infty)$ with $\eta^{(k)}(0) = 0$ for every $k = 2,3,\ldots,n$, such that for every choice of $n+2$ mutually different points $x_0,x_1,\ldots,x_{n+1}$ in J the function $t \mapsto [x_0,x_1,\ldots,x_{n+1};\eta_t]$ is exponentially convex on I.*

Consider the functional $\phi \mapsto \digamma(\phi)$ as given in (2.5). Then,

(i) $t \mapsto \digamma(\eta_t)$ is an exponentially convex function in the Jensen sense on I and the matrix $\left[\digamma\left(\eta_{\frac{p_j+p_k}{2}}\right)\right]_{j,k=1}^{m}$ is positive semi definite matrix for all $m \in \mathbb{N}$, $p_1,p_2,\ldots,p_m$ in I.

In particular,

$$det\left[\digamma\left(\eta_{\frac{p_j+p_k}{2}}\right)\right]_{j,k=1}^{m} \geq 0$$

for all $m \in \mathbb{N}$, $m = 1,2,\ldots,l$.

(ii) If $t \mapsto \digamma(\eta_t)$ is continuous on I then it is exponentially convex on I.

Theorem 2.6. *Let $I \subset \mathbb{R}$ be an open interval, and $\Gamma = \{\eta_t | t \in I\}$ is a family of $n+1$ time continuously differentiable functions defined on $J = [0,\infty)$ with $\eta^{(k)}(0) = 0$ for every $k = 2,3,\ldots n$, such that for every choice of $n+2$ mutually different points $x_0,x_1,\ldots,x_{n+1}$ in J the function $t \mapsto [x_0,x_1,\ldots,x_k;\eta_t]$ is*

2-*exponentially convex.*

Consider the functional $\phi \mapsto \digamma(\phi)$ *as given in* (2.5)*. Then,*

(i) If $t \mapsto \digamma(\eta_t)$ *is continuous on* I*, then it is* 2*-exponentially convex function on* I*. If, in addition,* $t \mapsto \digamma(\eta_t)$ *is is strictly positive then it is* log*-convex as well. Moreover, following inequality holds:*

$$\Big(\digamma(\eta_s)\Big)^{t-r} \leq \Big(\digamma(\eta_r)\Big)^{t-s}\Big(\digamma(\eta_t)\Big)^{s-r} \tag{2.8}$$

for every $r, s, t \in I$ *where* $r < s < t$*.*

(ii) If $t \mapsto \digamma(\eta_t)$ *is strictly positive and differentiable on* I*, then for every* $p, q, u, v \in I$ *such that* $p \leq u$*,* $q \leq v$*, we have:*

$$\mu_{p,q}(\digamma, \Gamma) \leq \mu_{u,v}(\digamma, \Gamma) \tag{2.9}$$

where

$$\mu_{p,q}(\digamma, \Gamma) = \begin{cases} \left(\frac{\digamma(\eta_p)}{\digamma(\eta_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\digamma(\eta_p)}{\digamma(\eta_p)}\right), & p = q. \end{cases} \tag{2.10}$$

Proof. (i) Follows from Theorem 2.4 and Remark 1.7.

(ii) Since, in particular, $t \mapsto \digamma(\eta_t)$ is strictly positive and continuous, so by (i) we have that $t \mapsto \digamma(\eta_t)$ is log-convex on I, that is, the function $t \mapsto \log \digamma(\eta_t)$ is convex on I. Therefore, we have:

$$\frac{\log \digamma(\eta_p) - \log \digamma(\eta_q)}{p-q} \leq \frac{\log \digamma(\eta_u) - \log \digamma(\eta_v)}{u-v} \tag{2.11}$$

for $p \leq u$, $q \leq v$, $p \neq q$, $u \neq v$. So, we get:

$$\mu_{p,q}(\digamma, \Gamma) \leq \mu_{u,v}(\digamma, \Gamma)$$

Case $p = q$ and $u = v$ follows from equation (2.11) as limit cases.

□

3 Applications

In this section, we shall give applications of the results given in the previous section.

Example 3.1. *The following family $\Gamma' = \{\phi_p : p \in (n+1,\infty)\}$, where*

$$\phi_p(t) = \frac{t^p}{\prod\limits_{i=0}^{n}(p-i)} \tag{3.12}$$

satisfies the assumptions of Theorem 2.6. Moreover, Ω' is linear functional defined with (2.5).*By using (2.10) we compute the mean $\mu_{p,q}$ and (2.9) holds.*

$$\mu_{p,q}(F',\Gamma') = \begin{cases} \left(\dfrac{\omega_{0,p}(\psi(a)) - \omega_{0,p}(\psi(b)) + \sum\limits_{k=1}^{n-1} \omega_{k,p}(\psi(a))\kappa + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum\limits_{k=1}^{n} \omega_{k,p}(t)\lambda_{n,k}(t)\,dt}{\omega_{0,q}(\psi(a)) - \omega_{0,q}(\psi(b)) + \sum\limits_{k=1}^{n-1} \omega_{k,q}(\psi(a))\kappa + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum\limits_{k=1}^{n} \omega_{k,q}(t)\lambda_{n,k}(t)\,dt}, \right)^{\frac{1}{p-q}} & p \neq q \\ \exp\left(\dfrac{\chi_0(\psi(a)) - \chi_0(\psi(b)) + \sum_{k=1}^{n-1} \chi_k(\psi(a))\kappa + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum\limits_{k=1}^{n} \chi_k(t)\lambda_{n,k}(t)\,dt}{\omega_{0,p}(\psi(a)) - \omega_{0,p}(\psi(b)) + \sum\limits_{k=1}^{n-1} \omega_{k,p}(\psi(a))\kappa + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} \sum\limits_{k=1}^{n} \omega_{k,p}(t)\lambda_{n,k}(t)\,dt} \right), & p = q, \end{cases}$$

where $\lambda_{i,k}(s) = B_{i,k}(\psi^{(1)}(s), \ldots, \psi^{(i-k+1)}(s))$, $\omega_{k,p}(s) = \frac{s^{p-k}}{\prod\limits_{i=k}^{n}(p-i)}$,

$$\chi_k(s) = \frac{\prod\limits_{i=k}^{n}(p-i)s^{p-k}\ln(s) - s^{p-k}\sum\limits_{j=k}^{n}\prod\limits_{i=k, i\neq j}^{n}(p-i)}{\prod\limits_{i=k}^{n}(p-i)^2}$$

and

$$\kappa = \sum_{i=k}^{n-1} \lambda_{i,k}(a)\frac{(b-a)^i}{i!}$$

where ψ is as in (2.5). *By using* (2.8) *we obtain following inequality:*

$$\left(\omega_{0,q}(\psi(a))-\omega_{0,q}(\psi(b))+\sum_{k=1}^{n-1}\omega_{k,q}(\psi(a))\kappa+\int_a^b\frac{(b-t)^{n-1}}{(n-1)!}\sum_{k=1}^{n}\omega_{k,q}(t)\lambda_{n,k}(t)\,dt\right)^{r-p}\leq$$
$$\left(\omega_{0,p}(\psi(a))-\omega_{0,p}(\psi(b))+\sum_{k=1}^{n-1}\omega_{k,p}(\psi(a))\kappa+\int_a^b\frac{(b-t)^{n-1}}{(n-1)!}\sum_{k=1}^{n}\omega_{k,p}(t)\lambda_{n,k}(t)\,dt\right)^{r-q}$$
$$\left(\omega_{0,r}(\psi(a))-\omega_{0,r}(\psi(b))+\sum_{k=1}^{n-1}\omega_{k,r}(\psi(a))\kappa+\int_a^b\frac{(b-t)^{n-1}}{(n-1)!}\sum_{k=1}^{n}\omega_{k,r}(t)\lambda_{n,k}(t)\,dt\right)^{q-p}$$

where $n<p<q<r$.

We can proceed with our construction also on inequalities given in Corollary 1.4 and Theorem 1.5. We define the linear functionals

$$\phi\mapsto\digamma_1(\phi)=\int_0^b\frac{(b-t)^{n-1}}{(n-1)!}\sum_{k=1}^{n}\phi^{(k)}(t)B_{n,k}(h(t),h^{(1)}(t),\dots,h^{(n-k)}(t))\,dt - \phi\left(\int_0^b h(t)\,dt\right). \quad (3.13)$$

where h is as in (a)-part of the Corollary 1.4. From Corollary 1.4 it follows that if ϕ is $(n+1)$-convex and $\phi^{(k)}(0)=0$, for $k=1,2,\dots,n$, then $\digamma_1(\Phi)\geq 0$.

$$\phi\mapsto\digamma_2(\phi)=\int_0^b K_n(t)\sum_{k=1}^{n}\Phi^{(k)}(t)B_{n,k}(h(t),h^{(1)}(t),\dots,h^{(n-k)}(t))\,dt - \int_0^b k(x)\phi\left(\int_0^x h(t)\,dt\right)dx \quad (3.14)$$

where k and K_n are as in Theorem 1.5 and h is as in Theorem 1.5(a). From Theorem 1.5 it follows that if ϕ is $(n+1)$-convex and $\phi^{(k)}(0)=0$, for $k=1,2,\dots,n$, then $\digamma_2(\Phi)\geq 0$.

Corollary 3.2. *Let $I\subset\mathbb{R}$ be an open interval, and $\Gamma'=\{\eta_t|t\in I\}$ is a family of $n+1$ time continuously differentiable functions defined on $J=[0,\infty)$ with $\eta^{(k)}(0)=0$ for every $k=1,2,\dots,n$, such that for every choice of $n+2$ mutually different points $x_0,x_1,\dots,x_{n+1}$ in J the function $t\mapsto[x_0,x_1,\dots,x_{n+1};\eta_t]$ is*

2-exponentially convex on I.

Consider the functional $\phi \mapsto \digamma_i(\phi)$, for $i = 1,2$ as given in (3.13) *and* (3.14). *Then,*

(i) If $t \mapsto \digamma_i(\eta_t)$ is continuous on I, then it is 2-exponentially convex function on I. If, in addition, $t \mapsto \digamma_i(\eta_t)$ is is strictly positive then it is log-convex as well. Moreover, following inequality holds:

$$\left(\digamma_i(\eta_s)\right)^{t-r} \le \left(\digamma_i(\eta_r)\right)^{t-s}\left(\digamma_i(\eta_t)\right)^{s-r} \tag{3.15}$$

for every $r,s,t \in I$ where $r < s < t$.

(ii) If $t \mapsto \digamma_i(\eta_t)$ is strictly positive and differentiable on I, then for every $p,q,u,v \in I$ such that $p \le u$, $q \le v$, we have:

$$\mu_{p,q}(\digamma_i,\Gamma') \le \mu_{u,v}(\digamma_i,\Gamma') \tag{3.16}$$

Example 3.3. *Let h be as in Corollary 1.4. The following family $\Gamma_1' = \{\phi_p : p \in (n+1,\infty)\}$, where ϕ_p is defined by (3.12) and satisfies the assumptions of Corollary 3.2. Moreover, Ω_1' is linear functional defined with* (3.13). *For the given family, we compute the mean and (3.16) holds:*

$$\mu_{p,q}(\digamma_1',\Gamma_1') = \begin{cases} \left(\dfrac{\int_0^b \frac{(b-t)^{n-1}}{(n-1)!} \sum\limits_{k=1}^{n} \frac{t^{p-k}}{\prod_{i=k}^{n}(p-i)} B_{n,k}(h(t),h^{(1)}(t),\dots,h^{(n-k)}(t))\,dt - \frac{\left(\int_0^b h(t)\,dt\right)^p}{\prod_{i=0}^{n}(p-i)}}{\int_0^b \frac{(b-t)^{n-1}}{(n-1)!} \sum\limits_{k=1}^{n} \frac{t^{q-k}}{\prod_{i=k}^{n}(q-i)} B_{n,k}(h(t),h^{(1)}(t),\dots,h^{(n-k)}(t))\,dt - \frac{\left(\int_0^b h(t)\,dt\right)^q}{\prod_{i=0}^{n}(q-i)}} \right)^{\frac{1}{p-q}}, & p \ne q > n, \\ \exp\left(\dfrac{\int_0^b \frac{(b-t)^{n-1}}{(n-1)!} \sum\limits_{k=1}^{n} \left[\chi_k(t) B_{n,k}(h(t),h^{(1)}(t),\dots,h^{(n-k)}(t))\,dt\right] - \chi_0(\int_0^b h(t)\,dt)}{\int_0^b \frac{(b-t)^{n-1}}{(n-1)!} \sum\limits_{k=1}^{n} \frac{t^{p-k}}{\prod_{i=k}^{n}(p-i)} B_{n,k}(h(t),h^{(1)}(t),\dots,h^{(n-k)}(t))\,dt - \frac{\left(\int_0^b h(t)\,dt\right)^p}{\prod_{i=0}^{n}(p-i)}} \right), & p = q > n, \end{cases}$$

where χ_k is as Example 3.1. By using (3.15), *we get the following inequality:*

$$\left(\int_0^b \frac{(b-t)^{n-1}}{(n-1)!}\sum_{k=1}^{n}\frac{t^{q-k}}{\prod\limits_{i=k}^{n}(q-i)}B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt-\frac{\left(\int_0^b h(t)\,dt\right)^q}{\prod\limits_{i=0}^{n}(q-i)}\right)^{r-p} \leq$$

$$\left(\int_0^b \frac{(b-t)^{n-1}}{(n-1)!}\sum_{k=1}^{n}\frac{t^{p-k}}{\prod\limits_{i=k}^{n}(p-i)}B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt-\frac{\left(\int_0^b h(t)\,dt\right)^p}{\prod\limits_{i=0}^{n}(p-i)}\right)^{r-q}$$

$$\cdot\left(\int_0^b \frac{(b-t)^{n-1}}{(n-1)!}\sum_{k=1}^{n}\frac{t^{r-k}}{\prod\limits_{i=k}^{n}(r-i)}B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt-\frac{\left(\int_0^b h(t)\,dt\right)^r}{\prod\limits_{i=0}^{n}(r-i)}\right)^{q-p}$$

where $n<p<q<r$.

Example 3.4. *The following family $\Gamma_2'=\{\phi_p : p\in(n+1,\infty)\}$, where ϕ_p is defined by (3.12) and satisfies the assumptions of Corollary 3.2. Moreover, Ω_2' is linear functional defined with* (3.14)*. For the given family, we compute the mean and (3.16) holds:*

$$\mu_{p,q}(F'_2,\Gamma'_2)=\begin{cases}\left(\dfrac{\int_0^b K_n(t)\sum\limits_{k=1}^{n}\frac{t^{p-k}}{\prod\limits_{i=k}^{n}(p-i)}B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt-\int_0^b k(x)\frac{\left(\int_0^x h(t)\,dt\right)^p}{\prod\limits_{i=0}^{n}(p-i)}\,dx}{\int_0^b K_n(t)\sum\limits_{k=1}^{n}\frac{t^{q-k}}{\prod\limits_{i=k}^{n}(q-i)}B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt-\int_0^b k(x)\frac{\left(\int_0^x h(t)\,dt\right)^q}{\prod\limits_{i=0}^{n}(q-i)}\,dx}\right)^{\frac{1}{p-q}}, & p\neq q,\\[2ex] \exp\left(\dfrac{\int_0^b K_n(t)\sum\limits_{k=1}^{n}\chi_k(t)(p-i)B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt-\int_0^b k(x)\chi_0(\int_0^x h(t)\,dt)\,dx}{\int_0^b K_n(t)\sum\limits_{k=1}^{n}\frac{t^{p-k}}{\prod\limits_{i=k}^{n}(p-i)}B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt-\int_0^b k(x)\frac{\left(\int_0^x h(t)\,dt\right)^p}{\prod\limits_{i=0}^{n}(p-i)}\,dx}\right), & p=q,\end{cases}$$

By using (3.15)*, we get following intersting inequalitiy.*

If $n<p<q<r$ then,

$$\left(\int_0^b K_n(t)\sum_{k=1}^n \frac{t^{q-k}}{\prod\limits_{i=k}^n (q-i)} B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt - \int_0^b k(x)\frac{(\int_0^x h(t)\,dt)^q}{\prod\limits_{i=0}^n (q-i)}\,dx\right)^{r-p}$$
$$\leq \left(\int_0^b K_n(t)\sum_{k=1}^n \frac{t^{p-k}}{\prod\limits_{i=k}^n (p-i)} B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt - \int_0^b k(x)\frac{(\int_0^x h(t)\,dt)^p}{\prod\limits_{i=0}^n (p-i)}\,dx\right)^{r-q}$$
$$\left(\int_0^b K_n(t)\sum_{k=1}^n \frac{t^{r-k}}{\prod\limits_{i=k}^n (r-i)} B_{n,k}(h(t),h^{(1)}(t),\ldots,h^{(n-k)}(t))\,dt - \int_0^b k(x)\frac{(\int_0^x h(t)\,dt)^r}{\prod\limits_{i=0}^n (r-i)}\,dx\right)^{q-p}.$$

The next inequality can be found in [3]. For $p>1$, $l>1$ and a nonnegative function h such that $x^{1-p/l}h \in L^p(0,b)$,

$$\int_0^b x^{-l}\left(\int_0^x h(t)\,dt\right)^p dx \leq \left(\frac{p}{l-1}\right)^p \int_0^b x^{p-l}h(x)^p\,dx. \tag{3.17}$$

Observe that the classical Hardy's inequality is inequality (3.17) with $b=\infty$.

The following inequality (see [3]) is similar to (3.17):

$$\int_0^b \left(\frac{1}{x}\int_0^x h(t)\,dt\right)^p dx \quad \leq \quad \frac{p}{p-1}\left[\int_0^b h(t)^p\,dt\right]^{\frac{1}{p}} \left[\int_0^b \left(1-\left(\frac{t}{b}\right)^{p-1}\right)^{\frac{p}{p-1}} dt\right]^{\frac{p-1}{p}}. \tag{3.18}$$

where h is as in Theorem 1.5.

The following example give us improvement of (3.18).

Example 3.5. *By taking $\phi(t)=t^p$, $p>1$, $n=1$ and $k(x)=x^{-p}$ in* (3.14) *and using* (3.15) *for $i=2$, we get*

$$\frac{p}{p-1}\int_0^b \left(1-\left(\frac{t}{b}\right)^{p-1}\right)h(t)\,dt - \int_0^b\left(\frac{1}{x}\int_0^x h(t)\,dt\right)^p dx \geq$$
$$\left(\frac{q}{q-1}\int_0^b \left(1-\left(\frac{t}{b}\right)^{q-1}\right)h(t)\,dt - \int_0^b\left(\frac{1}{x}\int_0^x h(t)\,dt\right)^q dx\right)^{\frac{r-p}{r-q}}$$
$$\left(\frac{r}{r-1}\int_0^b \left(1-\left(\frac{t}{b}\right)^{r-1}\right)h(t)\,dt - \int_0^b\left(\frac{1}{x}\int_0^x h(t)\,dt\right)^r dx\right)^{\frac{p-q}{r-q}}$$

where $1 < p < q < r$.

By using Hölder's inequality on left hand side, we get

$$\frac{p}{p-1}\left[\int_0^b h(t)^p\,dt\right]^{\frac{1}{p}}\left[\int_0^b\left(1-\left(\frac{t}{b}\right)^{p-1}\right)^{\frac{p}{p-1}}dt\right]^{\frac{p-1}{p}}-\int_0^b\left(\frac{1}{x}\int_0^x h(t)\,dt\right)^p dx \geq$$
$$\left(\frac{q}{q-1}\int_0^b\left(1-\left(\frac{t}{b}\right)^{q-1}\right)h(t)\,dt-\int_0^b\left(\frac{1}{x}\int_0^x h(t)\,dt\right)^q dx\right)^{\frac{r-p}{r-q}}$$
$$\left(\frac{r}{r-1}\int_0^b\left(1-\left(\frac{t}{b}\right)^{r-1}\right)h(t)\,dt-\int_0^b\left(\frac{1}{x}\int_0^x h(t)\,dt\right)^r dx\right)^{\frac{p-q}{r-q}}$$

which is improvement of (3.18).

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Solution of Reaction Diffusion Problem using Homotopy Perturbation Method and Differential Transformation Method: A Comparative Study

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Abstract

In recent years, a new difference scheme with high accuracy has been applied for solving convection-diffusion equation [3]. In this paper an application of Homotopy perturbation method (HPM) is used to solve linear and non-linear diffusion-reaction problem (NDRP). Diffusion-Reaction equations have special importance in engineering and sciences and constitute a good model for many systems in various fields.We tried to compare the differential transform method (DTM) and HPM for solving time dependent reaction-diffusion equations and found that the proposed method HPM are comparable with the results of DTM for small parameter values but differed at large parameters.The proper implementation of He's Homotopy perturbation method can extremely minimize the size of work if compared with the existing differential transformation method.

Keywords: Homotopy Perturbation Method, Differential Transformation Method, Reaction Diffusion Problem.

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1 Introduction

In the recent years, with the rapid development of nonlinear science, the development of numerical techniques for solving non-linear equations is a subject of considerable interest, and many scientists and engineers have done the excellent work [8]. The applications of Homotopy theory have become a powerful mathematical tool in the nonlinear problems [14]. The HPM has been widely used by scientists and engineers to study the linear and Non-linear problems [13]. As we all know, there are many effective methods that are applied to investigate the explicit solutions of various equations. Compared with other methods, the HPM always deforms the difficult problem into a simple and easily solvable one, which is a coupling of the traditional perturbation method and Homotopy in topology. With this method, a series solution can be obtained that is usually rapidly convergent and with easily computable components [5]. Geng and Cui [4] first pointed out the effective approach to such problems using the Homotopy perturbation, variational iteration, and numerical methods. Exponential methods are also applied for one-dimensional reaction diffusion problems where non-linearity treatment is made by Taylor series expansion [12]. In this paper, we considered reaction diffusion models with convective terms.

We apply the differential transform method as an alternative to other existing methods in solving linear and non-linear systems of partial differential equations. The concept of differential transform method is first introduced by Zhou [16] in solving linear and non-linear initial value problems in electrical circuit analysis.

The performance of DTM to solve initial value problem was studied by Jang et al. [9]. They have applied two-dimensional differential transform method to solve partial differential equations. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method computationally takes a long time for large orders. The differential transform method is an iterative method for obtaining analytic Taylor series solutions of ordinary or partial differential equations.

DTM and HPM are very efficient and useful methods to find the numerical and analytic solutions of linear, non-linear differential equations, delay differential equations as well as integral equations.

2 Mathematical Formulation

In this study we are going to discuss one dimensional time dependent Cauchy reaction diffusion problems such as [9]:

$$\left(\frac{\partial u(x,t)}{\partial t}\right) = D\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) + r(x,t)u(x,t), \quad (x,t) \in \Omega \subset R^2 \tag{2.1}$$

where u(x, t) is the concentration, r(x, t) is the reaction parameter and $D > 0$ is the diffusion coefficient, are subject to the initial and boundary conditions

$$u(x,0) = g(x), x \in R \tag{2.2}$$

$$u(0,t) = f_0(t) \ \ t \in R, \ \frac{\partial u(1,t)}{\partial x} = f_1(t) \ \ t \in R \tag{2.3}$$

The problem given by Eq. (1) and (2) is called the characteristic Cauchy reaction diffusion problem in domain $\Omega = R \times R_+$, while the problem given by Eq. (1) and Eq. (3) is called the non-characteristic Cauchy reaction diffusion problem in the domain $\Omega = R \times R_+$. Eq. (1) is referred as Kolmogorov PetrovskyPiskunov equation for $r(x,t) = 1$ and the prescribed initial and boundary conditions are chosen based on the model given by Bataineh et al. [2]. Various values of $f_0(t)$ and $f_1(t)$ are considered based on the case studies made by Bataineh et al. [2].

The solution of these problems is attempted by using the DTM and HPM. Although DTM is a good approach for application in various fields of science and engineering, but it has some drawbacks as DTM provides the series solution of the given problem, which does not explain the real behavior of the problem [6]. HPM has the flexibility to find the solution of linear & non-linear boundary value problems for small as well as large parameters [7].

The considered reaction terms are specified in different special class of reaction terms defined by Othman et al. [11] and Yildirim [15].

2.1 Applications

2.1.1 Case-(i)

Let us consider a steady reaction equation (i.e. reaction at a fixed time level), $r(x,t) = r(x)$ only, where $r(x) = (-1-4x^2)$. The reduced form of equation (1) can be written as

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + (-1-4x^2)u(x,t),\ (x,t) \in \Omega \subset R^2 \tag{2.4}$$

Initial and boundary conditions are given by [2]

$$u(x,0) = e^{x^2},\ x \in R \tag{2.5}$$

$$u(0,t) = e^t,\ t \in R \ and\ \frac{\partial u(1,t)}{\partial x} = 0,\ t \in R \tag{2.6}$$

The initial condition is considered based on the case studies made by Bataineh et al. [2] with $D = 1$ and the boundaries are due to Arrhenius combustion nonlinearity condition or combustion nonlinearity with activation energy but no ignition temperature cutoff in a rector algorithm.

2.1.2 Case-(ii)

In this case we have considered a time dependent reaction term (Yildirim [15]), $r(x,t) = r(x,t)$, where $r(x,t) = 2t-2-4x^2$. In this case, Eq. (1) will be reduced in the form:

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + (2t-2-4x^2)u(x,t),\ (x,t) \in \Omega \subset R^2 \tag{2.7}$$

Initial and boundary conditions are described by-

$$u(x,0) = e^{x^2},\ x \in R \tag{2.8}$$

$$u(0,t) = e^{t^2},\ t \in R \ and\ \frac{\partial u(1,t)}{\partial x} = 0,\ t \in R \tag{2.9}$$

In this work, we relies mainly on two of the most recently methods, the HPM and DTM. The two methods, which accurately compute the solutions in a series form, are of great interest to applied sciences. The effectiveness and the usefulness of both methods are demonstrated by finding exact solutions to the above problems described in case (i) and case (ii) that will be investigated. However, each method has its own characteristics and significance that will be examined.

2.2 Solution Methods

The Cauchy reaction diffusion equations stated above are 1st approximated by DTM.

2.2.1 One dimensional differential transformation

The differential transformation of k^{th} derivative of a function $s(x)$ is defined as

$$S(k) = \frac{1}{k!}\left[\frac{d^k s(x)}{dx^k}\right]_{(x=x_0)} \tag{2.10}$$

The transformed value of $s(x)$ is denoted by $S(k)$ the inverse transformation of $S(k)$ is defined as

$$s(x) = \sum_{k=0}^{\infty} S(k)(x-x_0)^k \tag{2.11}$$

By using (10) and (11) we get

$$s(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k s(x)}{dx^k}\right]_{(x=x_0)} (x-x_0)^k \tag{2.12}$$

The transformed term $s(x)$ represented by Eq. (12) can be derived and approximated by Taylor series expansion at $x = x_0$.

2.2.2 Two dimensional differential transformation

Suppose a function of two variables $u(x,y)$ analytic in the domain R, and let $(x,y) = (x_0,y_0)$ be a fixed point in this domain. The function $u(x,y)$ is then represented by a power series whose center at located is (x_0,y_0). The differential transformation of function $u(x,y)$ is of the form

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} u(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)} \tag{2.13}$$

The inverse transform is defined as

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)(x-x_0)^k(y-y_0)^h \tag{2.14}$$

By Eq. (13) and (14) we get

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k\partial y^h}\right](x-x_0)^k(y-y_0)^h \tag{2.15}$$

After using $(x_0,y_0)=(0,0)$, in Eq. (15) we get a finite series as

$$u(x,y)=\sum_{k=0}^{m}\sum_{h=0}^{n}U(k,h)x^ky^h \ \ at\ \ (0,0) \tag{2.16}$$

The final form of the solution gives the approximate solution in a series form which will be comparable with Adomian decomposition method (Lesnic [10]).

2.3 Application of DTM

The application of the differential transformation method (DTM) and its operation on solving the stated reaction diffusion problem Eq. (1) can be represented as,

$$\frac{\partial(\sum_{k=0}^{m}\sum_{h=0}^{n}U(k,h)x^kt^h)}{\partial t}=D\frac{\partial^2(\sum_{k=0}^{m}\sum_{h=0}^{n}U(k,h)x^kt^h)}{\partial x^2}+\sum_{k=0}^{m}\sum_{h=0}^{n}R(k,h)U(k,h)x^kt^h \tag{2.17}$$

$$\sum_{k=0}^{m}\sum_{h=0}^{n}U(k,h)x^kt^{h-1}h=D\left[\sum_{k=0}^{m}\sum_{h=0}^{n}U(k,h)k(k-1)x^{k-2}t^h)\right]+\sum_{k=0}^{m}\sum_{h=0}^{n}R(k,h)U(k,h)x^kt^h) \tag{2.18}$$

Comparing both side coefficients of x^h and t^k we get following transformation

$$U(k,h+1)=\frac{1}{h+1}\left[D(k+1)(k+2)U(k+2,h)\right]+\frac{1}{h+1}\left[\sum_{a=0}^{k}\sum_{b=0}^{h}U(k-a,b)R(a,h-b)\right] \tag{2.19}$$

Considering the differential transformation to the initial and boundary conditions (2) and (3) respectively, we get

$$U(k,0)=G(k) \tag{2.20}$$

$$U(0,h)=F_0(h) \tag{2.21}$$

$$U(1,h) = F_1(h) \tag{2.22}$$

By using Eq. (20)-(22) into Eq. (19), we can calculate $U(k,h)$. We substitute all $U(k,h)$ into Eq. (16) as $m \to \infty$ and $n \to \infty$, the solution of $u(x,y)$ can be consequently readily obtained.

2.4 Homotopy Perturbation Method (HPM)

To illustrate HPM consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \;\; r \in \Omega \tag{2.23}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial x}\right) = 0, \;\; r \in \Gamma \tag{2.24}$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω. The operator A can be generally divided into two parts F and N, Where F is linear,and N is non-linear. Therefore, Eq. (1) can be rewritten as follows:

$$F(u) + N(u) - f(r) = 0. \tag{2.25}$$

He constructed a Homotopy: $\Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)[F(v) - F(v_0)] + p[A(v) - f(r)] = 0 \tag{2.26}$$

or

$$H(v,p) = F(v) - F(v_0) + pF(v_0) + p[N(v) - f(r)] = 0 \tag{2.27}$$

Where $r \in \Omega$ and $p \in [0,1]$ that is called Homotopy parameter, and v_0 is an initial approximation of (1). Hence, it is obvious that-

$$H(v,0) = F(v) - F(v_0) = 0, \; H(v,1) = A(v) - f(r) = 0 \tag{2.28}$$

And the changing process of p from 0 to 1, is just that of $H(v,p)$ from $F(v) - F(v_0)$ to $A(v) - f(r)$. In topology, this is called deformation, $F(v) - F(v_0)$ and $A(v) - f(r)$ are called Homotopic. Applying the perturbation technique, due to the fact that $p \in [0\,1]$ can be considered as a small parameter, we can assume that the solution of (25) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots \tag{2.29}$$

When $p \to 1$, (25) corresponds to (23) and becomes the approximate solution of (23), i.e.

$$u = lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots \tag{2.30}$$

The series (29) is convergent for most cases, and the rate of convergence depends on $A(v)$. For applying this method He suggested following conditions:

(1) The second derivative of $N(v)$ with respect to v must be small relative to the parameter p.

(2) The norm $||F^{-1}\frac{\partial N}{\partial v}||$ must be smaller than 1 so that series converges.

2.5 Results and tested Problems

Here we will solve the problem of case (i) with $D = 1$ and $r(x,t) = (-1 - 4x^2)$.

2.5.1 Solutions obtained by DTM

The transformed form of equation (4) with $D = 1$ is given below

$$U(k, h+1) = \frac{1}{h+1}\left[(k+1)(k+2)U(k+2,h) - U(k,h) - \sum_{a=0}^{k}\sum_{b=0}^{h} U(k-a,b)R(a-2,h-b)\right] \tag{2.31}$$

By initial conditions-

$$U(k,0) = \{0 \ if \ k = 1,3,5,...\}, \ U(k,0) = \left\{\frac{1}{(\frac{k}{2})!} \ if \ k = 0,2,4...\right\} \tag{2.32}$$

By boundary conditions we can write-

$$U(0,h) = \frac{1}{h!} \ \ \forall h \geq 0 \tag{2.33}$$

$$U(1,h) = 0 \ \forall h \geq 0 \tag{2.34}$$

Using Eq. (32)-(34) into Eq. (31) by recursive method, we have the following values in Table (1) with $U(k,h) = 0$ for $k = 3,5,7...$ And $h \geq 1$ When we substitute all values $U(k,h)$ into (16) as $m \to \infty$ and $n \to \infty$, we obtain series for $u(x,t)$. Then when we rearrange the solution, we get the following closed form solution:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)(x)^k(t)^h = e^{(x^2+t)} \tag{2.35}$$

$U(2,1) = 1$	$U(4,1) = 1/2$	$U(6,1) = 1/6$	$U(8,1) = 1/24$
$U(2,2) = 1/2$	$U(4,2) = 1/4$	$U(6,2) = 1/12$	$U(8,2) = 1/48$
$U(2,3) = 1/6$	$U(4,3) = 1/12$	$U(6,3) = 1/36$	$U(8,3) = 1/144$
$U(2,4) = 1/24$	$U(4,4) = 1/48$	$U(6,4) = 1/144$	$U(8,4) = 1/576$
$U(2,5) = 1/120$	$U(4,5) = 1/240$	$U(6,5) = 1/720$	$U(8,5) = 1/2880$
$U(2,6) = 1/720$	$U(4,6) = 1/1440$	$U(6,6) = 1/4320$	$U(8,6) = 1/17280$
$U(2,7) = 1/5040$	$U(4,7) = 1/10080$	$U(6,7) = 1/30240$	$U(8,7) = 1/120960$
...	...	...	...
...	...	...	...

Table 1: Values of $U(k,h)$ for $k = 3,5,7...$ And $h \geq 1$

2.5.2 Solutions obtained by HPM

Consider the equation-

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + (-1-4x^2)u(x,t),\ (x,t) \in R^2 \tag{2.36}$$

Initial and boundary conditions are given below-

$$u(x,0) = e^{x^2},\ x \in R \tag{2.37}$$

$$u(0,t) = e^t,\ t \in R \ and\ \frac{\partial u(1,t)}{\partial x} = 0,\ t \in R \tag{2.38}$$

Creating Homotopy of the above given Eq. (36) We get the Eq. given below-

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = p\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u_0}{\partial t} - (1+4x^2)u\right] \tag{2.39}$$

Suppose the solution of Eq.(39) Will be in the form

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 ... \tag{2.40}$$

Substituting Eq. (40) in Eq. (39) and compare the distinct powers of p, follows that-

$$p^0 : \tfrac{\partial u_0}{\partial t} - \tfrac{\partial u_0}{\partial t} = 0$$
$$p^1 : \tfrac{\partial u_1}{\partial t} = \tfrac{\partial^2 u_0}{\partial x^2} - \tfrac{\partial u_0}{\partial t} - (1+4x^2)u_0$$
$$p^2 : \tfrac{\partial u_2}{\partial t} = \tfrac{\partial^2 u_1}{\partial x^2} - (1+4x^2)u_1$$
$$p^3 : \tfrac{\partial u_3}{\partial t} = \tfrac{\partial^2 u_2}{\partial x^2} - (1+4x^2)u_2$$
$$... ...$$

Similarly other terms we can obtain, now by choosing $u_0(x,t) = u(x,0) = e^{x^2}$,and solving above equations we will obtain following approximations-

$u_1 = e^{x^2} t$

$u_2 = \frac{1}{2!}(e^{x^2} t^2)$

$u_3 = \frac{1}{3!}(e^{x^2} t^3)$ . . .

Similarly we can obtain other terms Further the series solution by HPM will written in the form-

$$u = u_0 + u_1 + u_2 + ... \tag{2.41}$$

$$u = (e^{x^2}) + (e^{x^2} t) + \frac{1}{2!}(e^{x^2} t^2) + \frac{1}{3!}(e^{x^2} t^3)... \tag{2.42}$$

$$u = e^{x^2+t} \tag{2.43}$$

Hence the solution of the problem in equation (36) we obtained by Homotopy Perturbation Method is same as the solution we get from the Differential Transformation Method.

2.5.3 Simulated result through MATLAB

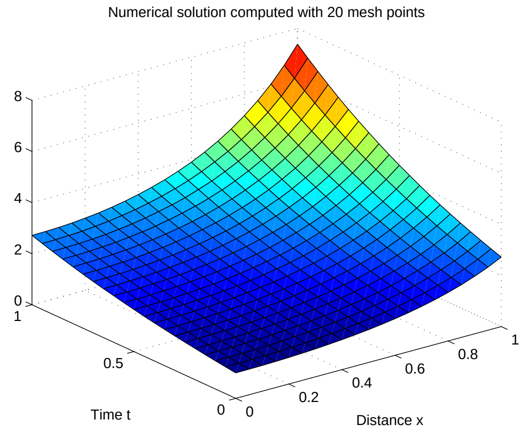

Figure 1: Exact solution obtained by DTM and HPM.

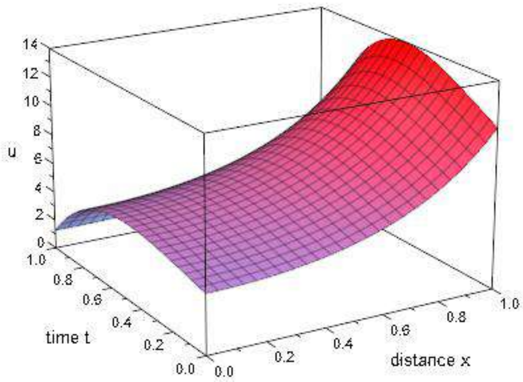

Figure 2: Approximate solution obtained by VIM (up to fourth order).

In computational form we tried to validate the simulated result using MATLAB. Fig. 1 represents the simulated solutions for exact results obtained by HPM and DTM. From the figure it is observed that both solution methods provide same solutions. To find the better solution method of these two methods, some special cases of Homotopy analysis method i.e. variational iteration method (VIM) is used for the simulation. Fig. 2 shows the approximate solution obtained by VIM which shows a minute difference with the results obtained by DTM and HPM and the solutions are obtained by considering the expanded

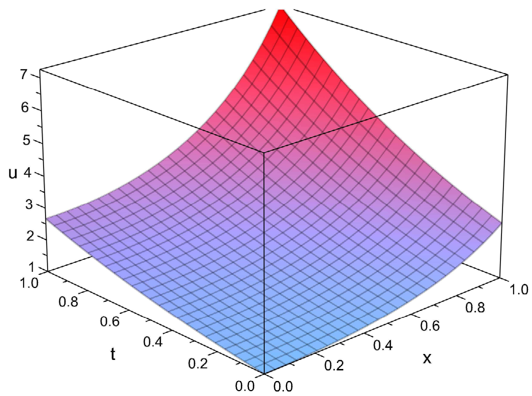

Figure 3: Approximate solution obtained by HPM and DTM (up to fourth order).

terms upto fourth order.

2.6 Test problem for non-linear Transient equation

Now the problem of case (ii), By taking $D = 1$ and $r(x,t) = 2t - 2 - 4x^2$.

2.6.1 Solution obtained by DTM

The transformed form of Eq. 7 is given below-

$$\begin{aligned} U(k,h+1) &= \frac{1}{h+1}\left[(k+1)(k+2)U(k+2,h) - 2U(k,h)\right] \\ &+ \frac{1}{h+1}\left[2\sum_{a=0}^{k}\sum_{b=0}^{h}U(k-a,b)R(a,h-b-1)\right] \\ &- \frac{1}{h+1}\left[4\sum_{a=0}^{k}\sum_{b=0}^{h}U(k-a,b)R(a-2,h-b)\right] \end{aligned} \tag{2.44}$$

By initial conditions-

$$U(k,0) = \{0 \ if \ k = 1,3,5,...\}, \ U(k,0) = \left\{\frac{1}{(\frac{k}{2})}! \ if \ k = 0,2,4...\right\} \tag{2.45}$$

By boundary conditions we can write-

$$U(k,0) = \{0 \ if \ k = 1,3,5,...\}, \ U(k,0) = \left\{\frac{1}{(\frac{h}{2})!} \ if \ k = 0,2,4...\right\} \tag{2.46}$$

$$U(1,h) = 0 \ \forall h \geq 0 \tag{2.47}$$

Using Eq. (45)-(47) into Eq. (44) by recursive method, we have the following values in Table (2) with $U(k,h) = 0$ for $k = 1,3,5,7...$ And $h \geq 0$ When we substitute all values $U(k,h)$ into (16) as $m \to \infty$ and $n \to \infty$, we obtain series for u(x, t). Then when we rearrange the solution, we get the following closed form solution:

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k,h)(x)^k(t)^h = e^{(x^2+t^2)} \tag{2.48}$$

U(2,2) =1	U(4,2)=1/2	U(6,2)=1/2	U(8,2)=1/24
U(2,4) =1/2	U(4,4)=1/4	U(6,4)=1/2	U(8,4)=1/48
U(2,6) =1/6	U(4,6)=1/12	U(6,6)=1/6	U(8,6)=1/144
U(2,8) =1/24	U(4,8)=1/48	U(6,8)=1/24	U(8,8)=1/576
...	...	...	...

Table 2: Values of $U(k,h)$ for $k = 1,3,5,7...$ and $h \geq 0$

2.6.2 Solution obtained by HPM

Consider the equation-

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + (2t - 2 - 4x^2)u(x,t), \ (x,t) \in \Omega \subset R^2 \tag{2.49}$$

Initial and boundary conditions are given below-

$$u(x,0) = e^{x^2}, \ x \in R \tag{2.50}$$

$$u(0,t) = e^{t^2}, \ t \in R \ and \ \frac{\partial u(1,t)}{\partial x} = 0, \ t \in R \tag{2.51}$$

Creating Homotopy of the above given Eq. (49) We get the Eq. given below-

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = p\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u_0}{\partial t} + (2t - 2 - 4x^2)u\right] \tag{2.52}$$

Suppose the solution of Eq.(49) Will be in the form

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3... \tag{2.53}$$

Substituting Eq. (53) in Eq. (52) and compare the distinct powers of p, follows that-

$p^0 : \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0$

$p^1 : \frac{\partial u_1}{\partial t} = \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial u_0}{\partial t} + (2t - 2 + 4x^2)u_0$

$p^2 : \frac{\partial u_2}{\partial t} = \frac{\partial^2 u_1}{\partial x^2} + (2t - 2 + 4x^2)u_1$

$p^3 : \frac{\partial u_3}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} + (2t - 2 + 4x^2)u_2$

... ...

Similarly other terms we can obtain, now by choosing $u_0(x,t) = u(x,0) = e^{x^2}$ And solving above equations we will obtain following approximations-

$u_1 = e^{x^2}t^2$

$u_2 = \frac{1}{2!}(e^{x^2}t^4)$

$u_3 = \frac{1}{3!}(e^{x^2}t^6)\ \ldots$

Similarly we can obtain other terms Further the series solution by HPM will written in the form-

$$u = u_0 + u_1 + u_2 + ... \tag{2.54}$$

$$u = (e^{x^2}) + (e^{x^2}t^2) + \frac{1}{2!}(e^{x^2}t^4) + \frac{1}{3!}(e^{x^2}t^6)... \tag{2.55}$$

$$u = e^{x^2+t^2} \tag{2.56}$$

Hence the solution of the problem in equation (49) we obtained by Homotopy Perturbation Method is same as the solution we get from the Differential Transformation Method.

2.6.3 Simulated result through MATLAB

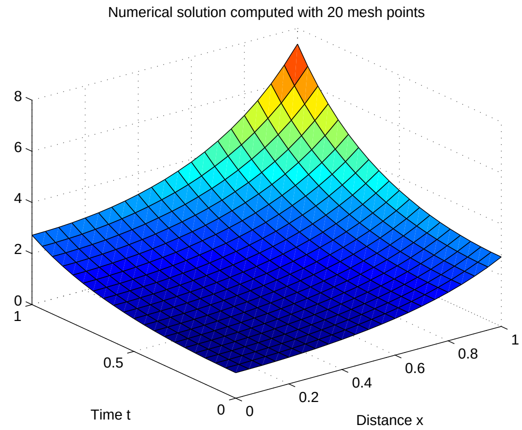

Figure 4: Exact solution obtained by DTM and HPM.

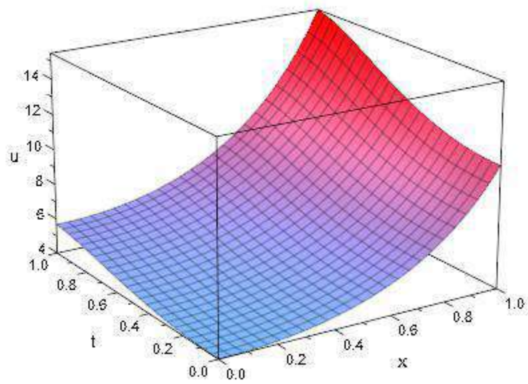

Figure 5: Approximate solution obtained by VIM (up to fourth order).

The solutions obtained by both these methods (DTM) and (HPM) (Fig. 4) are same. The accuracy of HPM for both problems is controllable and absolute errors are very small with the present choice of t and x. Fig. 5 represents the approximate solution obtained by VIM considering the series up to fourth order terms. The exact solution obtained by HPM or DTM are same as that obtained by Variation Iteration Method [1].

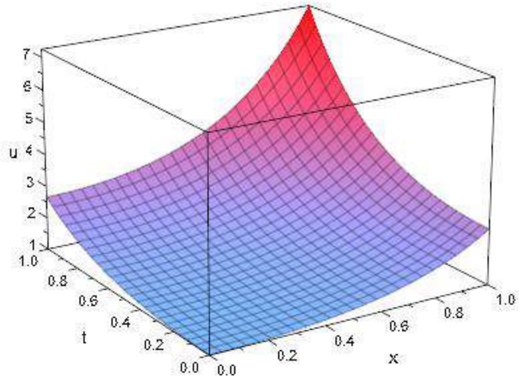

Figure 6: Approximate solution obtained by DTM and HPM (up to fourth order).

3 Conclusion

The main goal of this work is to conduct a comparative study between the Differential transformation method (DTM) and He's Homotopy Perturbation method (HPM). These two methods are so powerful and efficient that they both give approximations of higher accuracy and closed form solutions if existing. Differential transformation method (DTM) provides the components of the exact solution when these components follow the summation given in Eq. (12) and Eq. (16). However, He's Homotopy Perturbation method (HPM) is a powerful mathematical tool for solving linear, nonlinear partial differential equations. Moreover, the differential transformation method (DTM), which is based on the Taylor series expansion, constructs an analytical solution in the form of polynomial series solution by means of an iterative procedure. The solved examples presented in this paper always represent a closed-form solution. But the method can be extended by considering HPM analysis to deal with the family of nonlinear reaction diffusion equations. We tried to validate the result of HPM with DTM and VIM and concluded that HPM is more acceptable and accurate than other methods.

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Some Geometric Studies on the Classes of Bi-Univalent Functions

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Abstract

This article is a survey, in which we analyze certain aspects of the class of bi-univalent complex-valued functions defined on the unit disk. After the appearance of the paper by Lewin in the 1967, the class of bi-univalent functions did begin to attract interest among function theorists. He proposed a coefficient conjecture for the class of bi-univalent analytic functions like Bieberbach. In this article we begin with the basic definitions and some examples of bi-univalent functions. After a brief look at the literature, we focus our attention on the coefficient bounds for several geometric subclasses and discuss the recent developments along this line.

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Keywords: Univalent Functions, Bi-Univalent Functions, Starlike and Bi-Starlike Functions, Coefficient Bounds, Faber Polynomial.

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1 Introduction

Let $\mathscr{H}$ denote the class of analytic functions in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathscr{A} \subset \mathscr{H}$ be the class of normalized functions $f(z)$ in $\mathbb{U}$ such that $f(0) = f'(0) - 1 = 0$. Let $\mathscr{S}$ be the class of functions $f(z)$ of $\mathscr{A}$ which are univalent in $\mathbb{U}$. That is, $f \in \mathscr{S}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U} \tag{1.1}$$

One leading example of $\mathscr{S}$ is Koebe function,

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n$$

which maps $\mathbb{U}$ onto the full complex plane with a slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$. It plays an important role in the theory of univalent functions as it is extremal to many problems in univalent function theory. In 1916, Bieberbach proved his second coefficient theorem that $|a_2| \leq 2$ for $f \in \mathscr{S}$ and stated a historical conjecture that $|a_n| \leq n$ for $n \geq 3$ and $f \in \mathscr{S}$, with equality only for Koebe function. Bieberbach's second coefficient theorem was the first concrete evidence for the general conjecture. But it took almost seven decades to prove the Bieberbach conjecture. Finally, in 1985 this conjecture was proved by L. de Branges. During that period of 70 years, the geometric function theory developed very fast and several new areas of research emerged. One such area of research is to study the geometric properties of the function according to the bounds for the coefficients. The distortion theorem [21], one of the most important corollary of second coefficient theorem, which was one of the the basic tools for further study on the second coefficient of the functions in $\mathscr{S}$. That gave rise to the growth and covering theorems for the class of univalent functions. A basic way to obtain a coefficient inequality like Bieberbach type inequality is to relate the coefficients of f, to the area of some region in the complex plane. The first such result was the area theorem, proved in 1914 by T.H Gronwal see[21].

The basic principles observed in the proofs of theorems like growth theorems, covering theorems or area theorems was to start with a function in $\mathscr{S}$, carry out some algebraic transformations and then apply a known coefficient theorem to the results which will give several interesting consequences of Bieberbach theorem. One such result is Koebe's one-quarter theorem. The Koebe one quarter theorem [21], ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathscr{S}$ contains a disk of radius $1/4$. Thus, every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z,\ z \in \mathbb{U}$, and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right). \tag{1.2}$$

Infact, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.3}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(w)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Here we cite some examples of bi-univalent functions in $\mathbb{U}$ are

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

Whereas

$$k(z) = \frac{z}{(1-z)^2},\ z - \frac{z^2}{2},\ \frac{z}{1-z^2} \notin \Sigma.$$

We add, one family of functions defined by

$$\bar{\lambda}(e^{\lambda z} - 1) \quad (\lambda \in \mathbb{C}, |\lambda| = 1; z \in \mathbb{U})$$

are univalent in the larger disk $|z| < \pi$ and their inverse functions are univalent in $\mathbb{U}$. Therefore the functions are also bi-univalent.

In late 60's, it attracted interest among function theorists, to settle some coefficient inequalities like Bieberbach conjecture for the class of bi-univalent functions and study their geometrical consequences. Lewin is the first function theorist, who investigated the bi-univalent function class Σ and conjectured a coefficient inequality like Bieberbach type inequality. In 1967, by using Grunsky inequalities, Lewin [41] showed that $|a_2| < 1.51$. He also proved that $\Sigma_1 \subset \Sigma$, where Σ_1 is the class of all functions $f = \phi o \psi^{-1}$, where ϕ and ψ map from $\mathbb{U}$ onto the domain containing $\mathbb{U}$ and $\phi'(0) = \psi'(0)$. In an example it was shown that $\Sigma_1 \neq \Sigma$ see [64]. Subsequently, in 1967 Brannan and Clunie [13] conjectured that $|a_2| < \sqrt{2}$. In 1969, Suffridge [56] showed that a function in Σ_1 satisfies $|a_2| = 4/3$ and conjectured that $|a_2| \leq 4/3$ for all functions in Σ. Also in 1972, Jenson and Waadeland [36] proved that if $f(z) \in \Sigma$, then $|a_3| \leq 2.51$, but this result was not sharp. In 1976, Smith [54] showed the function $f(z) = z + a_2z^2 + a_3z^3 \in \Sigma$, with real coefficients satisfies $|a_2| \leq 2/\sqrt{27}$ and $|a_3| \leq 4/27$ and the later inequality was sharp. He also showed that if $f(z) = z + a_nz^n \in \Sigma$, then $|a_n| \leq \frac{(n-1)^{n-1}}{n^n}$ which will be sharp for $n = 2,3$. In 1988, Kedzierawski and Waniurski [39] proved the conjecture of Smith [56] for $n = 3,4$ by taking the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots + a_nz^n \in \Sigma$.

In 1984, Tan [72], improved Lewin's result and proved $|a_2| \leq 1.485$ for the functions in Σ. In 1985 Kedzierawski [37] proved the conjecture $|a_2| < \sqrt{2}$ for a special case when both f and f^{-1} are starlike functions. It was belived by many function theorists, that the bound $|a_n| \leq 1$ was true for every n for the class Σ. However, in 1969, in a very difficult paper E. Netyananhu [48], ruined this conjecture by proving that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. Also in [10], Busklein tried to prove $\max |a_n| > n/4$, for the functions in Σ. He tried by considering an interesting example to show this but later it was shown that the function was not in Σ because it had a pole in $\mathbb{U}$. Later, Styler and Wright, disproved the conjecture by showing $|a_2| > \frac{4}{3}$, for some functions in Σ, see [64].

By that time nothing was known about the $\max |a_n|$, for the functions in Σ when $n > 3$. But many function theorists tried to find some sharp coefficient bounds for the functions in different subclasses of Σ. In the next section we consider some of the developments in this area.

2 Some Classes of Bi-Univalent Analytic Functions

Analogous to $\mathscr{S}$, many subclasses of Σ have been introduced to settle the Lewin's, Brannan and Clunie conjectures. In 1985, Kedzierawski [37] first introduce a subclass of Σ that is, the special class of functions $f(z)$ such that both f and f^{-1} are starlike functions. He obtained the following result:

Theorem [37]: Let $f \in \mathscr{S}^*_\Sigma$, the class of functions $f(z)$ such that both f and f^{-1} are starlike functions and $g = f^{-1}$. Then

$$|a_2| \leq \begin{cases} 1.5894 & if \quad f \in \mathscr{S},\ g \in \mathscr{S}, \\ \sqrt{2} & if \quad f \in \mathscr{S}^*,\ g \in \mathscr{S}^*, \\ 1.507 & if \quad f \in \mathscr{S}^*,\ g \in \mathscr{S}, \\ 1.224 & if \quad f \in \mathscr{C},\ g \in \mathscr{S}. \end{cases}$$

Some more subclasses of Σ like the subclasses of $\mathscr{S}$, were developed and studied. Here we list some important subclasses and some results on these classes.

Bi-Starlike Functions of order α:

A function $f(z) \in \mathscr{A}$ is said to be bi-starlike of order α $(0 \leq \alpha \leq 1)$ if both $f(z)$ and f^{-1} are starlike functions of order α. Let $\mathscr{S}^*_\Sigma(\alpha)$ denotes class of functions of bi-starlike functions of order α. That is, $f(z) \in \mathscr{S}^*_\Sigma(\alpha)$ satisfies the following conditions:

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$$

and

$$\operatorname{Re}\left(\frac{wg'(w)}{g(w)}\right) > \alpha \quad (w \in \mathbb{U}),$$

where the function g is defined by (1.3).

Example: $f(z) = z + a_2 z^2 \in \mathscr{S}^*_\Sigma(\alpha)$ if $|a_2| \leq \frac{1-\alpha}{4(2-\alpha)}$.

Bi-Strongly Starlike Functions of order α:

A function $f(z)$, is said to be in the class $\mathscr{S}^{*\alpha}_\Sigma$, $0 < \alpha \leq 1$, the class of strongly starlike functions of order α, if each of the following conditions are satisfied:

$$f \in \Sigma, \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U})$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}),$$

where the function g is defined by (1.3).

In 1985, Branan and Taha [15], introduced the classes $\mathscr{S}_\Sigma^*(\alpha)$ and $\mathscr{S}_\Sigma^{*\alpha}$ and found the non-sharp bounds for $|a_2|$ and $|a_3|$ stated in the following theorem.

Theorem[15]: Let the function $f(z)$ in Σ and $(0 \le \alpha < 1)$, then

$$|a_2| \le \begin{cases} \dfrac{2\alpha}{\sqrt{1+\alpha}} & if \quad f \in \mathscr{S}_\Sigma^{*\alpha}, \\[2ex] \sqrt{2(1-\alpha)} & if \quad f \in \mathscr{S}_\Sigma^*(\alpha). \end{cases} \tag{2.4}$$

Similarly,

$$|a_3| \le \begin{cases} 2\alpha & if \quad f \in \mathscr{S}_\Sigma^{*\alpha}, \\ 2(1-\alpha) & if \quad f \in \mathscr{S}_\Sigma^*(\alpha). \end{cases} \tag{2.5}$$

Later in 2014, Mishra and Soren [45], improved the result. They found better estimation of $|a_3|$ for the functions $f(z)$ both in the class $\mathscr{S}_\Sigma^{*\alpha}$ and the class $\mathscr{S}_\Sigma^*(\alpha)$. They also found the bounds for $|a_4|$ of the functions in the classes which are included in the following theorems.

Theorem[45]: For $0 < \alpha \le 1$, let the function $f(z)$ be in $\mathscr{S}_\Sigma^{*\alpha}$, then

$$|a_3| \le \begin{cases} \alpha, & 0 < \alpha \le \frac{1}{3}, \\ \frac{4\alpha^2}{1+\alpha}, & \frac{1}{3} \le \alpha \le 1 \end{cases} \tag{2.6}$$

and

$$|a_4| \le \begin{cases} \frac{2\alpha}{3}\left(1 - \frac{2}{3}\frac{16\alpha^2-3\alpha-1}{(1+\beta)^{\frac{1}{3}}}\right), & 0 < \alpha < \frac{3+\sqrt{73}}{32}, \\[2ex] \frac{2\alpha}{3}\left(1 + \frac{2}{3}\frac{16\alpha^2-3\alpha-1}{(1+\beta)^{\frac{1}{3}}}\right), & \frac{3+\sqrt{73}}{32} \le \alpha < \frac{2}{5}, \\[2ex] \frac{2\alpha}{3}\left(\frac{15\alpha}{5\alpha+4} + \frac{2}{3}\frac{16\alpha^2-3\alpha-1}{(1+\beta)^{\frac{1}{3}}}\right), & \frac{2}{5} \le \alpha \le 1. \end{cases} \tag{2.7}$$

Theorem[45]: For $0 \leq \alpha < 1$, let the function $f(z)$ be in $\mathscr{S}^*_\Sigma(\alpha)$, then

$$|a_4| \leq \begin{cases} \frac{2(1-\alpha)}{3}[1+2\sqrt{2(1-\alpha)}], & 0 < \alpha \leq \frac{1}{2}, \\ \\ \frac{2(1-\alpha)}{3}[1+4(1-\alpha)], & \frac{1}{2} \leq \alpha \leq 1 \end{cases} \tag{2.8}$$

Srivastava et. al [67], introduced and investigated two novel subclasses $\mathscr{R}_\Sigma(\beta)$ and $\mathscr{R}^*_\Sigma(\alpha)$ of Σ and found non-sharp bounds.

The Class $\mathscr{R}_\Sigma(\beta)$: A function $f(z)$ given by (1.1), is said to be in the class $\mathscr{R}_\Sigma(\beta)$, $(0 \leq \beta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \operatorname{Re}(f'(z)) > \beta \quad z \in \mathbb{U}$$

and

$$\operatorname{Re}(g'(w)) > \beta \quad w \in \mathbb{U},$$

where $g(w)$ is given by (1.3).

The Class $\mathscr{R}^*_\Sigma(\alpha)$: A function $f(z)$ given by (1.1), is said to be in the class $\mathscr{R}^*_\Sigma(\alpha)$, $(0 < \alpha \leq 1)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \arg|(f'(z)| \leq \frac{\alpha\pi}{2} \quad z \in \mathbb{U}$$

and

$$\arg|(g'(w)| \leq \frac{\alpha\pi}{2} \quad w \in \mathbb{U},$$

where $g(w)$ is given by (1.3).

We next cite some of the results of Srivastava at.el:

Theorem[67]: Let $f(z)$ given by (1.1) be in the class $\mathscr{R}^{*}_{\Sigma}(\alpha),\ (0<\alpha\leq 1)$. Then

$$|a_2|\leq \alpha\sqrt{\frac{2}{\alpha+2}} \quad \text{and} \quad |a_3|\leq \frac{\alpha(3\alpha+2)}{3}. \tag{2.9}$$

Theorem[67]: Let $f(z)$ given by (1.1) be in the class $\mathscr{R}_{\Sigma}(\beta),\ (0\leq\beta<1)$. Then

$$|a_2|\leq \sqrt{\frac{2(1-\beta)}{3}} \quad \text{and} \quad |a_3|\leq \frac{(1-\beta)(5-3\beta)}{3}. \tag{2.10}$$

Starlike and Bi-Starlike Functions with respect to symmetric points:

In [53], Sakaguchi introduced and investigated the class $\mathscr{S}^{*}_{s}$, the class of starlike functions with respect to symmetric points in $\mathbb{U}$; consisting of functions $f\in\mathscr{A}$ that satisfy

$$\operatorname{Re}\frac{zf'(z)}{f(z)-f(-z)}>0, \quad z\in\mathbb{U}$$

The class of functions univalent and starlike with respect to symmetric points includes the classes of convex functions and odd starlike functions. Later in [52], Ravichandran has extended this class like Ma-Minda type (see [44]). He defined, the class $\mathscr{S}^{*}_{s}(\phi)$, by

$$\operatorname{Re}\frac{2zf'(z)}{f(z)-f(-z)}\prec\phi, \quad z\in\mathbb{U}, \tag{2.11}$$

where the function ϕ is analytic with positive real part in $\mathbb{U}$; $\phi(0)=1$ and $\phi'(0)>0$ and with the property that ϕ maps $\mathbb{U}$ onto a domain starlike with respect to 1 and symmetric with respect to the real axis. So $\phi(z)$ has the series form

$$\phi(z)=1+c_1z+c_2z^2+\cdots, \quad c_1>0. \tag{2.12}$$

In [18], Crisan defined similar type of class for the bi-univalent functions. A function $f\in\Sigma$ is said to be in the class $\mathscr{S}^{*}_{s,\Sigma}(\phi)$ if both f and f^{-1} are $\mathscr{S}^{*}_{s}(\phi)$, where $\mathscr{S}^{*}_{s}(\phi)$ is the class of functions given in

(2.11). He derived the following estimations.

Theorem[18]: Let $f(z)$ given by (1.1) be in the class $\mathscr{S}^{*}_{s,\Sigma}(\phi)$, then

$$|a_2| \leq \frac{c_1\sqrt{c_1}}{\sqrt{2|c_1^2+2c_1-2c_2|}} \quad \text{and} \quad |a_3| \leq \frac{1}{2}c_1\left(1+\frac{1}{2}c_1\right). \tag{2.13}$$

For $\phi(z) = \frac{1+(1-2\alpha)}{1-z}$, $0 \leq \alpha < 1$, then the above theorem gives the following results.

Corollary[18]: Let $0 \leq \alpha < 1$ and $f(z)$ given by (1.1). If the following conditions are satisfied

$$\mathrm{Re}\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) > \alpha, \quad z \in \mathbb{U}$$

and

$$\mathrm{Re}\left(\frac{2wg'(w)}{g(w)-g(-w)}\right) > \alpha, \quad z \in \mathbb{U},$$

where $g = f^{-1}$ defined by (1.3), then

$$|a_2| \leq \sqrt{1-\alpha} \quad \text{and} \quad |a_3| \leq (1-\alpha)(2-\alpha).$$

Also similar type of the coefficient estimations of the bi-univalent functions in the class of convex functions and strongly starlike functions with respect to the symmetric points and Ma-Minda starlike and convex functions have been derived in [4, 18, 19]. Recently, in [8, 7], Altinkaya and Yalcin have introduced some more generalized classes of Ma-Minda type and derived the coefficient bounds of the initial coefficients.

3 Some More Generalized Subclasses of Bi-Univalent Analytic Functions

In fact, the aforecited work of Srivastava et al.[67] essentially revived the investigation of various subclasses of the bi-univalent function class in recent years;it was followed by such works as those by Frasin and Aouf [25].

In this section we are going to highlight the studies on some more generalized subclasses of bi-univalent analytic functions. Extending the results of Srivastava et.al [67], Frasin and Aouf [25] obtained estimate of $|a_2|$ and $|a_3|$ for the functions $f(z) \in \Sigma$ for a more generalized subclass similar to the subclass of $\mathscr{S}$ introduced by Ding et. al.[23]. They defined the following subclasses:

Definition (The Class $\mathscr{B}_\Sigma(\alpha,\lambda)$): A function $f(z)$ given by (1.1) is said to be in the class $\mathscr{B}_\Sigma(\alpha,\lambda)$ $(0 \le \alpha < 1,\ \lambda \ge 1)$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \operatorname{Re}\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \alpha, \quad z \in \mathbb{U}$$

and

$$\operatorname{Re}\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right) > \alpha, \quad w \in \mathbb{U},$$

where g is given by (1.3).

Definition (The Class $\mathscr{B}^*_\Sigma(\alpha,\lambda)$): A function $f(z)$ given by (1.1) is said to be in the class $\mathscr{B}^*_\Sigma(\alpha,\lambda)$ $(0 < \alpha \le 1,\ \lambda \ge 1)$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{U},$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right| < \frac{\alpha\pi}{2}, \quad w \in \mathbb{U}$$

where g is given by (1.3).

For $\lambda = 1$ the above classes reduces to $\mathscr{R}_\Sigma(\alpha)$ and $\mathscr{R}^*_\Sigma(\alpha)$ introduced and studied in [67]. They proved the following results which generalizes the results in [67].

Theorem[25]: Let $f(z)$ given by (1.1) be in the class $\mathscr{B}_{\Sigma}(\alpha,\lambda),\ (0 \leq \alpha < 1, \lambda \geq 1)$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{2\lambda+1}}, \tag{3.14}$$

and

$$|a_3| \leq \frac{4(1-\alpha)^2}{(\lambda+1)^2} + \frac{2(1-\alpha)}{2\lambda+1}. \tag{3.15}$$

Theorem[25]: Let $f(z)$ given by (1.1) be in the class $\mathscr{B}^{*}_{\Sigma}(\alpha,\lambda),\ (0 < \alpha \leq 1, \lambda \geq 1)$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda+1)^2+\alpha(1+2\lambda+\lambda^2)}}, \tag{3.16}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}. \tag{3.17}$$

Later in 2013, Caglar et.al.[20] introduced and studied following two more generalized classes similar to the classes of Bazilveic functions in $\mathscr{S}$ defined by Zhu in [77].

Definition (The Class $\mathscr{M}^{\mu}_{\Sigma}(\alpha,\lambda)$): A function $f(z)$ given by (1.1), is said to be in the class $\mathscr{M}^{\mu}_{\Sigma}(\alpha,\lambda)$ $(0 \leq \alpha \leq 1,\ \lambda \geq 1)$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \operatorname{Re}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > \alpha, \quad z \in \mathbb{U}$$

and

$$\operatorname{Re}\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) > \alpha, \quad w \in \mathbb{U},$$

where g is given by (1.3).

Definition (The Class $\mathscr{N}^{\mu}_{\Sigma}(\alpha,\lambda)$): A function $f(z)$ given by (1.1) is said to be in the class $\mathscr{N}^{\mu}_{\Sigma}(\alpha,\lambda)$ $(0 < \alpha \leq 1,\ \lambda \geq 1,\ \mu \geq 0)$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \quad z \in \mathbb{U}$$

and

$$\left|\arg\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right|<\frac{\alpha\pi}{2},\quad w\in\mathbb{U},$$

where g is given by (1.3).

For $\lambda = 1$ and $\mu = 1$ the above classes reduces to $\mathscr{R}_{\Sigma}(\alpha)$ and $\mathscr{R}^{*}_{\Sigma}(\alpha)$ introduced and studied in [67].

Further in 2016, these classes have been extended and studied by Sahoo and Singh [63] for $\mu < 0$, which are similar to the subclasses of non-Bazelevic functions in $\mathscr{S}$ introduced and studied in [69].

Bounds for for initial coefficients of several subclasses of bi-univalent functions similar to many subclasses of $\mathscr{S}$ were also investigated in [4, 11, 20, 25, 26, 28, 29, 31, 32, 40, 45, 47, 46, 51, 63, 65, 67, 68, 70, 75, 76]. Fekete Szego problems on some subclasses of bi-univalent analytic functions were also investigated in [5, 34, 78].

4 Use of Faber Polynomials in the class of Bi-Univalent Analytic and Meromorphic Functions

In the literature, there exists only a few works of determining the general coefficient bounds $|a_n|$ for the bi-univalent functions by using Faber polynomial expansions. The Faber polynomial introduced by Faber [24], plays an important role in various areas of mathematical sciences,especially in geometric function theory.

Let σ denotes the class of all meromorphic univalent functions q of the form

$$q(z) = z + \sum_{n=0}^{\infty}\frac{b_n}{z^n}, \tag{4.18}$$

defined on the domain $\mathbb{U}^* = \{z\in\mathbb{C} : 1 < |z| < \infty\}$. So

$$q(z) = \frac{1}{f(1/z)} \in \sigma,\quad z\in\mathbb{U}^*,$$

where $f(z)$ defined by (1.1). In [73], P.G.Todorov defined the Faber polynomials $\phi_n(t)$ of degrees $n = 1,2,\ldots$ in terms of a_n, the coefficients of $f(z)$ defined in (1.1) as follows:

$$\phi_n(t) = nd_n + n\sum_{k=1}^{n}(k-1)!D_{n,k}(a_1,a_2,\ldots,a_{n-k+1})t^k, \quad n = 1,2,\ldots, \tag{4.19}$$

where $a_1 = 1$ and

$$d_n = \sum_{k=1}^{n}(-1)^{k-1}(k-1)!D_{n,k}(a_2,a_3,\ldots,a_{n-k+2}) \quad n = 1,2,\ldots. \tag{4.20}$$

and

$$D_{n,k}(a_1,a_2,\ldots,a_{n-k+1}) = \sum_{n=1}^{\infty}\frac{(a_1)^{i_1}\ldots(a_{n-k+1})^{i_{n-k+1}}}{i_1!\ldots i_{n-k+1}!}, \quad 1 \le k \le n, \tag{4.21}$$

$a_1 = 1$ and sum is taken over all non-negative integers $i_1,\ldots i_n$ satisfying $i_1+i_2+\cdots+i_{n-k+1} = k$, $i_1 + 2i_2+\cdots+(n-k+1)i_{n-k+1} = n$. Also $\phi_n(t)$ can be written as b_n, the coefficients of $q(z)$ defined in (4.18) as follows [74]:

$$\phi_n(t) = -n\psi_n(t) + n\sum_{m=1}^{n}\frac{1}{m}\psi_{n-m}(m)t^n, \quad n = 1,2,\ldots, \tag{4.22}$$

where

$$\psi_n = \sum_{k=1}^{n}(-1)^{k-1}(k-1)!D_{n,k}(b_0,b_1,\ldots,b_{n-k}), \tag{4.23}$$

and where

$$\psi_{n-m} = \sum_{k=0}^{n-m}(-m)_k!D_{n-m,k}(b_0,b_1,\ldots,b_{n-m-k}). \tag{4.24}$$

Later in 2006, Airault and Bouali [1], derived the inverse of an analytic function in terms of Faber polynomial as follows:

Theorem: Let $f(z) \in \mathscr{A}$ given by (1.1). Then inverse function of f, is expressed as

$$f^{-1}(w) = w + \sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots)w^n,$$

where

$$K_{n-1}^{-n} = \tfrac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \tfrac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 + \tfrac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 + \tfrac{(-n)!}{(2(-n+2))!(n-5)!} \times$$

$$a_2^{n-5}[a_5 + (-n+2)a_3^2] + \tfrac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6 + (-2n+5)a_3a_4] + \textstyle\sum_{j\geq 7} a_2^{n-j}V_j,$$

such that V_j with $7 \leq j \leq n$, is a homogeneous polynomial in variables $a_2, a_3, \dots$ see [3]. In particular,

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3(a_2^2 - a_3), \quad K_3^{-4} = (5a_2^3 - 5a_2a_3 + a_4).$$

In general, for any $p \in \mathbb{Z}$,

$$K_n^p = pa_n + \frac{p(p-1)}{2}D_{n,2} + \frac{p!}{(p-3)!3!}D_{n,3} + \cdots + \frac{p!}{(p-n)!n!}D_{n,n},$$

where $D_{n,p} = D_{n,p}(a_1, a_2, \dots, a_n)$, and $D_{n,k}$ is defined in (4.21) [74]. Let σ denotes the class of all meromorphic univalent functions q of the form given by (4.18). Since $q \in \sigma$ is univalent, it has inverse q^{-1} that satisfies

$$q^{-1}(q(z)) = z \quad (z \in, \mathbb{U}^*)$$

and

$$q(q^{-1}(w)) = w \quad (M < |w| < \infty, M > 0).$$

Furthermore, the inverse function q^{-1} has a series expansion of the form

$$h(w) = q^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}, \tag{4.25}$$

where $M < |w| < \infty, M > 0$.

The estimates of the coefficients of the meromorphic univalent functions were investigated by many authors which are existed in the literature. For example, in 1938, Schiffer [58] obtained the estimates $|b_2| \leq 2/3$ for the function $q(z)$ of the form (4.18) in σ for $b_0 = 0$. In 1971, Duren [22] proved the inequality $|b_n| \leq 2/(n+1)$ for the functions $q(z)$ of the form (4.18) in σ for $b_k = 0$ for $1 \leq k < n/2$. He then proved that this bound also holds for meromorphic starlike univalent functions q of order zero. In

1951, Springer [61] proved the following coefficient inequalities for inverse of the meromorphic univalent functions

$$|B_3| \leq 1 \quad \text{and} \quad |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2}$$

and

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \quad (n = 1, 2, \ldots).$$

Kapoor and Mishra [38] found the coefficient estimates for the inverse of the meromorphic functions which are starlike of positive order in $\mathbb{U}^*$ and for its inverse functions they obtained the bound $2(1-\alpha)/(n+1)$ when $((n-1)/n) \leq \alpha < 1$. More recently, Srivastava et al. [66] found sharp bounds for the coefficients of starlike univalent functions of order α for $0 \leq \alpha < 1$, having m- fold gaps in their series representation in $\mathbb{U}^*$ and also for their inverse functions.

The above two articles [38, 66] settled the coefficient bounds for starlike functions and their inverses but they have not considered the bi-starlike case.

A function $q(z) \in \sigma$ is said to be meromorphic bi-univalent if both $q(z)$ and $q^{-1}(z)$ are in σ. Let $\sigma_{\mathscr{B}}$ denotes the class of all meromorphic bi-univalent functions.

Example: The function $q(z) = z + 1/z$ is a meromorphic univalent function in $\mathbb{U}^*$. As a calculation shows as

$$q^{-1}(w) = \frac{w + \sqrt{w^2 - 4}}{2} = w - \frac{1}{w} - \frac{1}{w^3} - \frac{2}{w^5} - \cdots.$$

The problem arises when the bi-univalency condition is imposed on the meromorphic functions $q(z)$. The bi-univalency requirement makes the task of finding bounds for the coefficients of q and its inverse map $h = q^{-1}$ more involved. In [32], Halim et.al found bounds of $|b_0|$ and $|b_1|$ on the class of bi-starlike meromorphic functions and bi-strongly starlike meromorphic functions, whereas in [30], for the first time, S.G. Hamidi et.al used the Faber polynomial expansions to study the coefficients of meromorphic bi-starlike functions and found a more improved bound of the coefficients b_n of $q(z)$.

Definition: A function q given by (4.18) is said to be a meromorphic bi-starlike function of order α $(0 \le \alpha < 1)$, if

$$\operatorname{Re}\left(\frac{zq'(z)}{q(z)}\right) > \alpha \quad (z \in \mathbb{U}^*)$$

and

$$\operatorname{Re}\left(\frac{wh'(w)}{h(w)}\right) > \alpha \quad (w \in \mathbb{U}^*),$$

where $h = q^{-1}$.

Definition: A function q given by (4.18) is said to be a meromorphic bi-strongly starlike function of order α $(0 < \alpha \le 1)$, if

$$\left|\arg\left(\frac{zq'(z)}{q(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}^*)$$

and

$$\left|\arg\left(\frac{wh'(w)}{h(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}^*),$$

where $h = q^{-1}$.

Theorem [32]: Let $q(z)$ defined by (4.18) be meromorphic bi-starlike of order α $(0 \le \alpha < 1)$, then the coefficients b_0 and b_1 satisfy the inequalities

$$|b_0| \le 2(1-\alpha) \quad \text{and} \quad |b_1| \le (1-\alpha)\sqrt{4\alpha^2 - 8\alpha + 5}.$$

Theorem [32]: Let $q(z)$ defined by (4.18) be a meromorphic bi-strongly starlike function of order α $(0 < \alpha \le 1)$, then the coefficients b_0 and b_1 satisfy the inequalities

$$|b_0| \le 2\alpha \quad \text{and} \quad |b_1| \le \sqrt{5}\alpha^2.$$

In [30] S.G. Hamidi et.al proved the following theorem:

Theorem [30]: Let $q(z)$ defined by (4.18) be meromorphic bi-starlike of order α $(0 \leq \alpha < 1)$ in $\mathbb{U}^*$. If $b_1 = b_2 = \cdots = b_{n-1} = 0$ for n being odd or if $b_0 = b_1 = \cdots = b_{n-1} = 0$, for n being even, then

$$|b_n| \leq \frac{2(1-\alpha)}{n+1}, \quad n \in \mathbb{N}.$$

Theorem [30]: Let $q(z)$ defined by (4.18) be meromorphic bi-starlike of order α $(0 \leq \alpha < 1)$, then the coefficients b_0 and b_1 satisfy the inequalities

$$|b_0| \leq \sqrt{2(1-\alpha)} \quad \text{and} \quad |b_1| = 1-\alpha.$$

Motivated by the classes $\mathscr{M}_{\Sigma}^{\mu}(\alpha,\lambda)$ and $\mathscr{N}_{\Sigma}^{\mu}(\alpha,\lambda)$ introduced by M. Cağlar et. al [20] for the bi-univalent analytic functions (see section 3), Hamidi et.al introduced similar classes $\mathscr{M}_{\sigma}(\alpha,\mu,\lambda)$ and $\mathscr{N}_{\sigma}(\alpha,\mu,\lambda)$ for the bi-univalent meromorphic functions in [33] and extended the results in [30, 32]. They obtained

Theorem[33]: For $0 \leq \lambda \leq 1$ and $0 \leq \alpha < 1$, let $q(z)$ given by (4.18) and let $q \in \mathscr{M}_{\sigma}(\alpha,\mu,\lambda)$. If $b_k = 0$; $(0 \leq k \leq n-1)$, then

$$|b_n| \leq \left|\frac{2(1-\alpha)}{\mu-(n+1)\lambda}\right|.$$

Theorem[33]: For $0 \leq \lambda \leq 1$ and $0 \leq \alpha < 1$ let $q(z)$ given by (4.18) and let $q \in \mathscr{N}_{\sigma}(\alpha,\mu,\lambda)$. Then

$$|b_0| \leq \begin{cases} \sqrt{\frac{4(1-\alpha)}{|(\mu-1)(\mu-2\lambda)|}}, & 0 \leq \alpha < 1 - \frac{(\mu-\lambda)^2}{|(\mu-1)(\mu-2\lambda)|}, \\ \\ \frac{2(1-\alpha)}{|\mu-\lambda|}, & 1 - \frac{(\mu-\lambda)^2}{|(\mu-1)(\mu-2\lambda)|} \leq \alpha < 1 \end{cases}$$

and

$$|b_1| \leq \frac{2(1-\alpha)}{|\mu-\lambda|}.$$

The recent publications [30] and [33], the application of the Faber polynomial expansions to meromorphic bi-univalent functions motivated to function theorists to apply the same technique to classes of analytic bi-univalent functions. In 2014, first time Hamidi and Jahangiri [29] considered the class of analytic bi-close-to-convex functions under certain gap series condition and derived bound of $|a_n|$ which are not yet appeared in the literature. They also demonstrated the unpredictability of the coefficient behavior of bi-starlike functions.

Close-to-convex function and Bi-Close-to-convex function: A function f is said to close to convex function if there exists a convex function $\phi(z)$ such that Re $\left(\frac{f'(z)}{\phi'(z)}\right) > 0$ for all $z \in \mathbb{U}$. A function is said to be bi-close-to-convex if both f and f^{-1} are close-to-convex.

Example [29]: For $n \geq 3$, $f(z) = z + \frac{1}{n-1}z^n$ is a bi-close-to-convex function.

We cite some of the results on this class as follows:

Theorem [29]: For $0 \leq \alpha < 1$ let $f(z) \in \mathscr{S}$ be a bi-close-convex function of order α. If $a_k = 0;\ 2 \leq k \leq n-1$, then

$$|a_n| \leq 1 + \frac{2(1-\alpha)}{n}.$$

Theorem [29]: Let $0 \leq \alpha < 1$ and $f \in \mathscr{S}^*_\Sigma(\alpha)$. Then

$$|a_2| \leq \begin{cases} \sqrt{2(1-\alpha)}, & 0 \leq \alpha < \frac{1}{2}, \\ 2(1-\alpha), & \frac{1}{2} \leq \alpha < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} 2(1-\alpha), & 0 \leq \alpha < \frac{1}{2}, \\ (1-\alpha)(3-2\alpha), & \frac{1}{2} \leq \alpha < 1 \end{cases}$$

This estimations improved the results of [15].

In the same year, S. Bulut considered the class $\mathscr{M}^\mu_\Sigma(\alpha, \lambda)$ defined by M. Cağlar in [20] (see section 3) and derived the coefficient estimations which improved the coefficient bounds derived by M. Cağlar in [20].

Theorem [9]: For $\lambda \geq 1$, $\mu \geq 0$ and $0 \leq \alpha < 1$, let $f(z) \in \mathscr{M}^\mu_\Sigma(\alpha, \lambda)$ be given by (1.1). If $a_k =;\ 2 \leq k \leq n-1$, then

$$|a_n| \leq \frac{2(1-\alpha)}{\mu + (n-1)\mu} \quad (n \geq 4).$$

Theorem [9]: For $\lambda \geq 1$, $\mu \geq 0$ and $0 \leq \alpha < 1$, let $f(z) \in \mathscr{M}_{\Sigma}^{\mu}(\alpha, \lambda)$ be given by (1.1). Then

$$|a_2| \leq \begin{cases} \sqrt{4\frac{1-\alpha}{(1+\mu)(2\lambda+\mu)}}, & 0 \leq \alpha < \frac{\mu+2\lambda-\lambda^2}{(1+\mu)(2\lambda+\mu)}, \\ \frac{2(1-\alpha)}{\lambda+\mu}, & \frac{\mu+2\lambda-\lambda^2}{(1+\mu)(2\lambda+\mu)} \leq \alpha < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \min\left\{\frac{4(1-\alpha)^2}{(\lambda+\mu)^2} + \frac{2(1-\alpha)}{2\lambda+\mu}, \frac{4(1-\alpha)}{(2\lambda+\mu)(1+\mu)}\right\}, & 0 \leq \mu < 1, \\ \frac{2(1-\alpha)}{\lambda+\mu}, & \mu \geq 1. \end{cases}$$

The estimation for $|a_2|$ is an improvement of the bound found by M. Cağlar in [20] and for particular values of $\mu = 1$ and $\lambda = 1$, it improved the result in [50]. Recently, Jahangiri et. al [35] considered the class of analytic bi-univalent functions with positive real-part derivatives and found the estimations of $|a_n|$ by using Faber polynomial.

Also [2, 16] are some recent references in which Faber polynomial used to derive the estimation of $|a_n|$ for some new class of bi-univalent functions.

Remark: Most of the studies on the classes of bi-univalent analytic functions and bi-univalent meromorphic functions, which are only in finding the estimations on bounds of some of the initial coefficient whose sharpness are still open. Although the classes of bi-univalent analytic functions have been extensively studied in this line but there are many problems like subordination, convolution preserving problems and many more are still untouched. So many new problems can be developed and solved on these classes.

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Fixed Points for Contractive Mappings on a Metric Space with a Graph: A Survey

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Abstract

In this paper we discuss and relate some important fixed point theorems and best proximity point theorems for contractions on a metric space endowed with a graph proved by various authors in recent times. We establish an existence theorem on best proximity point for generalized contractive mappings on a metric space endowed with a graph. Moreover, our theorem subsumes and generalizes many recent fixed point and best proximity point results.

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1 Introduction

Fixed point theory plays an important role in supplying a uniform treatment for solving equations of the form $f(x) = x$ where f is a mapping from a set K into a set X containing K. An element $x \in K$ is said

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to be a fixed point of the mapping f if $f(x) = x$. Fixed point theorems deal with sufficient conditions on the set $K \subseteq X$ and the mapping $f : K \to X$ which ensure the existence of a fixed point of f. Fixed point theorems can be classified as metric fixed point theorems, topological fixed point theorems and order-theoretic fixed point theorems. The well known Banach contraction principle is a fundamental theorem in metric fixed point theory. Its significance lies in its vast applicability in a number of branches of mathematics such as differential equations, eigenvalue problems, integral equations, numerical analysis and complex analysis. The Banach contraction principle states that if (X,d) is a complete metric space and f is a mapping from X into itself such that

$$d(f(x), f(y)) \leq kd(x,y)$$

for some $k \in (0,1)$ and for all $x,y \in X$, then the mapping f has a unique fixed point x_0 and the successive approximation $\{f^n(x)\}$ converges to x_0 for any $x \in X$.

Nadler [12] in 1969 generalized the contraction principle to set-valued mappings. Let $CB(X)$ be the family of all non-empty closed and bounded subsets of X and H be the Hausdorff metric induced by d on $CB(X)$. Nadler [12] proved that a mapping F from X into $CB(X)$ has a fixed point (i.e., there exists $x_0 \in X$ such that $x_0 \in F(x_0)$) if (X,d) is complete and $H(F(x),F(y)) \leq kd(x,y)$ for some $k \in (0,1)$ and for all $x,y \in X$.

After that, Mizoguchi and Takahashi [11] established the following result as an interesting extension of Nadler's theorem.

Theorem 1. *[11] Let (X,d) be a complete metric space and $F : X \to CB(X)$ be a mapping such that*

$$H(F(x),F(y)) \leq \alpha(d(x,y))d(x,y) \quad \forall x,y \in X, \tag{1.1}$$

where $\alpha : [0,\infty) \to [0,1)$ satisfying $\limsup_{s \to t+} \alpha(s) < 1$ for all $t \geq 0$. Then F has a fixed point.

Recently in 2008, Jachymski [8] extended the Banach contraction principle in a different direction. He obtained a fixed point theorem for single-valued mappings on a metric space endowed with a graph. Let

(X,d) be a metric space. Consider a directed graph G where the set $V(G)$ of its vertices coincides with X, the set $E(G)$ of its edges is such that $E(G) \supseteq \Delta$ (where $\Delta = \{(x,x) : x \in X\}$) and $E(G)$ has no parallel edges. Jachymski [8] proved the following theorem for mappings on (X,d) endowed with the graph G.

Theorem 2. *(see [8].) Let (X,d) be complete and $f : X \to X$ be a mapping such that for all $x,y \in X$ with $(x,y) \in E(G)$, $(f(x), f(y)) \in E(G)$ and $d(f(x), f(y)) \leq kd(x,y)$ where $k \in [0,1)$. Assume that for any $\{y_n\}_{n\in\mathbb{N}}$ in X with $y_n \to y^*$ and $(y_{n+1}, y_n) \in E(G)$ $\forall n \geq 1$, there exists a subsequence $\{y_{n_p}\}_{p\in\mathbb{N}}$ such that $(y_{n_p}, y^*) \in E(G)$ for all $p \in \mathbb{N}$. Then the following statements hold:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges to a fixed point of f if $(x, f(x)) \in E(G)$;

(ii) For each $x \in X$, $\{f^n(x)\}_{n\in\mathbb{N}}$ converges to a unique fixed point of f if G is weakly connected and there exists $x_0 \in X$ with $(x_0, f(x_0)) \in E(G)$.

Extension of the results due to Jachymski [8] for set-valued mappings can be found in [3]. For more fixed point results on a metric space with a graph, one can refer to [10, 2].

The paper is organized as follows. In Section 2, we have given some basic notations and definitions required for the paper. In Section 3, we have discussed and related few important fixed point results for set-valued contractions on a metric space with a graph proved by various authors in recent times. Then, in Section 4.1, we derive the existence of best proximity points for non-self Mizoguchi-Takahashi G-contraction.

2 Preliminaries

In this section, we now recall some definitions and notations which are needed and related to the context of our results.

Let (X,d) be a metric space, $CB(X)$ be the family of all non-empty closed and bounded subsets of X and $Cl(X)$ be the family of all non-empty closed subsets of X. For $A, B \in CB(X)$, let

$$H(A,B) = \max\left\{\sup_{b\in B} D(b,A), \sup_{a\in A} D(a,B)\right\},$$

where $D(a,B) = \inf_{b\in B} d(a,b)$. The mapping H is a metric on $CB(X)$ and it is said to be the *Hausdorff metric* induced by d.

Let us consider a directed graph G in which the set of its vertices coincides with X (that is, $V(G) = X$) and the set of its edges $E(G)$ contains all diagonal elements, that is, $E(G) \supseteq \Delta$ where $\Delta = \{(x,x) : x \in X\}$. We also suppose that G has no parallel edges. Thus the graph G can be identified with the pair $(V(G),E(G))$. Let G^{-1} denotes the graph derived from G by reversing the direction of the edges (that is, $E(G^{-1}) = \{(x,y) \in X\times X : (y,x) \in E(G)\}$). Let us denote by $\tilde{G}$ the undirected graph deduced from G by ignoring the direction of edges (i.e., $E(\tilde{G}) = E(G)\cup E(G^{-1})$).

Following [9], we now introduce some basic notions concerning the connectivity of graphs. Let $x,y \in X$. A *path* in G of length N (where $N \in \mathbb{N}\cup\{0\}$) from x to y is a sequence $(x^i)_{i=0}^N$ of points in X such that $x^0 = x$, $x^N = y$ and $(x^{i-1},x^i) \in E(G)$ $\forall i = 1,2,\ldots,N$. We call that the graph G is connected if there exists a path between any two vertices and weakly connected if $\tilde{G}$ is connected. Let us denote

$$[x]_G^N = \{y \in X : \text{there is a path in } G \text{ of length } N \text{ from } x \text{ to } y\},$$

$$[x]_G = \bigcup_{n\in\mathbb{N}} [x]_G^N.$$

Throughout this articl the notation S denotes the class of functions $\alpha : (0,\infty) \to [0,1)$ satisfying $\limsup_{s\to t+} \alpha(s) < 1$ for every $t \in [0,\infty)$.

3 Fixed Points of Set-Valued Contractions on a Metric Space with a Graph

Throughout this section we assume that (X,d) is a metric space and G is a directed graph such that $V(G) = X$, $E(G) \supseteq \Delta$ and G has no parallel edges. Now, we recall the notion of Mizoguchi-Takahashi G contraction from the paper due to Sultana and Vetrivel [15].

Definition 1. *[15] A set-valued mapping $F : X \to CB(X)$ is said to be Mizoguchi-Takahashi G-contraction if for all $x,y \in X$ with $(x,y) \in E(G)$,* *(i)* $H(F(x),F(y)) \leq \alpha(d(x,y))d(x,y)$ *for some* $\alpha \in S$;

(ii) if $u \in F(x)$ *and* $v \in F(y)$ *is such that* $d(u,v) \leq d(x,y)$, *then the pair* $(u,v) \in E(G)$.

Remark 1. Let (X,d) be a metric space and $F : X \to CB(X)$ be a set-valued mapping satisfying the contractive condition (1.1) due to Mizoguchi-Takahashi [11]. Then the mapping F is a Mizoguchi-Takahashi G-contraction for the graph G where $V(G) = X$ and $E(G) = X \times X$. But the converse need not be true which is illustrated by the following example.

Example 1. *Let* $X = \{\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots\} \cup \{0,1\}$, $d(x,y) = |x-y|$ *for* $x,y \in X$. *Define a map* $F : X \to CB(X)$ *as follows:*

$$F(x) = \begin{cases} \left\{0, \frac{1}{2}\right\} & \text{for } x = 0, \\ \left\{\frac{1}{2^{n+1}}, \frac{1}{2}\right\} & \text{for } x = \frac{1}{2^n},\ n = 1,2,\cdots, \\ \{1\} & \text{for } x = 1. \end{cases}$$

Consider a graph G *with* $V(G) = X$ *and* $E(G) = \{(x,y) \in X \times X : d(x,y) < \frac{1}{2}\}$. *Clearly* $E(G) \supseteq \Delta$ *and* G *has no parallel edges. For* $x = 0$ *and* $y = \frac{1}{2^n}$, $n \geq 2$, *the pair* $(x,y) \in E(G)$ *and*

$$H(F(x),F(y)) = H\left(\left\{0, \frac{1}{2}\right\}, \left\{\frac{1}{2^{n+1}}, \frac{1}{2}\right\}\right) = \frac{1}{2^{n+1}} \leq \frac{1}{2}d(x,y).$$

Also, for $x = \frac{1}{2^n}$ *and* $y = \frac{1}{2^m}$, $m \geq n \geq 1$, *the pair* $(x,y) \in E(G)$ *and*

$$H(F(x),F(y)) = H\left(\left\{\frac{1}{2^{n+1}}, \frac{1}{2}\right\}, \left\{\frac{1}{2^{m+1}}, \frac{1}{2}\right\}\right) = \frac{1}{2^{n+1}} - \frac{1}{2^{m+1}} \leq \frac{1}{2}d(x,y).$$

Hence, for all $x,y \in X$ *with* $(x,y) \in E(G)$, $H(F(x),F(y)) \leq \alpha(d(x,y))d(x,y)$ *where* $\alpha(t) = \frac{1}{2}$ *for all* $t \in [0,\infty)$ *and if* $u \in F(x)$ *and* $v \in F(y)$ *is such that* $d(u,v) \leq d(x,y)$, *then* $(u,v) \in E(G)$. *Thus* F *is a Mizoguchi-Takahashi* G*-contraction. However, for* $x = 0$ *and* $y = 1$,

$$H(F(x),F(y)) = H\left(\left\{0, \frac{1}{2}\right\}, \{1\}\right) = 1 = d(0,1) > \phi(d(0,1))d(0,1),$$

for any $\phi : [0,\infty) \to [0,1)$ *with* $\limsup_{s \to t+} \phi(s) < 1$ *for all* $t \geq 0$. *This proves that* F *does not satisfy the contractive condition (1.1).*

In [15], the authors proved the following theorem that gives the sufficient conditions for the existence of fixed points for Mizoguchi-Takahashi G-contractions.

Theorem 3. *[15] Let (X,d) be complete and $F : X \to CB(X)$ be a Mizoguchi-Takahashi G-contraction. For some $N \in \mathbb{N}$, suppose that*

(a) there exists $x_0 \in X$ such that $[x_0]_G^N \cap F(x_0) \neq \emptyset$;

(b) for any sequence $\{z_n\}_{n\in\mathbb{N}}$ in X, if $z_n \to z$ and $z_{n+1} \in [z_n]_G^N \cap F(z_n)$ $\forall n \in \mathbb{N}$, then there is a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ such that $(z_{n_k}, z) \in E(G)$ for $k \in \mathbb{N}$.

Then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X where $x_n \in [x_{n-1}]_G^N \cap F(x_{n-1})$ for all $n \in \mathbb{N}$ converging to a fixed point of F.

Remark 2. *It is worth to note that the above theorem yields Theorem 1 by considering the graph G where $V(G) = X$ and $E(G) = X \times X$.*

Sultana and Vetrivel [15] also studied the existence of unique fixed point for single-valued mappings.

Theorem 4. *[15] Let (X,d) be complete and $f : X \to X$ be a single-valued map such that for all $(x,y) \in E(G)$,*

$$(f(x), f(y)) \in E(G) \text{ and } d(f(x), f(y)) \leq \alpha(d(x,y))d(x,y), \tag{3.2}$$

where $\alpha \in S$. For some natural number N, assume that

(I) there exists $x_0 \in X$ such that $f(x_0) \in [x_0]_G^N$;

(II) for any sequence $\{z_n\}_{n\in\mathbb{N}}$ in X, if $z_n \to z \in X$ and $z_{n+1} \in [z_n]_G^N$ $\forall n \in \mathbb{N}$, then there exists $\{z_{n_k}\}_{k\in\mathbb{N}}$ such that $(z_{n_k}, y) \in E(G)$ for $k \in \mathbb{N}$.

Then for all $x \in X$, $\{f^n(x)\}_{n\in\mathbb{N}}$ converges to a unique fixed point of f if G is weakly connected.

Remark 3. *We have Theorem 2, the fixed point theorem due to Jachymski [8] as a corollary to the above result by considering $\alpha(t) = k$ for all $t \in [0,\infty)$.*

After that, Sultana and Vetrivel [17] extended the notion of Mizoguchi-Takahashi G-contraction without involving the Hausdorff metric and study the existence of fixed points for such contractions. By following [17], we introduce the notion of generalized Mizoguchi-Takahashi G-contraction.

Definition 2. *A set-valued map $F: X \to Cl(X)$ is called generalized Mizoguchi-Takahashi G-contraction if for any $x \in X$ and $y \in F(x)$ with $(x,y) \in E(G)$,*

(i) $D(y,F(y)) \leq \alpha(d(x,y))d(x,y)$ for some $\alpha \in S$;

(ii) if $z \in F(y)$ with $d(z,y) \leq d(x,y)$, then $(z,y) \in E(G)$.

Remark 4. *Any Mizoguchi-Takahashi G-contraction is a generalized Mizoguchi-Takahashi G-contraction. In fact, for any $x \in X$ and $y \in F(x)$ with $(x,y) \in E(G)$,*

$$D(y,F(y)) \leq H(F(x),F(y)) \leq \alpha(d(x,y))d(x,y) \text{ for some } \alpha \in S,$$

and if $z \in F(y)$ is such that $d(z,y) \leq d(x,y)$, then $(z,y) \in E(G)$. But the converse need not be true which is illustrated by the following example.

Example 2. *Let (X,d) be a metric space where $X = [0,1]$ and $d: X \times X \to \mathbb{R}$ is the standard metric. Define a function $F: X \to CB(X)$ as*

$$F(x) = \begin{cases} \{0\} & \text{for } x = 0, \\ \{\frac{1}{2}x^2, 1\} & \text{for } x \in (0, \frac{1}{4}], \\ \{\frac{1}{2}x^2\} & \text{for } x \in (\frac{1}{4}, 1), \\ \{\frac{1}{2}, 1\} & \text{for } x = 1. \end{cases}$$

Consider a graph G where $V(G) = X$ and $E(G) = \{(x,y) \in X \times X : d(x,y) < \frac{1}{2}\}$. It is clear that $E(G) \supseteq \Delta$ and G has no parallel edges. Let us define a function $\alpha: [0,\infty) \to [0,1)$ by

$$\alpha(t) = \begin{cases} \frac{3}{2}t, & \text{for } t \in [0, \frac{1}{2}), \\ 0 & \text{for } t \geq \frac{1}{2}. \end{cases}$$

Thus the function $\alpha \in S$. Moreover, it is easy to verify that for any $x \in X$ and $y \in F(x)$ with $(x,y) \in E(G)$,

$$D(y,F(y)) \leq \alpha(d(x,y))d(x,y)$$

and for any $z \in F(y)$ with $d(z,y) \leq d(x,y)$, $(z,y) \in E(G)$. Hence F is a generalized Mizoguchi-Takahashi G-contraction. However, for $(0,\frac{1}{4}) \in E(G)$,

$$H\left(F(0), F\left(\frac{1}{4}\right)\right) = 1 > \alpha\left(d\left(0,\frac{1}{4}\right)\right) d\left(0,\frac{1}{4}\right),$$

for any $\alpha \in S$. Thus F is not a Mizoguchi-Takahashi G-contraction.

The authors in [17] established the below stated theorem for giving the existence of fixed points for a generalized Mizoguchi-Takahashi G-contraction.

Theorem 5. *Let (X,d) be complete and $F: X \to Cl(X)$ be a generalized Mizoguchi-Takahashi G-contraction. Assume that*

(a) there exists $x_0 \in X$ such that $[x_0]_G^1 \cap F(x_0) \neq \emptyset$;

(b) for any $\{z_n\} \subseteq X$, if $z_n \to z$ and $z_{n+1} \in [z_n]_G^1 \cap F(z_n)$ $\forall n \in \mathbb{N}$, then there exists $\{z_{n_k}\} \subseteq \{z_n\}$ such that $D(z,F(z)) \leq \liminf_{k\to\infty} D(z_{n_k}, F(z_{n_k}))$.

Then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in [x_{n-1}]_G^1 \cap F(x_{n-1})$ for any $n \in \mathbb{N}$ converging to a fixed point of F.

It is worth to see that Theorem 5 yields the following result which is same as Theorem 3 when $N = 1$.

Corollary 1. *Let (X,d) be complete and $F: X \to CB(X)$ be a Mizoguchi-Takahashi G-contraction. Assume that*

(i) there exists $x_0 \in X$ such that $[x_0]_G^1 \cap F(x_0) \neq \emptyset$;

(ii) for any $\{z_n\}$ in X, if $z_n \to z$ and $z_{n+1} \in [z_n]_G^1 \cap F(z_n)$ for all $n \in \mathbb{N}$, then there exists $\{z_{n_k}\} \subseteq \{z_n\}$ such that $(z_{n_k}, z) \in E(G)$ for $k \in \mathbb{N}$.

Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \in [x_{n-1}]_G^1 \cap F(x_{n-1})$ for all $n \in \mathbb{N}$ converging to a fixed point of F.

Proof. According to Remark 4, the map F is a generalized Mizoguchi-Takahashi G-contraction. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $y_n \to y$ and $y_{n+1} \in [y_n]_G^1 \cap F(y_n)$ $\forall n \in \mathbb{N}$. Then according to the condition (ii), there exists a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ such that $(y_{n_k}, y) \in E(G)$ for $k \in \mathbb{N}$. For any $p \in F(y_{n_k})$ (where $k \in \mathbb{N}$),

$$\begin{aligned} D(y,F(y)) &\leq d(y,y_{n_k}) + d(y_{n_k},p) + D(p,F(y)) \\ &\leq d(y,y_{n_k}) + d(y_{n_k},p) + H(F(y_{n_k}),F(y)). \end{aligned}$$

Since $p \in F(y_{n_k})$ is arbitrary, we get

$$\begin{aligned} D(y,F(y)) &\leq d(y,y_{n_k}) + D(y_{n_k},F(y_{n_k})) + H(F(y_{n_k}),F(y)) \\ &\leq d(y,y_{n_k}) + D(y_{n_k},F(y_{n_k})) + d(y_{n_k},y), \end{aligned}$$

for any $k \in \mathbb{N}$. In the last inequality, we have used the fact that F is a Mizoguchi-Takahashi G-contraction and $(y_{n_k}, y) \in E(G)$ for all $k \in \mathbb{N}$. As $y_{n_k} \to y \in X$, we have from the above inequality that $D(y,F(y)) \leq \liminf_{k\to\infty} D(y_{n_k},F(y_{n_k}))$. This implies that the condition (b) of Theorem 5 holds. Again, it is obvious from condition (i) that the condition (a) of Theorem 5 is also satisfied. Therefore the proof follows by Theorem 5. $\square$

Remark 5. *M. Frigon and T. Dinevari [3] extended the Nadler's fixed point theorem for mappings on a metric space endowed with a graph. On the other hand, Eldred et al. proved that the fixed point theorem due to Nadler [12] is equivalent to Theorem 1 due to Mizoguchi and Takahashi [11] on a metrically convex complete metric space. It will be of interest to see whether one can derive that the equivalent between Theorem 3 and the fixed point theorem due to M. Frigon and T. Dinevari [3] for mappings on a metrically convex complete metric space endowed with a graph.*

4 Best Proximity Points for Non-Self Contractions

Let A and B be two non-empty subsets of a metric space (X,d). For a non-self single valued map $f : A \to B$, if $f(A) \cap A = \emptyset$, there does not exist a solution of the equation $f(x) = x$. This means that the map $f : A \to B$ does not have any fixed point. Then it is interesting to find a point $x \in A$ that is closest to $f(x)$ in some sense. Best approximation and best proximity point results have been established in this direction. The well-known best approximation theorem due to Ky Fan [6] states that for a given non-empty compact convex subset C of a normed linear space E and a continuous mapping $S : C \to E$, there exists $x^* \in C$ such that $\|x^* - S(x^*)\| = D(S(x^*), C) = \inf\{\|S(x^*) - x\| : \ x \in C\}$. Though this result gives the existence of an approximate solution of $S(x) = x$, such solution need not be optimal in the sense that $\|x - S(x)\|$ is minimum.

Naturally for the map $f : A \to B$, one can think of finding an element $x^* \in A$ such that $d(x^*, f(x^*)) = \min\{d(x, f(x)) : x \in A\}$. Since for all $x \in A$, $d(x, f(x)) \geq dist(A,B) = \inf\{d(a,b) : a \in A, \ b \in B\}$, an optimal solution of $\min\{d(x, f(x)) : x \in A\}$ is one for which the value $dist(A,B)$ is attained. An element $x^* \in A$ is called a *best proximity point* for the map f if $d(x^*, f(x^*)) = dist(A,B)$. Hence a best proximity point of the map f is not only an approximate solution of $f(x) = x$, but also optimal in the sense that $d(x, f(x))$ is minimum. For some interesting best proximity point results one can refer to [1, 4, 7, 16].

Example 3. *Consider $X = \mathbb{R}^2$ with usual metric and suppose that*

$$A = \left\{ \left(0, \frac{1}{n}\right) : n \in \mathbb{N} \right\} \cup \{(0,0)\},$$
$$B = \left\{ \left(1, \frac{1}{n}\right) : n \in \mathbb{N} \right\} \cup \{(1,0)\}.$$

Let us define a map $f : A \to B$ as follows:

$$f((0,x)) = \left(1, \frac{x}{2}\right), \quad \textit{for all } (0,x) \in A.$$

It is clear that $d((0,0), f((0,0)) = dist(A,B) = 1$ and consequently, $(0,0)$ is a best proximity point of f.

For given two non-empty subsets A and B of a metric space (X,d), we denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x,y) = dist(A,B) \text{ for some } y \in B\}$$
$$B_0 = \{y \in B : d(x,y) = dist(A,B) \text{ for some } x \in A\}.$$

The pair (A,B) is said to have the *P-property* [13] if and only if

$$\left.\begin{array}{l} d(x_1,y_1) = dist(A,B) \\ d(x_2,y_2) = dist(A,B) \end{array}\right\} \quad \Rightarrow \quad d(x_1,x_2) = d(y_1,y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is easy to check that for a non-empty subset A of (X,d), the pair (A,A) has the P-property.

Example 4. *Let the metric space (X,d) and the sets A and B be as in Example 3. Note that in this case $A_0 = A$, $B_0 = B$ and $dist(A,B) = 1$. Suppose that*

$$d((0,x_1),(1,y_1)) = \sqrt{1+(x_1-y_1)^2} = dist(A,B) = 1,$$
$$d((0,x_2),(1,y_2)) = \sqrt{1+(x_2-y_2)^2} = dist(A,B) = 1,$$

where $(0,x_1),(0,x_2) \in A_0$ and $(1,y_1),(1,y_2) \in B_0$. Then $x_1 = y_1$ and $x_2 = y_2$. Thus

$$d((0,x_1),(0,x_2)) = |x_1 - x_2| = |y_1 - y_2| = d((1,y_1),(1,y_2)).$$

Hence the pair (A,B) has the P-property.

Example 5. *[13] Let A and B be two non-empty closed convex subsets of a real Hilbert space H. Then the pair (A,B) has the P-property.*

Example 6. *[1] Let A and B be two non-empty bounded closed convex subsets of a uniformly convex Banach space X. Then the pair (A,B) has the P-property.*

The following best proximity point theorem proved by Sankar Raj [14] generalizes the Banach contraction principle.

Theorem 6. *[14] Let (X,d) be a complete metric space and A and B be two non-empty closed subsets of (X,d) such that $A_0 \neq \emptyset$ and the pair (A,B) satisfies the P-property. Suppose that $f : A \to B$ is a map such that $f(A_0) \subseteq B_0$ and*

$$d(f(x),f(y)) \leq kd(x,y) \quad \textit{for all } x,y \in A \textit{ and for some } k \in [0,1). \tag{4.3}$$

Then there exists a unique x^* *in* A *such that* $d(x^*, f(x^*)) = dist(A,B)$. *Further, for any fixed* $x_1 \in A_0$, *there exists a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *with* $d(x_{n+1}, f(x_n)) = dist(A,B)$ *for* $n \in \mathbb{N}$, *converging to* x^*.

After that, Sultana and Vetrivel [16] established the below stated theorem which gives sufficient conditions for the existence of a unique best proximity point for generalized contractions.

Theorem 7. *Let* A *and* B *be two non-empty closed subsets of a complete metric space* (X,d) *such that the pair* (A,B) *has the P-property and* $A_0 \neq \emptyset$. *Let* $f : A \to B$ *be a map with* $f(A_0) \subseteq B_0$. *Suppose that* f *satisfies any one of the following contractive conditions:*

(I) $d(f(x), f(y)) \leq \phi(d(x,y))$ *for all* $x,y \in A$, *where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is non-decreasing and satisfies* $lim_{n\to\infty}\phi^n(t) = 0$ *for any* $t > 0$;

(II) $d(f(x), f(y)) \leq d(x,y) - \psi(d(x,y))$ *for all* $x,y \in A$, *where* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is either non-decreasing or continuous with* $\psi^{-1}(0) = \{0\}$.

Then there exists a unique x^* *in* A *such that* $d(x^*, f(x^*)) = dist(A,B)$. *Moreover, if* $x_0 \in A_0$ *and* x_n *is defined by* $d(x_n, f(x_{n-1})) = dist(A,B)$ $\forall n \in \mathbb{N}$, *then* $x_n \to x^*$ *as* $n \to \infty$.

4.1 Best Proximity Point Theorem on a Metric Space with a Graph

In this section, we introduce the concept of non-self Mizoguchi-Takahashi G-contraction and study the existence of best proximity points for such contractions.

Definition 3. *Let* A *and* B *be two non-empty subsets of* (X,d). *A map* $f : A \to B$ *is called a non-self Mizoguchi-Takahashi G-contraction if for all* $x,y \in A$ *with* $(x,y) \in E(G)$:

(a) $d(f(x), f(y)) \leq \alpha(d(x,y))d(x,y)$ *for some* $\alpha \in S$;

(b) $$\left.\begin{array}{l} d(x_1, f(x)) = dist(A,B) \\ d(y_1, f(y)) = dist(A,B) \end{array}\right\} \Rightarrow (x_1,y_1) \in E(G), \quad \text{for all } x_1, y_1 \in A.$$

The following theorem gives sufficient conditions for the existence of a best proximity point for a non-self Mizoguchi-Takahashi G-contraction.

Theorem 8. *Let (X,d) be complete and A and B be two non-empty closed subsets of (X,d) such that (A,B) has the P-property. Let $f : A \to B$ be a non-self Mizoguchi-Takahashi G-contraction such that $f(A_0) \subseteq B_0$. For some $N \in \mathbb{N}$, assume that*

(i) there exist x_0 and x_1 in A_0 such that there is a N-length path $(y_0^i)_{i=0}^N \subseteq A_0$ in G between them and $d(x_1, f(x_0)) = dist(A,B)$;

(ii) for any sequence $\{s_n\}_{n\in\mathbb{N}}$ in A with $s_n \to s$ and $s_{n+1} \in [s_n]_G^N$ $\forall n \in \mathbb{N}$, there is a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ such that $(s_{n_k}, s) \in E(G)$ $\forall k \in \mathbb{N}$.

Then f has a best proximity point x^* and further, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $d(x_{n+1}, f(x_n)) = dist(A,B)$ for $n \in \mathbb{N}$, converging to x^*.

Proof. It follows from (i) that there is two points x_0 and x_1 in A_0 such that

$$d(x_1, f(x_0)) = dist(A,B)$$

and a sequence $(y_0^i)_{i=0}^N$ containing points of A_0 such that $y_0^0 = x_0$, $y_0^N = x_1$ and $(y_0^{i-1}, y_0^i) \in E(G)$ $\forall i = 1,\dots,N$. Since $y_0^1 \in A_0$ and $f(y_0^1) \in f(A_0) \subseteq B_0$, there exists $y_1^1 \in A_0$ such that

$$d(y_1^1, f(y_0^1)) = dist(A,B). \tag{4.4}$$

In a similar fashion, for $i = 2,\cdots,N$, there exists $y_1^i \in A_0$ such that

$$d(y_1^i, f(y_0^i)) = dist(A,B). \tag{4.5}$$

As $(y_0^0 = x_0, y_0^1) \in E(G)$ and $d(x_1, f(x_0)) = d(y_1^1, f(y_0^1)) = dist(A,B)$, it is apparent from the definition of non-self Mizoguchi-Takahashi G-contraction that $(x_1, y_1^1) \in E(G)$. Also, for each $i = 2,\cdots,N$, $(y_0^{i-1}, y_0^i) \in E(G)$ and

$$d(y_1^{i-1}, f(y_0^{i-1})) = d(y_1^i, f(y_0^i)) = dist(A,B). \qquad \text{[by (4.4) and (4.5)]}$$

Hence, by the definition, it follows that $(y_1^{i-1}, y_1^i) \in E(G)$ for each $i = 2, \cdots, N$. Let $x_2 = y_1^N$. Thus $(y_1^i)_{i=0}^N$ is a path from $x_1 (= y_1^0)$ to $x_2 (= y_1^N)$.

Further, for each $i = 1, 2, \cdots, N$, since $y_1^i \in A_0$ and $f(y_1^i) \in f(A_0) \subseteq B_0$, there exists $y_2^i \in A_0$ such that $d(y_2^i, f(y_1^i)) = dist(A,B)$. Also, we have $d(x_2, f(x_1)) = dist(A,B)$. Similar to the previous paragraph, it appears that $(x_2, y_2^1) \in E(G)$ and $(y_2^{i-1}, y_2^i) \in E(G)$ for each $i = 2, \cdots, N$. Set $x_3 = y_2^N$. Therefore $(y_2^i)_{i=0}^N$ is a path from $x_2 (= y_2^0)$ to $x_3 (= y_2^N)$.

By continuing in this manner for all $n \in \mathbb{N}$, we obtain a sequence $\{x_n\}_{n \in \mathbb{N}}$ where $x_{n+1} \in [x_n]_G^N$ and $d(x_{n+1}, f(x_n)) = dist(A,B)$ by producing a path $(y_n^i)_{i=0}^N$ from $x_n (= y_n^0)$ to $x_{n+1} (= y_n^N)$ in such way that

$$d(y_n^i, f(y_{n-1}^i)) = dist(A,B) \quad \forall i = 0, \ldots, N. \tag{4.6}$$

Using the P-property of (A,B), it is evident from equation (4.6) that for each $n \in \mathbb{N}$,

$$d(y_n^{i-1}, y_n^i) = d(f(y_{n-1}^{i-1}), f(y_{n-1}^i)) \quad \forall i = 1, \ldots, N. \tag{4.7}$$

Since for all $n \in \mathbb{N}$ and $i = 1, 2, \cdots, N$, $(y_{n-1}^{i-1}, y_{n-1}^i) \in E(G)$ and f is a non-self Mizoguchi-Takahashi G-contraction, it is clear that for any positive integer n and for each $i = 1, 2, \ldots, N$,

$$\begin{aligned} d(y_n^{i-1}, y_n^i) &= \alpha(d(y_{n-1}^{i-1}, y_{n-1}^i)) d(y_{n-1}^{i-1}, y_{n-1}^i) \\ &< d(y_{n-1}^{i-1}, y_{n-1}^i). \end{aligned} \tag{4.8}$$

For $1 \le i \le N$, we denote $d_n^i = d(y_n^{i-1}, y_n^i)$ for $n \ge 1$. By the above inequality, it is obvious that for every i, $\{d_n^i\}_{n \in \mathbb{N}}$ is a monotonically non-increasing sequence of non-negative real numbers. Let $d_n^i \to s^i \ge 0$ as $n \to \infty$. For each $i = 1, \ldots, N$, since $\limsup_{t \to s^i+} \alpha(t) < 1$, there exist positive integer M^i and real number $h^i \in [0,1)$ such that $\alpha(d_n^i) < h^i \; \forall n \ge M^i$ where $\limsup_{t \to s^i+} \alpha(t) < h^i < 1$. Hence for each i,

$$\alpha(d_n^i) < h \quad \forall n \ge M, \quad \text{where } h = \max_{1 \le i \le N} h^i \text{ and } M = \max_{1 \le i \le N} M^i.$$

Hence for $1 \le i \le N$ and $n \ge M+1$,

$$\begin{aligned} d_n^i &\le \alpha(d_{n-1}^i)d_{n-1}^i \cdots \le \prod_{s=0}^{n-1} \alpha(d_s^i)d_0^i \\ &\le h^{n-M} \prod_{s=0}^{M-1} \alpha(d_s^i)d_0^i = C^i h^n, \end{aligned}$$

where C^i is a non-negative real number. Subsequently, for $n \ge M+1$,

$$d(x_n, x_{n+1}) = d(y_n^0, y_n^N) \le \sum_{i=1}^{N} d_n^i \le \sum_{i=1}^{N} C^i h^n.$$

Hence, for $n \ge M+1$ and $m \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_{n+m}) &\le d(x_n, x_{n+1}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\le \sum_{i=1}^{N} C^i \left[h^n + \cdots + h^{n+m-1}\right] \le \sum_{i=1}^{N} C^i \frac{h^n}{1-h}. \end{aligned}$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore $\{x_n\}_{n\in\mathbb{N}}$ converges to some point $x^* \in A$ as $n \to \infty$. Since $x_n \to x^*$ and $x_{n+1} \in [x_n]_G^N$ for each $n \in \mathbb{N}$, according to the condition (ii), there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(x_{n_k}, x^*) \in E(G)$ $\forall k \in \mathbb{N}$. Hence,

$$d(f(x_{n_k}), f(x^*)) \le \alpha(d(x_{n_k}, x^*))d(x_{n_k}, x^*) \le d(x_{n_k}, x^*) \quad \text{for } k \in \mathbb{N}.$$

Thus as $k \to \infty$, $f(x_{n_k}) \to f(x^*)$. Using the continuity of the metric function, we get $d(x_{n_{k+1}}, f(x_{n_k})) \to d(x^*, f(x^*))$ as $k \to \infty$. Now $\{d(x_{n_{k+1}}, f(x_{n_k}))\}$ is nothing but a constant sequence with value $dist(A,B)$. Therefore $d(x^*, f(x^*)) = dist(A,B)$. This completes the proof. □

Remark 6. *It will be of interest to see whether one can derive the existence of best proximity points for mappings satisfying the generalized contractive conditions (I) or (II) on a metric space endowed with a graph.*

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Generalized Convexity in Mathematical Programming

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Abstract

The purpose of the paper is to give a brief review of some generalized convex functions existing in the literature. It also contains some unpublished definitions and results.

Keywords: Convex Function, Convexity, Mathematical Programming.

2010 AMS classification:90 C 30.

1 Introduction

Convexity plays a key role in mathematical programming. Though many significant results in mathematical programming have been derived under convexity assumptions, yet most of the real world problems are nonconvex in nature. Therefore a systematic attempt is being made by several authors to introduce and discuss various new kinds of generalized convex functions.

The purpose of this note is to give a brief review of various generalizations of convexity existing in the literature. The definitions of generalized convex functions are given in a tabular form and some results are quoted which give relationship among these concepts. This review also mentions some unpublished works of the authors and suggests several open problems for further study.

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Name of the function /set	Abbreviation	Definition of the functions	References
Convex set	Cs	$S \subset \mathbb{R}^n$ is convex if $x_1, x_2 \in S$ and $0 \le \lambda \le 1 \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in S$.	Fenchel[20], Valentine[59]
Convex	C	f is convex if $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$ for $\lambda \in [0,1]$ and x_1, x_2 in the domain. If f is differentiable an alternative definition is given by $f(x_1) - f(x_2) \ge (x_1 - x_2)^t \bigtriangledown f(x_2)$	Rockafellar[51], Mond[41], Blumberg[2], Fenchel [20], Kenyon [30],
Strictly convex	Sc	If strict inequality holds in the above definition for each distinct $x_1, x_2 \in \ S$ then f is strictly convex.	
Logarithmic Convex	Lc	A positive function f defined on a convex $S \subset \mathbb{R}^n$ is LC if for $x_1, x_2 \in S$, $0 < \lambda < 1$, $f(\lambda x_1 + (1-\lambda)x_2) \le f(x_1)^{\lambda}(f(x_2))^{1-\lambda}$	Klinger and Mangasarian [31]
Harmonic Convex	Hc	A positive function f defined on a convex $S \subset \mathbb{R}^n$ is Hc if for $x_1, x_2 \in S$, $0 < \lambda < 1$, $f(\lambda x_1 + (1-\lambda)x_2) \le \frac{1}{\frac{\lambda}{f(x_1)} \ \frac{1-\lambda}{f(x_2)}}$	Das [17]
Quasi-Convex	Qc	f is Qc if for all x_1, x_2 in its domain S and $\lambda \in [0,1]$, $f(\lambda x_1 + (1-\lambda)x_2) \le \max\,(f(x_1), f(x_2))$ f is differentiable quasi convex function if and only if $f(x_1) - f(x_2) \le 0$ $\Rightarrow (x_1 - x_2)^t \nabla f(x_2) \le 0$	Ponstein [49], Greemburg and Pierskatta [22]
Strictly Quasi-convex	Sqc	If strict inequality holds in the above definition for each distinct $x_1, x_2 \in \ S \subset \mathbb{R}^n$ then f is strictly quasi-convex	Mond[41]
Strongly Quasi-convex function	Sqc	f is strongly quasi-convex if for all x_1, x_2 $x_1 \neq x_2$ in its domain and all $\lambda \in (0,1)$, $\lambda x_1 + (1-\lambda)x_2$ is in its domain and $f(\lambda x_1 + (1-\lambda)x_2) < max(f(x_1), f(x_2))$	Ponstein[49]
Convex Like	Cl	$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is Cl if for $y \in \mathbb{R}^m$, $x_1, x_2 \in \mathbb{R}^n$, $\exists x_3 \in \mathbb{R}^n$ such that $f(x_3, y) \le \frac{f(x_1,y)+f(x_2,y)}{2}$	Simons [53]
Logarithmic Convex Like	Lcl	$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is Lcl if for $y \in \mathbb{R}^m$, $x_1, x_2 \in \mathbb{R}^n, \exists x_3 \in R^n$ such that $f(x_3, y) \le f(x_1, y)^{\frac{1}{2}} f(x_1, y)^{\frac{1}{2}}$	Kar [26]

Name of the function /set	Abbreviation	Definition of the functions	References
Harmonic Convex Like	Hcl	$f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is Hcl if for $y\in\mathbb{R}^m$, $x_1,\ x_2\in\mathbb{R}^n, \exists\ x_3\in\mathbb{R}^n$ such that $f(x_3,y)\le\frac{2f(x_1,y)f(x_2,y)}{f(x_1,y)+f(x_2,y)}$	Kar [26]
Quasi Convex Like	Qcl	$f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is Qcl if for $y\in\mathbb{R}^m$, $x_1,\ x_2\in\mathbb{R}^n, \exists\ x_3\in\mathbb{R}^n$ such that $f(x_3,y)\le\max\left(f(x_1,y),f(x_2,y)\right)$	Kar [26]
Convex type	Ct	$f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is Ct if for $y\in\mathbb{R}^m$, $x_1,\ x_2\in\mathbb{R}^n, \exists\ x_3\in\mathbb{R}^n$ such that $x_3\le\frac{x_1+x_2}{2}, f(x_3,y)\le\frac{f(x_1,y)+f(x_2,y)}{2}$	Kar [26]
Logarithmic Convextype	Lct	$f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is LCt if for $y\in\mathbb{R}^m$, $x_1,\ x_2\in\mathbb{R}^n, \exists\ x_3\in\mathbb{R}^n$ such that $x_3\le\frac{x_1+x_2}{2}, f(x_3,y)\le f(x_1,y)^{\frac{1}{2}}f(x_1,y)^{\frac{1}{2}}$	Kar [26]
Harmonic Convex type	Hct	$f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is Hct if for $y\in\mathbb{R}^m$, $x_1,\ x_2\in\mathbb{R}^n, \exists\ x_3\in\mathbb{R}^n$ such that $x_3\le\frac{x_1+x_2}{2}\ f(x_3,y)\le\frac{2f(x_1,y)f(x_2,y)}{f(x_1,y)+f(x_2,y)}$	Kar [26]
Quasi Convex type	Qct	$f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is Qct if for $y\in\mathbb{R}^m$, $x_1,\ x_2\in\mathbb{R}^n, \exists\ x_3\in\mathbb{R}^n$ such that $x_3\le\frac{x_1+x_2}{2}\ f(x_3,y)\le\max\left(f(x_1,y),f(x_2,y)\right)$	Kar [26]
Pseudoconvex	Pc	A differentiable function f is Pc if for x,y in the domain $(y-x)^t\nabla f(x)\ge 0$ $\Rightarrow f(y)-f(x)\ge 0$	Mangasarian [34, 35]
Strongly Pseudoconvex	Spc	A real differentiable function f is Spc with respect to a positive real function K(x,y) if $K(x,y)[f(y)-f(x)]\ge(y-x)^t\nabla f(x)$	Chandra [12] Mond [41]
Invex	I	f is invex w.r.t. a vector function h(x,y) if $f(y)-f(x)\ge\eta^t(x,y)\nabla f(x)$ for all x, y in the domain	Craven [13] Mond [41]
Pseudo-invex	Pi	f is Pi w.r.t.a vector function $\eta(x,y)$ if $\eta^t(x,y)\nabla f(x)\ge 0\Rightarrow f(y)-f(x)\ge 0$	Mond [41]
Quasi-invex	Qi	f is Qi w.r.t $\eta(x,y)$ if $\eta^t(x,y)\nabla f(x)\le 0$ $\Rightarrow f(y)-f(x)\le 0$	Mond [41]

Name of the function /set	Abbreviation	Definition of the functions	References
Harmonic Quasiconvex	Hqc	A differentiable function f is Hqc if for x_1, x_2 in the domain $\frac{f(x_2)-f(x_1)}{f(x_2)} \leq 0$ $\Rightarrow \frac{\nabla f(x_1)(x_2-x_1)}{f(x_1)} \leq 0$	Kar[26], Kar et al.[28]
Harmonic Pseudoconvex	Hpc	A differentiable function f is Hpc if for x_1, x_2 in the domain $\frac{\nabla f(x_1)(x_2-x_1)}{f(x_1)} \geq 0$ $\Rightarrow \frac{f(x_2)-f(x_1)}{f(x_2)} \leq 0$	Kar[26], Kar et al.[28]
Locally star-shaped set	Lsss	$S \subseteq \mathbb{R}^n$ is locally star-shaped at $\bar{x} \in S$ if corresponding to $\bar{x} \in S$ and each $X \in S$, $\exists$ a maximum positive number a $(\bar{x}, x) \leq 1$ such that $(1-\lambda)\bar{x} + \lambda x \in S, 0 < \lambda < a(\bar{x}, x)$	Kaul and Kaur [29]
Semilocally convex	Slc	A real function f defined on$S \subseteq \mathbb{R}^n$ is Slc at $\bar{x} \in S$ if S is locally star-shaped at$\bar{x}$ and corresponding to $\bar{x}$ and each $x \in S$, $\exists$ a positive number $d(\bar{x}, x) \leq a(\bar{x}, x)$ such that for$0 < \lambda < d(\bar{x}, x)$, $f((1-\lambda)\bar{x} + \lambda x) \leq (1-\lambda)f(\bar{x}) + \lambda f(x)$	Kaul and Kaur [29]
Semilocally Logarithmic convex	Silc	f defined on $S \subseteq \mathbb{R}^n$ is Silc at $\bar{x} \in S$ if S is locally star-shaped at $\bar{x}$ and corresponding to $\bar{x}$ and each $x \in S$, $\exists$ a positive number $d(\bar{x}, x) \leq a(\bar{x}, x)$ such that for $0 < \lambda < d(\bar{x}, x)$, $f((1-\lambda)\bar{x} + \lambda x) \leq (f(\bar{x})^(1-\lambda) f(\bar{x})^{\lambda})$	Kar[26], Kar et al.[28], Mishra[39]
Semilocally Harmonic convex	Sihc	f is Sihc at $\bar{x} \in S$ is S is locally star-shaped and corresponding to $\bar{x}$ and each $x \in S$,$\exists$ a positive number $d(\bar{x}, x) \leq a(\bar{x}, x)$ such that for $0 < \lambda < d(\bar{x}, x), f((1-\lambda)\bar{x} + \lambda x) \leq \frac{1}{\frac{1-\lambda}{f(\bar{x})} + \frac{\lambda}{f(x)}}$	Kar[26], Kar et al.[28], Mishra[39]
Semilocally Quasi convex	Siqc	f is Siqc at $\bar{x} \in S$ is S is locally star-shaped at $\bar{x}$ and corresponding to $\bar{x}$ and each $x \in S$,$\exists$ a positive number$d(\bar{x}, x) \leq a(\bar{x}, x)$such that for $0 \leq \lambda a \leq a(\bar{x}, x), f(x) \leq f(\bar{x})$ $\Rightarrow f((1-\lambda)\bar{x} + \lambda x) \leq f(\bar{x})$	Kaul and Kaur[29]

Name of the function /set	Abbreviation	Definition of the functions	References
Semilocally invex	Sli	A real function f defined on a set $S \subseteq \mathbb{R}^n$ is Sli at $\bar{x} \in S$ if for each $x \in S$ the right differential $(df)^+$ $(\bar{x}, x-\bar{x})$ of f at $\bar{x}$ in the direction of $x-\bar{x}$ exists and $f(y) - f(x) \geq h^t(x,y)$ $(df)^+(\bar{x}, x-\bar{x})$ for all x,y in the domain and some vector function $h(x,y)$	Kar[26], Kar and Nanda[27]
B-vex	Bv	$f : X \to \mathbb{R}$ is B-vex at $u \in X$ if there exists a function $b(x,u,\lambda) : X \times X \times [0,1] \to \mathbb{R}_+$ such that $f(\lambda x + (1-\lambda)u) \leq \lambda b(x,u,\lambda) f(x)$ $(1-\lambda b(x,u,\lambda)) f(u)$, for $0 \leq \lambda \leq 1$ and for every $x \in X$. f is said to be B-vex on X if it is B-vex at each $u \in X$.	Bector and Singh[1]
Quasi B-vex	QBv	A function f is said to be quasi B-vex at $u \in X$ if there exists a function $b(x,u,\lambda)$ such that $f(x) \leq f(u) \Leftrightarrow b(x,u,\lambda)$ $f[\lambda x + (1-\lambda)u] \leq b(x,u,\lambda) f(u)$, for $0 \leq \lambda \leq 1$ and for every $x \in X$. f is said to be quasi B-vex on X if it is quasi B-vex at each point $u \in X$.	Bector et al.[2]
Pseudo B-vex	PBv	A function f is said to be pseudo B-vex at $u \in X$ if there exists a function $\bar{b}(x,u)$ such that $(x-u)^T \nabla_x f(u) \geq 0$ $\Leftrightarrow \bar{b}(x,u) f(x) \geq \bar{b}(x,u) f(u), \quad \forall\ x \in\ X$ where $\bar{b}(x,u) = \lim_{\lambda \to 0_+} b(x,u,\lambda)$ f is said to be pseudo B-vex on X if it is pseudo B-vex at each $u \in\ X$.	Bector et al.[2]
B-preinvex	BPi	A numerical function f defined on a nonempty subset X of $\mathbb{R}^n$ which is invex at $u\ \in\ X$, is said to be B-preinvex with respect to η at $u\ \in\ X$ if there exists $b : X \times X \times [0,1] \to R_+$ such that $f[u + \lambda\eta(x,u)] \leq \lambda b(x,u,\lambda) f(x)$ $+(1-\lambda b(x,u,\lambda)) f(u)$ $\forall\ x\ \in\ X$ and $0 \leq \lambda \leq 1$ f is said to be B-preinvex with respect to η on X if it is B-preinvex at each $u\ \in\ X$ with respect to the same η.	Bector et al.[2]

Name of the function /set	Abbreviation	Definition of the functions	References
B-invex	Bi	A differentiable function f is B-invex with respect to η at $u \in X$ if there exists a function $b(x,u) : X \times X \to \mathbb{R}_+$ such that $\eta(x,u)^T \nabla_x f(u) \leq b(x,u)[f(x) - f(u)]$, $\forall\ x \in X$ and $0 \leq \lambda \leq 1$. f is said to be B-invex with respect to η on X if it is B-invex at each $u \in X$ with respect to the same η.	Bector et al.[2]
Pseudo B-invex	PBi	A differentiable function f is pseudo B-invex with respect to η at $u \in X$ if there exists a function $b(x,u)$ such that $\eta(x,u)^T \nabla_x f(u) \geq 0$ $\Rightarrow b(x,u) f(x) \geq b(x,u) f(u),\ \forall\ x \in X$ f is said to be Pseudo B-invex with respect to η on X if it is Pseudo B-invex at each $u \in X$ with respect to the same η.	Bector et al.[2]
Quasi B-invex	QBi	A differentiable function f is quasi B-invex with respect to η at $u \in X$ if there exists a function $b(x,u)$ such that $f(x) \leq f(u)$ $\Rightarrow \eta(x,u)^T \nabla_x f(u) \leq 0 \quad \forall\ x \in X$ f is said to be quasi B-invex with respect to η on X if it is quasi B-invex at each $u \in X$ with respect to the same η.	Bector et al.[2]
Prequasi invex	Pqi	f is prequasiinvex at $u \in X$ w.r.t. η if X is invex at u w.r.t. η and for each $x \in X$, $f(x) \leq f(u)$ $\Rightarrow f(u + \lambda\eta(x,u)) \leq (1-\lambda) f(u) + \lambda f(x)$ $0 \leq \lambda \leq 1$. f is prequasiinvex on X w.r.t. η if X is invex at u w.r.t. η and f is prequasiinvex at each $u \in X$ w.r.t. the same η	Jeyakumar[24]

Name of the function /set	Abbreviation	Definition of the functions	References
Strictly Prequasi invex	Spqi	f is strictly prequasiinvex at $u \in X$ w.r.t. η if X is invex at u w.r.t. η and for each $x \in X, f(x) < f(u)$ $\Rightarrow f(u + \lambda\eta(x,u)) < f(u)$ $0 < \lambda < 1$. f is strictly prequasiinvex on X w.r.t. η if X is invex at u w.r.t. η and f is prequasi invex at each $u \in X$ w.r.t. the same η	Jeyakumar[24]
(α,λ)-Convex	(α,λ)C	f is (α,λ)-convex if X is a convex set and $f(\bar{\alpha}u + \alpha x) \leq \bar{\lambda} f(u) + \lambda f(x)$, $\alpha \in [0,1]$, $x,\ u\ \in X$, $\bar{\lambda} = 1-\lambda$, $\bar{\alpha} = 1-\alpha$; $\lambda = \lambda(a; x, y) \in [0,1]$ depends in general both on α and the pair of points x,y satisfying $\lambda(a,x,y) = \bar{\lambda}(\bar{\alpha}, y, x)$.	Castagnoli[12]
B-preincave	BPic	f is B-preincave on X w.r.t. η, b_1, b_2 if -f is B-preinvex on X w.r.t. η, b_1, b_2.	Suneja et al.[55]
(ρ,θ)-B-vex	(ρ,θ)-Bv	Let X be a real Banach space, and D be a non-empty open convex subset of X. A function $f : D \subseteq X \to \mathbb{R}$ is said to be (ρ,θ)-B-vex at $y\ \in\ D$, if there exists a function $b(x,u,\lambda) : D \times D \times [0,1] \to \mathbb{R}_+$, $\theta : D \times D \to X$, and $\rho\ \in\ \mathbb{R}$ such that $f(\lambda x + (1-\lambda)y) \leq$ $\lambda b(x,y,\lambda) f(x) + (1 - \lambda b(x,y,\lambda)) f(y)$ $+\rho\\|\theta(x,y)\\|^2$ for $0 \leq \lambda \leq 1$, and $\forall\ x \in X$. f is said to be (ρ,θ)-B-vex on D if it is (ρ,θ)-B-vex at each $y\ \in\ D$.	Behera et al.[3]

Name of the function /set	Abbreviation	Definition of the functions	References
(ρ,θ)-η-B-preinvex	(ρ,θ)-η-BPI	A function $f : D \subseteq X \to \mathbb{R}$ defined on a non empty subset D of X which is invex at $y \in D$ is said to be $(\rho-\theta), -\eta)$ -B-preinvex w.r.t. to η, at $y \in D$, if there exist $b : D \times D \times [0,1] \to \mathbb{R}_+$, and $\rho \in \mathbb{R}$ such that $f(y+\lambda\eta(x,y)) \leq \lambda b(x,y,\lambda) f(x)$ $+(1-\lambda b(x,y,\lambda)) f(y) + \rho\|\theta(x,y)\|^2$ $\forall\ x \in D, 0 \leq \lambda \leq 1.$ f is said to be (ρ,θ)-η-B-preinvex w.r.t. η on D if it is (ρ,θ)-η-B-preinvex at each $y \in D$ w.r.t. the same η.	Behera et al.[3]
(ρ,θ)-η-B-invex	(ρ,θ)-η-BI	A differentiable function $f : D \subseteq X \to \mathbb{R}$ is said to be $(\rho,\theta)-\eta$-B-invex w. r.t. η, θ at $y \in D$, if there exist $\eta : D \times D \to X$, $\theta : D \times D \to X$, $b : D \times D \to \mathbb{R}_+$, and $\rho \in \mathbb{R}$ such that $b(x,y)(f(x) - f(y)) \geq (\nabla f(y), \eta(x,y))$ $+\rho\|\theta(x,y)\|^2\ \forall\ x \in D.$ f is said to be (ρ,θ)-η-B-invex w. r.t. η, θ on D, if it is (ρ,θ)-η-B-invex at each $y \in D$ w. r. t.the same η, θ.	Behera et al.[3]
(ρ,θ)-η-prequasiinvex	(ρ,θ)-η-Pqi	A function $f : D \subseteq X \to \mathbb{R}$ is said to be (ρ,θ)-η-prequasi-invex w. r.t. η, θ at $y \in D$, if D is invex w. r. t. η, and $\theta : D \times D \to X$, $\rho \in \mathbb{R}$, and for each $x \in D$, $f(x) \leq f(y)$ $\Rightarrow f(y+\lambda\eta(x,y)) \leq (1-\lambda) f(y) + \lambda f(x)$ $\rho\|\theta(x,y)\|^2, 0 \leq \lambda \leq 1.$ f is said to be (ρ,θ)-η-prequasi-invex on D w. r. t. η, if D is invex w. r. t. η, at each $y \in D$ w. r. t. the same η.	Behera et al.[3]

Name of the function /set	Abbreviation	Definition of the functions	References
strictly(ρ,θ)-η-prequasiinvex	S(ρ,θ)-η-Pqi	A function $f : D \subseteq X \to \mathbb{R}$ is said to be strictly (ρ,θ)-η-prequasi-invex w. r. t. η, θ at $y \in D$, if D is invex w. r. t. η, and $\theta : D \times D \to X$, $\rho \in \mathbb{R}$, and for each $x \in D$, $f(x) < f(y) \Rightarrow f(y+\lambda\eta(x,y)) < f(y) + \rho\\|\theta(x,y)\\|^2$, $0 < \lambda < 1$. f is said to be strictly (ρ,θ)-η -prequasi-invex on D w. r. t. η, if D is invex w. r. t. η, and strictly prequasi-invex at each $y \in D$ w. r. t. the same η.	Behera et al.[3]
ρ-(η,θ)-invex	ρ-(η,θ)I	A differentiable function $f : X \to \mathbb{R}$ is said to be $\rho-(\eta,\theta)$-invex w. r. t. η, θ, if there exist $\eta : X \times X \to X$, $\theta : X \times X \to X$ and $\rho \in \mathbb{R}$ such that $f(x_0) - f(x_1) \geq \langle f'(x_1), \eta(x_0,x_1)\rangle + \rho \parallel \theta(x_0,x_1) \parallel^2$ for all $x_0, x_1 \in X$.	Behera et al.[4]
ρ-(η,θ) pseudo-invex	ρ-(η,θ)PI	A differentiable function $f : X \to \mathbb{R}$ is said to be $\rho-(\eta,\theta)$-pseudo-invex w. r. t. η, θ, if there exist $\eta : X \times X \to X$, $\theta : X \times X \to X$ and $\rho \in \mathbb{R}$ such that $\langle f'(x_1), \eta(x_0,x_1)\rangle \geq -\rho \parallel \theta(x_0,x_1) \parallel^2 \Rightarrow f(x_0) \geq f(x_1)$	Behera et al.[4]
ρ-(η,θ) quasi-invex	ρ-(η,θ)Qi	A differentiable function $f : X \to \mathbb{R}$ is said to be ρ-(η,θ)-quasi-invex w. r. t. η, θ, if there exist $\eta : X \times X \to X$ and $\theta : X \times X \to X$ and $\rho \in \mathbb{R}$ such that $f(x_0) \leq f(x_1) \Rightarrow \langle f'(x_1), \eta(x_0,x_1)\rangle \leq -\rho \parallel \theta(x_0,x_1) \parallel^2$	Behera et al.[4]
ρ-(η,θ) pseudo-B-invex	ρ-(η,θ)PBI	A differentiable function $f : X \to \mathbb{R}$ is said to be $\rho-(\eta,\theta)$-pseudo-B-invex at u w. r. t η, θ, if there exist $\eta : C \times C \to X$, $\theta : C \times C \to X$, where C is a closed convex subset of X $\overline{b_0} : X \times X \to \mathbb{R}_+$ and $\rho \in \mathbb{R}$ such that $\langle f'(u), \eta(x,u)\rangle + \rho \parallel \theta(x,u) \parallel^2 \geq 0 \Rightarrow \overline{b_0}(x,u)(f(x) - f(u)) \geq 0$, $\forall x \in X$.	Nahak et al.[44]

Name of the function /set	Abbreviation	Definition of the functions	References
ρ-(η,θ) quasi-B-invex	ρ-(η,θ)QBi	A differentiable function $f: X \to \mathbb{R}$ is said to be $\rho-(\eta,\theta)$-quasi-B-invex at u w. r. t η, θ, if there exist $\eta: C\times C \to X$, $\theta: C\times C \to X$, where C is a closed convex subset of X $\overline{b_0}: X \times X \to \mathbb{R}_+$ and $\rho \in \mathbb{R}$such that $\overline{b_0}(x,u)\left\{f(x)-f(u)\right\} \leq 0$ $\Rightarrow \langle f'(u), \eta(x,u)\rangle + \rho \parallel \theta(x,u) \parallel^2 \leq 0,$ $\forall\ x \in X.$	Nahak et al.[44]
Invex set	Is	$S \subset \mathbb{R}^n$ is said to be invex with respect to a given function $\eta: R^n \times \mathbb{R}^n \to \mathbb{R}$ iff $x, y \in S$, $0 \leq \lambda \leq 1 \Rightarrow y + \lambda\eta(x,y) \in S$.	Craven[14]
E-convex set	Ecs	$M \subset \mathbb{R}^n$ is said to be E-convex with respect to an operator $E: \mathbb{R}^n \to \mathbb{R}^n$ iff $\lambda E(x) + (1-\lambda)E(y) \in$ M for each x, y $\in$ M and $\lambda \in [0,\ 1]$.	Youness[63]
(F, E)convex set	(E,F)cs	M $\subset \mathbb{R}^n$ is said to be (E, F) convex iff there exist two points to set maps E, $F: \mathbb{R}^n \to 2^{\mathbb{R}^n}$ such that $\lambda E(x) + (1-\lambda)E(y) \subset$M for all y $\in$ M and $\lambda \in [0,\ 1]$.	Youness[64]
Semi invex Set	Sis	Let K $\subset \mathbb{R}^n$ and $\eta: K \times K \times [0,1] \to \mathbb{R}^n$. K is said to be semi-invex at x $\in$ K $\forall\ y\ \in\ K$, $t\ \in [0,1]$, $x + t\eta(y,x,t) \in$ K	Yang and Chem[62]
Sub convex set	Scs	S $\subset \mathbb{R}^n$ is said to be subconvex if for each x_1, x_2 $\in$S, there exists $x_3 \in$ S such that $x_3 \leq \frac{1}{2}(x_1 + x_2)$	Simons[54]
Mid-point quasiconvex or m - quasi convex	Mpqc	f is m-quasi convex if for all x_1, x_2 in the domain $\frac{x_1+x_2}{2}$ is in the domain and $f(\frac{x_1}{2} + \frac{x_2}{2}) \leq max(f(x_1), f(x_2))$	Behringer[5]
Strictly m -quasi convex	Smqc	f is strictly m- quasi convex if for $f(x_1) \neq f(x_2), f(\frac{x_1}{2} + \frac{x_2}{2})$ $< max(f(x_1), f(x_2))$	Behringer[5]

Name of the function /set	Abbreviation	Definition of the functions	References
Rational quasiconvex or r- quasi-convex	Rqc	f is r-quasi-convex if for x_1, x_2 in the domain, $\lambda \in [0,1] \bigcap Q$ such that $\lambda x_1 + (1-\lambda)x_2$ is in the domain and $f(\lambda x_1 + (1-\lambda)x_2) \leq \max (f(x_1), f(x_2))$	Behringer[5]
Strictly r-quasi convex	Srqc	f is strictly r -quasi convex if $f(x_2) < f(x_1)$ implies $f(\lambda x_1 + (1-\lambda)x_2) < f(x_1)$ for all $\lambda \in (0,1) \bigcap Q$	Behringer[5]
Cone-convex or S-convex	Cc	f is S-convex (where S is a cone in $\mathbb{R}^n$)if $-f(\lambda x + (1-\lambda)y) + \lambda + (x) + (1-\lambda)f(y) \in$ S for $0 \leq \lambda \leq 1$. If f is differentiable, this is equivalent to $f(y) - f(x) - (y-x)^t \Delta f(x) \in$ S	Mond[41]
S-pseudo convex and S-Quasi convex	Spc and Sqc	f is S-pseudo convex if $(y-x)^t \Delta f(x) \in$ S $\Longrightarrow f(y) - f(x) \in$ S. f is S-quasi convex if $-f(y) + f(x) \in$S $\Longrightarrow (y-x)^t \Delta f(x) \in S$	Mond[41]
g-convex	gc	Let $g : \mathbb{R} \to \mathbb{R}$. $f : \mathbb{R}^n \to \mathbb{R}$ is g-convex if gf is convex on a convex set K, that is if $gf(\lambda x + (1-\lambda)y) \leq \lambda gf(x) + (1-\lambda)gf(y)$ for all $x, y \in K$ and $\lambda \in [0,1]$.	Nanda[45]
Pre-invex	Pi	Let A $\subset \mathbb{R}^n$ be an invex set and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n . f : A \to \mathbb{R}$ is said to be pre-invex with respect to η if $x, y \in A,\ \lambda \in [0,1] \Longrightarrow f(y + \lambda \eta(x,y)) \leq \lambda f(x) + (1-\lambda)f(y)$.	Weir and Mond[61]
Strictly Pre invex	Spi	f is strictly pre-invex if strict inequality holds in the above definition for $x \neq y$.	Weir and Mond[61]
Pre quasi invex	Pqi	$f : A \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be pre quasi -invex on an invex set A w.r.t. $\eta \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ if $x, y \in A,\ \lambda \in [0,1] \Longrightarrow f(y + \lambda \eta(x,y)) \leq max(f(x), f(y))$.	Yang and Chem[62]

Name of the function /set	Abbreviation	Definition of the functions	References
Semi-strictly Pre-invex	Sspi	$f: K \to \mathbb{R}^n$ is said to be semi-strictly pre-invex on a invex set K with respect to $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ if for $x, y \in K$, $\lambda \in [0,1]$, $f(x) \neq f(y) \Longrightarrow f(y+\lambda\eta(x,y)$ $< \lambda f(x) + (1-\lambda) f(y))$.	Yang and Chem[62]
Strictly pre-quasi invex	Spqi	f is strictly pre-quasi invex if f is pre quasi-invex such that $x \neq y$ implies $f(y+\lambda\eta(x,y) < max(f(x), f(y))$	Yang and Chem[62]
Semi strictly prequasi invex	Sspqi	f is semi strictly prequasi invex if $f(x) \neq f(y)$ implies $f(y+\lambda\eta(x,y))$ $< max(f(x), f(y))$.	Yang and Chem[62]
Semi pre-invex	Spi	Let $K \subset \mathbb{R}^n$ be a semi invex set with respect to $\eta: K \times K \times [0,1] \to \mathbb{R}^n$. $f: K \to \mathbb{R}$ is said to be semi pre invex if for all $x, t \in [0,1]$, $f(x+t\eta(y,x,t))$ $\leq (1-t) f(x) + t f(y)$and $lim_{t\to 0}[t\eta(y,x,t)] = 0$	Yang and Chem[62]
Quasi semi pre-invex	Qspi	$f: K \to \mathbb{R}$ is said to be quasi semi pre invex if K is semi invex $\eta: K \times K \times [0,1] \to \mathbb{R}$, $x, y \in K, t \in [0,1] \Longrightarrow f(x+tn(y,x,t))$ $\leq max(f(x), f(y))$	Yang and Chem[62]
Semi invex	Si	$f: K \to \mathbb{R}^n$ is said to be semi-invex with respect to $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$at $x \in K$ if for all $y \in K$, there exists $\xi \in R^n$ such that $f(y) - f(x) \geq (\xi, \eta(y,x))$	Dutta et. al[19]
Quasi-semi -invex	Qsi	f is quasi-semi invex if $f(y) - f(x) \leq 0$ $\Longrightarrow (\xi, \eta(y,x)) \leq 0$	[Nanda (unpublished)]
Arc wise convex	Awc	f is said to be arc wise convex if there is a map $H: \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$such that $f(H(x,y,t) \leq t f(y) + (1-t) f(x)$.	
Generalized pre-invex	Gpi	f is said to be generalized pre-invex (g preinvex)if there is $\xi: \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that $f(x+\xi(y,x,t)) \leq t f(y) + (1-t) f(x)$	[Nanda (unpublished)]
Logarithmic arcwise convex	Lac	f is said to be logarithmic arc wise convex if $f(H(x,y,t)) \leq (f(y))^t + (f(x))^{1-t}$	[Nanda (unpublished)]

Name of the function /set	Abbreviation	Definition of the functions	References
Logarithmic pre-invex (L-pre-invex)	Lpi	f is said to be logarithmic pre-invex if $f(x+t\eta(y,x)) \leq (f(y))^t + (f(x))^{1-t}$	[Nanda (unpublished)]
E-convex	Ec	A function $f:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be E-convex with respect to an operator $E:\mathbb{R}^n \to \mathbb{R}^n$ on an E-convex set M iff for $x, y \in M$, $\lambda \in [0,1]$, $f(\lambda Ex + (1-\lambda)Ey \leq \lambda(f \circ E)x$ $+(1-\lambda)(f \circ E)y$	Youness[63]
Quasi E-convex	QEc	$f:\mathbb{R}^n \to \mathbb{R}$ is said to be quasi E-convex on an E-convex set $M \subset \mathbb{R}^n$ with respect to a map $E:\mathbb{R}^n \to \mathbb{R}^n$ iff for $x, y \in M$, $\lambda \in [0,1]$, $f(\lambda Ex + (1-\lambda)Ey) \leq max((f \circ E)x, (f \circ E)y)$	Youness[63]
(E,F)-convex	(E,F)c	A map $f:\mathbb{R}^n \to \mathbb{R}$ is (E,F)-convex on a set $M \subseteq \mathbb{R}^n$ if there exist two point-to-set maps, $E, F:\mathbb{R}^n \to 2^{\mathbb{R}^n}$ such that (E,F)-convex and $f((\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \lambda f(\bar{x}) + (1-\lambda)f(\bar{y})$ $\forall \bar{x} \in E(x)$, $x, y \in M$, $\lambda \in [0,1]$	Youness[63]
Sublinear convex	Slc	A differentiable function θ is said to be sublinear convex if $\theta(y) - \theta(x) \geq F_{yx}\nabla\theta(x)$ $\forall x, y$ and for some arbitrary given sublinear functional F.	Mond[41]
F-convex	Fc	Let T be a family of functions $F:\mathbb{R}^n \to \mathbb{R}$. Then θ is called F-convex if, for every x in the domain of θ, $\exists$ an $F \in T$ such that $\theta(x) = F(x)$ and $\theta(z) \geq F(z)$ $\forall x \neq z$, in which case F is support of θ at x.	Mond[41]
$(p-\theta)convex$	(p,θ)c	Since $f(\lambda x + (1-\lambda)y)$, $0 \leq \lambda \leq 1$, consists of the values of f at all points on the straight line between x and y that are in the domain of f, we now consider any path from x to y. Let $p_{xy}(\lambda)$, where $p_{xy}(0) = X$, $p_{xy}(1) = y$, be any continuous path from x to y in $\mathbb{R}^n$ such that $f(P_{xy}(\lambda))$, $0 \leq \lambda \leq 1$, is defined. f is said to be $(p-\theta convex)$ if $f(P_{xy}(\lambda)) \leq \theta^{-1}[\lambda\theta(f(y)) + (1-\lambda)\theta(f(x))]$ $\forall x, y$ the domain of f, $0 \leq \lambda \leq 1$.	Mond[41]

Name of the function /set	Abbreviation	Definition of the functions	References
Bonvex	BX	Let S be a convex subset of $\mathbb{R}^n$ and C^2 the class of all continuous functions $f: S \to \mathbb{R}$ such that all the second order partial derivatives of f exist and are continuous over the interior of S. Let ∇f denote the gradient and $\nabla^2 f$ the Hessian of f with respect to $x \in \mathbb{R}^n$. Let $f \in C^2$ and $x^0 \in S$. f is bonvex at $x^0 \in S$ if $\forall \eta \in R^n$, $f(x) - f(x^0)$ $\geq (x - x^0)^t [f(x)^0 + \nabla^2 f(x^0)\eta]$ $-\frac{1}{2}\eta^t \nabla^2 f(x^0)\eta$	Bector and Bector[2]
Strictly Bonvex	SBX	f is f is strictly bonvex at $x^0 \in S$ if $\forall \eta \in \mathbb{R}^n$, $x \neq x^0$, $f(x) - f(x^0) > (x - x^0)^t [f(x)^0$ $+\nabla^2 f(x^0)\eta]$ $-\frac{1}{2}\eta^t \nabla^2 f(x^0)\eta$	Bector and Bector[2]
Pseudo-Bonvex	PBX	f is Pseudo-bonvex at $x^0 \in S$ if $\forall \eta \in \mathbb{R}^n$, $(x - x^0)^t [\nabla f(x^0) + \nabla^2 f(x^0)] \geq 0$ $\Rightarrow f(x) \geq f(x^0) - \frac{1}{2}\eta^t \nabla^2 f(x^0)\eta$	Bector and Bector[2]
Strictly Pseudo-Bonvex	SPBX	f is strictly pseudo-bonvex at $x^0 \in S$ if $\forall \eta \in \mathbb{R}^n$, $x \neq x^0$, $(x - x^0)^t [\nabla f(x^0) + \nabla^2 f(x^0)] \geq 0$ $\Rightarrow f(x) > f(x^0) - \frac{1}{2}\eta^t \nabla^2 f(x^0)\eta$	Bector and Bector[2]
Quasi-Bonvex	QBX	f is quasi-bonvex at $x^0 \in S$ if $\forall \eta \in \mathbb{R}^n$, $f(x) \leq f(x^0) + \frac{1}{2}\eta^t \nabla^2 f(x^0)\eta$ $\Rightarrow (x - x^0)^t [\nabla f(x^0) + \nabla^2 f(x^0)\eta] \leq 0$ or $f(x) < f(x^0) + \frac{1}{2}\eta^t \nabla^2 f(x^0)\eta$ $\Rightarrow (x - x^0)^t [\nabla f(x^0) + \nabla^2 f(x^0)\eta] \leq 0$	Bector and Bector[2]
B-vex set	Bvs	Given $S \subseteq \mathbb{R}^n \times \mathbb{R}$, S is said to be B-vex set if $(x, \alpha), (u, \beta) \in S$ imply that $(\lambda x + (1 - \lambda)u, \lambda b\alpha + (1 - \lambda b)\beta) \in S$, $0 \leq \lambda \leq 1$	Bector et al.[2]
B-invex set	Bis	Given $S \subseteq \mathbb{R}^n \times \mathbb{R}$, S is said to be B-invex set w.r.t. η, b_1, b_2 if (x, α), $(u, \beta) \in S$ imply$(u + \lambda\eta(x, u), b_1\alpha + b_2\beta) \in S$ for $0 \leq \lambda \leq 1$, $b_1 + b_2 = 1$.	Bector et al.[2]

Name of the function /set	Abbreviation	Definition of the functions	References
(ρ,θ)-B-vex set	(ρ,θ)-Bvs	Given $S \subseteq D \times \mathbb{R}$, S is said to be (ρ,θ)-B-vex set if (x,α), $(y,\beta) \in S$ $\Rightarrow (\lambda x+(1-\lambda)y, \lambda b\alpha+(1-\lambda b)\beta)$ $+\rho\\|\theta(x,y)\\|^2 \in S, 0 \leq \lambda \leq 1$.	Behera et al.[3]
(ρ,θ)-η -invex set	(ρ,θ)-ηIs	The set D is said to be (ρ,θ)-η invex at $y \in D$ w.r.t. η if for each $x \in D$, $y+\lambda\eta(x,y)+\rho\\|\theta(x,y)\\|^2 \in D$, $0 \leq \lambda \leq 1$. D is said to be (ρ,θ)-η-invex set with respect to η if D is invex at each $y \in D$ with respect to the same η.	Behera et al.[3]
(ρ,θ)-η- B-invex set	(ρ,θ)-η-Bis	Let $S \subseteq D \times \mathbb{R}$. S is said to be (ρ,η)-$\theta)$-B-invex set w. r. t. η, b_1, b_2 if (x,α), $(y,\beta) \in$ S imply $(y+\lambda\eta(x,y), b_1\alpha+b_2\beta$ $+\rho\\|\theta(x,y)\\|^2) \in S$ for $0 \leq \lambda \leq 1$, $b_1+b_2=1$.	Behera et al.[3]
Condition C	CC	Let $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. We say that η satisfies condition C if for any x,y, $\eta(y, x+\lambda\eta(x,y)) = -\lambda\eta(x,y)$ $\eta(x, y+\lambda\eta(x,y)) = (1-\lambda)\eta(x,y)$ for all $0 \leq \lambda \leq 1$.	Mohan and Neogy[40]
Semi-strictly semi invex	Sssi	f is semi-strictly semi invex if $f(x) \neq f(y) \Rightarrow f(y)-f(x)$ $> (\xi, \eta(y,x))$ for some $\xi \in \mathbb{R}^n$	[Nanda (unpublished)]
Strictly semi invex	Ssi	f is strictly semi invex if $x \neq y \Rightarrow f(y)-f(x)$ $> (\xi, \eta(y,x))$ for some $\xi \in \mathbb{R}^n$	[Nanda (unpublished)]
Pseudo semi-invex	Psi	f is pseudo semi-invex if $(\xi, \eta(y,x)) \geq 0$ $\Rightarrow f(y)-f(x) \geq 0$	[Nanda (unpublished)]
Strictly(semi-strictly) semi quasi invex	S(s)sqi	f is strictly(semi-strictly) semi quasi-invex if $f(y) < f(x)$ $\Rightarrow (\xi, \eta(y,x)) < 0$	[Nanda (unpublished)]
L-geodesically convex on manifolds	Lgc	f is said to be L-geodesically convex on manifolds if $f(\gamma_{x,x_0}(t))$ $\leq (f(x_0))^{1-t}(f(x))^t$	[Nanda (unpublished)]

Name of the function /set	Abbreviation	Definition of the functions	References
γ-Quasiconvex	γqc	A function $g : D \subset \mathbb{R}^n \to \mathbb{R}$ is said to be γ-quasiconvex (γqc) if $g(\lambda x + (1-\lambda)y) \leq max(g(x), g(y))$ for $x,\ y\ \in\ D$ and $\lambda\ \in\ [0,1] \cap\ Q$	Nanda[45]
g-Preinvex	gpi	Let f be a real function defined on an invex subset A of $\mathbb{R}^n$ and let $g : \mathbb{R} \to \mathbb{R}$ f is said to be g-preinvex w.r.t. η if $gf(y+\lambda\eta(x,y)) \leq \lambda gf(x) + (1-\lambda)gf(y)$ $\forall\ x, y\ \in\ A$ and $\lambda \in [0,1]$.	Nanda[45]
Semistrictly g-preinvex	ssgpi	Let $f : K \subset \mathbb{R}^n \to \mathbb{R}$, K an invex set and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. f is said to be semistrictly g-preinvex (ssbpi) if for $x, y \in K$, $gf(x) \neq gf(y)$, $gf(x+\lambda\eta(x,y)) < \lambda gf(x) + (1-\lambda gf(y))$	Nanda[45]
g-semi-invex	gsi	f is said to be g-semi-invex at $x_0 \in X \subset \mathbb{R}^n$ if $\forall\ x \in\ X, \exists\ \xi \in\ \mathbb{R}^n$ such that $g[f(x) - f(x_0)] \geq (\xi, \eta(x, x_0))$	Nanda[45]
α-Quasi convex	αqc	f is α-quasi convex if for all x_1, x_2 in domain and given $\alpha \in\ [0,1]$ such that $\alpha x_1 + (1-\alpha)x_2$ in its domain and $f(\alpha x_1 + (1-\alpha)x_2) \leq max(f(x_1), f(x_2))$	Behringer[5]
Semilocally explicitly Quasiconvex	Sleqc	A real valued function θ defined on a set S in $\mathbb{R}^n$ is Sleqc at $\bar{x} \in S$ if S is locally starshaped at $\bar{x}$ and if for $\bar{x}$ and each $x \in S$, there exists a positive number $d(\bar{x}, x) \leq a(\bar{x}, x) \leq 1$ such that $\theta(x) < \theta(\bar{x})$ $\Rightarrow \theta((1-\lambda)\bar{x} + \lambda x) < \theta(\bar{x})$ $0 < \lambda < d(\bar{x}, x)$. If $d(\bar{x}, x) = a(\bar{x}, x) = 1$ for each $x \in S$, then θ is explicitly quasiconvex at $\bar{x}$. If θ is Sleqc at each $\bar{x} \in S$, then θ is Sleqc on S.	Kaul and Kaur[29]

Name of the function /set	Abbreviation	Definition of the functions	References
Semilocally strongly Quasiconvex	Slsqc	A real valued function θ defined on a set S in $\mathbb{R}^n$ is Slsqc at $\bar{x} \in S$ if S is locally starshaped at $\bar{x}$ and if corresponding to $\bar{x}$ and each $x \in S$, there exists a positive number $d(\bar{x}, x) \leq a(\bar{x}, x)$ such that $\theta(x) < \theta(\bar{x})$, $x \neq \bar{x}$ $\Rightarrow \theta((1-\lambda)\bar{x} + \lambda x) < \theta(\bar{x})$ $0 < \lambda < d(\bar{x}, x)$. If $d(\bar{x}, x) = a(\bar{x}, x) = 1$ for each $x \in S$, then θ is strongly quasiconvex at $\bar{x}$. If θ is SlSqc at each $\bar{x} \in S$, then θ is SlSqc on S.	Kaul and Kaur[29]
Semilocally Pseudoconvex	Slpc	A real valued function θ defined on a set S in $\mathbb{R}^n$ is Slpc at $\bar{x} \in S$ if for each $x \in S$, the right differential $(d\theta)^+ \ (\bar{x}, x - \bar{x})$ of θ at $\bar{x}$ in the direction $x - \bar{x}$ exists and $(d\theta)^+ \ (\bar{x}, \bar{x} - x) \geq 0 \Rightarrow \theta(x) \geq \theta(\bar{x})$	Kaul and Kaur[29]
K-convex	Kc	A function f on an interval I of the real line is K-convex, where K is a nonnegative real number if for any $x, y \in I$, $x < y$ and $0 \leq \lambda \leq 1$, $f(\lambda x + (1-\lambda)y)$ $\leq \lambda f(x) + (1-\lambda) f(y) + K$. If K=0, this becomes the usual definition of convexity.	Doeringer[18]
Condition A	CA	Let the set Γ be invex with respect to η, and let $f : \Gamma \to \mathbb{R}$. Then $f(y + \eta(x, y)) \leq f(x)$, for any $x, y \in \Gamma$.	Yang et al.[62]
Lipschitz (of rank k)	L	Let X is an open convex subset of $\mathbb{R}^n$. The function $f : X \to \mathbb{R}$ is said to be Lipschitz(of rank k) near $x \in X$ if there exist $\delta > 0$ and $k > 0$ such that $\lvert f(y) - f(z) \rvert \leq k \lVert y - z \rVert$, whenever $y, z \in x + \delta B$	Clarke[15]
Globally Lipschitz	Gl	f is asid to be globally lipschitz (of rank k) on X if $\lvert f(y) - f(z) \rvert \leq k \lVert y - z \rVert$, for some $k > 0$ and every $y, z \in X$	Clarke[15]

Name of the function /set	Abbreviation	Definition of the functions	References
Pre-pseudo invex	Ppi	A function f is pre-pseudo invex on an invex set A if there exists a function η and a strictly positive function b such that $f(x) < f(y)$ $\Rightarrow f(y+\lambda\eta(x,y)) \leq f(y)+\lambda(\lambda-1)b(x,y)$ for every $\lambda \in (0,1)$ and $x,y \in A$	Pini[50]

Theorem 1. $HC \Rightarrow LC \Rightarrow C \Rightarrow QC$

$\not\Leftarrow \quad \not\Leftarrow \quad \not\Leftarrow$

$f(x) = x^2$ on $[0,1]$ is C but not LC.

$f(x) = \frac{1}{x^2}$ on $[0,\infty]$ is LC but not HC.

Theorem 2. $C \Rightarrow Ct \Rightarrow CI$

$\not\Leftarrow \quad \not\Leftarrow$

Observe that

$f(x) = -x^2$ on $[0,1]$ is CI but not Ct.

$f(x) = -x^3$ on R is Ct but not C.

Theorem 3. $LC \Rightarrow LCt \Rightarrow CI$

$\not\Leftarrow \quad \not\Leftarrow$

The function defined by

$$f(x) = \begin{cases} 0, & \text{if } 0.5 \leq x \leq 1, \\ x, & \text{if } 0 < x < 0.5. \end{cases}$$

is LCI but not LCt where as

$f(x) = sin x$, $x \in [0,\pi/2]$ is LCt but not LC.

Theorem 4. $HC \Rightarrow HCt \Rightarrow HCl$

$\not\Leftarrow$ $\not\Leftarrow$

$$f(x) = \begin{cases} \frac{1}{1+x^2}, & \text{if } 0.5 \le x \le 1, \\ \frac{1}{1-x^2}, & \text{if } 0 < x < 0.5. \end{cases}$$

is HCl but not HCt.

$f(x) = x$ on $[0,1]$ is HCt but not HC.

Theorem 5. $QC \Rightarrow QCt \Rightarrow QCl$

$QC \not\Leftarrow QCt$, $St.QCt \not\Leftarrow QCl$

$f(x) = sinx$ on $[0,\pi]$ is QCI but not strictly QCt.

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is QCt but not QC.

Theorem 6. $HCl \Rightarrow LCl \Rightarrow Cl \Rightarrow QCl$

But $QCl \not\Rightarrow St.Cl$, $Cl \not\Rightarrow St.LCl$, $LCl \not\Rightarrow St.HCl.$

$f(x) = sinx$ on $[0,\pi]$ is QCl but not St. Cl.

$f(x) = x^2$ on $[0,1]$ is Cl but not St. LCl.

$f(x) = sinx$ on $[0,\pi/2]$ is LCl but not St. HCl.

Theorem 7. $HCt \Rightarrow LCt \Rightarrow Ct \Rightarrow QCt$

$St.HCt \not\Leftarrow LCt \not\Leftarrow Ct \not\Leftarrow QCt$

$f(x) = -x^2$, $x \ge 0$ is QCt but not Ct.

$f(x) = 1 - x$ on $(0,1)$ is Ct but not LCt.

$f(x) = sinx$ on $[0,\pi/2]$ is LCt but not St. HCt.

Theorem 8. $Hc \Rightarrow Hqc$

$\nLeftarrow$

Consider the function $f(x) = x$ in [0,1]. f is Hqc.

For, let $x_1, x_2 \in [0,1]$. Without any loss of generality we may suppose that $x_1 \geq x_2$. Then

$$\frac{f(x_2) - f(x_1)}{f(x_2)} \leq 0 \ and \ \frac{\nabla f(x_1)(x_2 - x_1)}{f(x_1)} \leq 0.$$

But f is not Hc. For if $x_1 = 0.5$ and $x_2 = 0.3$,

$$\frac{f(x_2) - f(x_1)}{f(x_2)} = -0.66 \ and \ \frac{\nabla f(x_1)(x_2 - x_1)}{f(x_1)} = -0.4.$$

Theorefore

$$\frac{f(x_2) - f(x_1)}{f(x_2)} < \frac{\nabla f(x_1)(x_2 - x_1)}{f(x_1)}$$

which implies that f is not Hc.

Theorem 9. $Hc \Rightarrow Hpc$

$\nLeftarrow$

Consider the function $f(x) = x$ in [0,1]. f is Hpc.

For, let $x_1, x_2 \in [0,1]$ and $x_1 \leq x_2$ then

$$\frac{f(x_2) - f(x_1)}{f(x_2)} \geq 0 \ and \ \frac{\nabla f(x_1)(x_2 - x_1)}{f(x_1)} \geq 0.$$

It has now already been prove that f is not Hc.

Theorem 10. For a positive function defined on a set $S \subseteq \mathbb{R}^n$, $SlHc \Rightarrow HlLc \Rightarrow Slc \Rightarrow SlQc$

$$SlHc \nLeftarrow HlLc \nLeftarrow Slc \nLeftarrow SlQc$$

Consider any positive linear function on $S \subseteq \mathbb{R}^n - 0$ which is not a positive constant.

Suppose it is SlHc on S. Then corresponding to any $\bar{x} \in S$ and each $x \in S$, there exists $d(\bar{x}, x) \leq \alpha(\bar{x}, x)$ such that for $0 < \lambda < d(\bar{x}, x)$,

$$f((1-\lambda)\bar{x} + \lambda x) \leq \frac{1}{\frac{1-\lambda}{f(\bar{x})} + \frac{\lambda}{f(x)}}$$

Since f is linear and $A.M \geq G.M \geq H.M$,

$$\begin{aligned} f((1-\lambda)\bar{x} + \lambda x) &= (1-\lambda) f(\bar{x}) + \lambda f(x) \\ &> (f(\bar{x}))^{1-\lambda} (f(x))^{\lambda} \\ &> \frac{1}{\frac{1-\lambda}{f(\bar{x})} + \frac{\lambda}{f(x)}} \end{aligned}$$

Therefore for $0 < \lambda < d(\bar{x}, x)$

$\frac{1}{\frac{1-\lambda}{f(\bar{x})} + \frac{\lambda}{f(x)}} < (f(\bar{x}))^{1-\lambda} (f(x))^{\lambda} < (1-\lambda) f(\bar{x}) + \lambda f(x)$.

But this is a contradiction. Therefore a positive linear non constant function is not SlHc, nor SlLc. Hence Slc $\nRightarrow$ SlLc.

Theorem 11. Every differentiable convex function is strongly Pseudo-convex but not conversely.

$$f(x) = x^2 \text{ on } [1, \infty] \text{ with } K(x, y) = y(x + y)$$

is strongly pseudo-convex but not convex at $x = 1$ and $y = 1.5$

Theorem 12. Every strongly Pseudo-convex function is invex but not conversely.

$$f(x) = -sinx \text{ in } [-\pi/2, \pi/2] \text{ and}$$

$$\eta(x, y) = \frac{sinx - siny}{cosy} \text{ is invex.}$$

But not strongly Pseudo-convex by considering $K(x, y) = -sinxsiny$ and $x = \pi/4$, $y = -\pi/2$.

Theorem 13. Quasi convex$\Rightarrow$ Quasi Invex

$\not\Leftarrow$

Consider

$$f(x) = sin^3 x \text{ on } [0,\pi]$$

f is quasi invex on $[0,\pi]$ with $\eta(x,y) = cosy(sinx - siny)$. But f is not quasi convex for $x = 3\pi/4,\ y = \pi/4$.

Theorem 14. Pseudo-convex$\Rightarrow$ Pseudo-Invex

$\not\Leftarrow$

Consider

$$f(x) = -cos^2 x \text{ in }]-\pi/2,\pi/2[\text{ and}$$

$$\eta(x,y) = siny(cosy - cosx)$$

f is pseudo-invex but not pseudo convex for $x = 0$ and $y = \pi/4$.

Theorem 15. Invex$\Rightarrow$ Quasi-Invex

$\not\Leftarrow$

Consider $f(x) = sin^3 x$ on $[0,\pi]$ and

$$\eta(x,y) = cosy(sinx - siny)$$

It is quasi-invex but not invex for $x = \pi/4$ and $y = \pi/2$.

Theorem 16. Invex$\Rightarrow$ Pseudo-Invex

$\not\Leftarrow$

Consider

$$f(x) = -cos^2 x \text{ on } (-\pi/2,\pi/2) \text{ and}$$

$$\eta(x,y) = siny(cosy - cosx)$$

f is pseudo-invex but not invex at $x = -\pi/6,\ y = -\pi/4$.

Theorem 17. Strictly invex$\Rightarrow$ invex

$\not\Leftarrow$

Consider

$f(x) = -sinx$ on $]-\pi/2, \pi/2[$ and

$\eta(x, y) = \frac{sinx - siny}{cosy}$

f is invex but not Strictly invex as equality holds good throughout the domain.

Theorem 18. Convex$\Rightarrow$ B-vex

$\not\Leftarrow$

The function defined by

$$f(x) = \begin{cases} 0, & \text{if } 0 < x < 1, \\ x, & \text{if } 1 \leq x < 2. \end{cases}$$

is B-vex with

$$b(x, u, \lambda) = \begin{cases} \frac{\lambda(u-x)}{u}, & \text{if } x \leq u, \\ \frac{[u+\lambda(x-u)]}{(\lambda x)}, & \text{if } x > u, \lambda \neq 0, \\ 1, & \text{if } x > u, \lambda = 0. \end{cases}$$

f is not convex at x=1/2, u=3/2, $\lambda = 1/2$.

Theorem 19. B-vex$\Rightarrow$ Quasi B-vex

$\not\Leftarrow$

Define a function $f : X(= [0, 2]) \rightarrow R$ by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x < 2, \\ 1, & \text{if } x = 2, \end{cases}$$

and $b : X \times X \times [0, 1] \rightarrow R_+$ by

$$b(x, u, \lambda) = \begin{cases} 0, & \text{if } x = 2 or u = 2, \\ (x+u)/(4\lambda), & \text{if otherwise} \end{cases}$$

f is quasi B-vex but not B-vex at $x = 1/2$, $u = 2$, $\lambda = 1/3$.

Theorem 20. Quasiconvex$\Rightarrow$ quasi B-vex

$$\not\Leftarrow$$

However, the converse is not necessarily true if $b(x,u,\lambda) = 0$ for some $x,u \in X$, $0 \le \lambda \le 1$.

Define a function $f : X(= [0,2]) \to R$ by

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < 2, \\ 1, & \text{if } x = 2, \end{cases}$$

f is quasi B-vex but not quasi convex at $x = 1/2$, $u = 2$, $\lambda = 1/4$.

Theorem 21. $B - vex \Rightarrow Pseudo\ \bar{B} - vex$

$$\not\Leftarrow$$

where

$$\bar{b}(x,u) = \lim_{\lambda \to 0_+} b(x,u,\lambda)$$

Define a function $f : X(=]-1,1[) \to R$ by $f(x) = x + x^3$ and

define $b : X \times X \times [0,1] \to R_+$ by

$$b(x,u,\lambda) = \begin{cases} 1-\lambda, & \text{if } xu \ge 0 \\ -xu, & \text{if } xu < 0, \end{cases}$$

Then

$$\bar{b}(x,u) = \begin{cases} 1, & \text{if } xu \ge 0 \\ -xu, & \text{if } xu < 0, \end{cases}$$

f is pseudo $\bar{B}$-vex but not B-vex at $x = -1/4$, $u = -1/2$, $\lambda = 1/2$.

Theorem 22. Pseudo convex$\Rightarrow$ Pseudo B-vex

$$\not\Leftarrow$$

where $b(x,u) = 0$ for some $x,u \in X$

Define a function $f : X(=]-1,1[) \to R$ by $f(x) = x^3$, and

define $b : X \times X \to R_+$ by

$$b(x,u) = \begin{cases} 1, & \text{if } xu \ne 0 \\ 0, & \text{if } xu = 0, \end{cases}$$

f is pseudo B-vex but not pseudo convex $x = -1/2,\ u = 0$.

Theorem 23. $Invex\ \ w.r.t.\,\eta \ \Rightarrow\ \ B - invex\ w.r.t.same\,\eta$

$$\nLeftarrow$$

where $b(x,u) = 1$. However, if $b(x,y) \neq 1$, the converse is not true.

Define a function $f : X(=]0,\pi/2[) \to R$ by $f(x) = x + sinx$, and

define $\eta : X \times X \to R$ by

$$\eta(x,u) = 2(sinx - sinu)/cosu$$

and $b : X \times X \to R_+$

$$b(x,u) = 2$$

f is B-invex with respect to η but is not invex with respect to η at $x = \pi/4,\ u = \pi/6$.

Theorem 24. Every B-invex function f with respect to η, where $b(x,u) > 0$, $\forall\ \ x,u\ \in\ X$, is invex with respect to some $\bar{\eta}$, where

$$\bar{\eta}(x,u) = \eta(x,u)/b(x,u)$$

Theorem 25. Every quasi B-invex function f with respect to η, is quasi-invex with respect to some $\bar{\eta}$, where

$$\bar{\eta}(x,u) = \eta(x,u)b(x,u),\ \ \forall\ x,\ u\ \in\ X.$$

Theorem 26. Pseudo-invex w.r.t. η $\Rightarrow$ Pseudo B-invex w.r.t.the same η

$$\nLeftarrow$$

when $b(x,u) = 0$, for some $x,u\ \in\ X$

Define a function $f : X(=]0,,\pi/2[) \to R$ by $f(x) = cosx$, and

define $\eta : X \times X \to R$ by

$$\eta(x,u) = u - x$$

and $b : X \times X \to R_+$ by

$$b(x,u) = \begin{cases} 0, & \text{if } x \geq u \\ xu, & \text{if } x < u, \end{cases}$$

f is pseudo B-invex w.r.t. η but not pseudo invex w.r.t. η at $x = \pi/3$, $u = \pi/6$.

Theorem 27. Quasi-invex w.r.t. η $\Rightarrow$ Quasi B-invex w.r.t. the same η

$$\not\Leftarrow$$

when $b(x,u) = 0$, for some $x, u \in X$

Define a function $f : X(=]0,,\pi/2[) \to R$ by $f(x) = sin^3 x$, and

define $\eta : X \times X \to R$ by

$$\eta(x,u) = cosu(sinu - sinx)$$

and $b : X \times X \to R_+$ by

$$b(x,u) = \begin{cases} 0, & \text{if } x \leq u \\ xu, & \text{if } x > u, \end{cases}$$

f is quasi B-invex w.r.t. η but not quasi invex w.r.t. η at $x = \pi/6$, $u = \pi/3$.

Theorem 28. Every differentiable B-vex function is $\bar{B}$-invex function with respect to some η, where

$$\bar{b}(x,u) = \lim_{\lambda \to 0_+} b(x,u,\lambda),$$

but not conversely.

Define a function $f : X(=]0,\pi/2[) \to R$ by $f(x) = sinx$, and

define $\eta : X \times X \to R$ by

$$\eta(x,u) = \begin{cases} (sinx - sinu)/cosu, & \text{if } x \geq u \\ 0, & \text{if } x < u, \end{cases}$$

and $b : X \times X \times [0,1] \to R_+$ by

$$b(x,u,\lambda) = \begin{cases} 1, & \text{if } x \geq u \\ \lambda, & \text{if } x < u, \end{cases}$$

Then,

$$\bar{b}(x,u) = \begin{cases} 1, & \text{if } x \geq u \\ 0, & \text{if } x < u, \end{cases}$$

f is $\bar{B}$-invex function with respect to η, but not B-vex at $x = \pi/3,\ u = \pi/6,\ \lambda = 1/2$.

Theorem 29. Pseudo B-vex $\Rightarrow$ Pseudo B-invex w.r.t. some η

$$\not\Leftarrow$$

when $b(x,u) = 0$, for some $x, u \in X$

Define a function $f : X(=]-\pi/2, \pi/2[) \to R$ by $f(x) = cos^2 x$, and

define $\eta : X \times X \to R$ by

$$\eta(x,u) = sinu(cosu - cosx)$$

and $b : X \times X \to R_+$ by

$$b(x,u) = \begin{cases} 1, & \text{if } xu \neq 0 \\ 0, & \text{if } xu = 0, \end{cases}$$

f is pseudo B-invex w.r.t. η but not pseudo B-vex at $x = -\pi/3,\ u = \pi/6$.

Theorem 30. Every differentiable quasi B-vex function is quasi $\bar{B}$-invex function with respect to some $\eta = x - u$, where

$$b(x,u) = \lim_{\lambda \to 0_+} b(x,u,\lambda),$$

but not conversely.

Define a function $f : X(=]0, \pi[) \to R$ by $f(x) = sin^3 x$, and

define $\eta : X \times X \to R$ by

$$\eta(x,u) = cos\ u(sin\ u - sin\ x)$$

and $b : X \times X \times [0,1] \to R_+$ by

$$b(x,u,\lambda) = \begin{cases} xu, & \text{if } x \geq u \\ \lambda, & \text{if } x < u, \end{cases}$$

Then

$$\bar{b}(x,u) = \begin{cases} xu, & \text{if } x \geq u \\ 0, & \text{if } x < u, \end{cases}$$

f is quasi $\bar{B}$-invex w.r.t. η but not quasi B-vex at $x = 3\pi/4,\ u = \pi/4,\ \lambda = 1/2$.

Theorem 31. B-invex w.r.t. η $\Rightarrow$ Pseudo B-invex w.r.t.the same η

$$\not\Leftarrow$$

Define a function $f : X(=]0,\pi[) \to R$ by $f(x) = sin^3 x$, and

define $\eta : X \times X \to R$ by

$$\eta(x,u) = cos\ u(sin\ u - sin\ x)$$

and $b : X \times X \to R_+$ by

$$b(x,u) = \begin{cases} xu, & \text{if } x \geq u \\ 0, & \text{if } x < u, \end{cases}$$

f is pseudo B-invex w.r.t. η but not B-invex w.r.t. η at $x = \pi/3$, $u = \pi/6$.

Theorem 32. $\bar{B}$-invex w.r.t. $\eta \Rightarrow$ Quasi $\bar{B}$-invex w.r.t. the same η

$$\not\Leftarrow$$

Define a function $f : X(=]0,\pi[) \to R$ by $f(x) = sin^3 x$, and

define $\eta : X \times X \to R$ by

$$\eta(x,u) = cos\ u(sin\ u - sin\ x)$$

and

$$\bar{b}(x,u) = \begin{cases} xu, & \text{if } x \geq u \\ 0, & \text{if } x < u, \end{cases}$$

f is quasi $\bar{B}$-invex w.r.t. η but not $\bar{B}$-invex w.r.t. η at $x = \pi/3$, $u = 5\pi/6$.

Theorem 33. Convex set $\Rightarrow$ Invex set w. r. t. $\eta(x,u) = x - u$

$$\not\Leftarrow$$

The set $S = R \sim]-1/2, 1/2[$ is invex with respect to η, where

$$\eta(x,u) = \begin{cases} x - u, & \text{if } x > 0,\ u > 0\ or\ x < 0,\ u < 0 \\ u - x, & \text{if } x < 0,\ u > 0\ or\ x > 0,\ u < 0, \end{cases}$$

But S is not convex.

Theorem 34. Every convex function is B-preinvex w.r.t. η, b_1, b_2, where

$\eta(x,u) = x - u$, $b_1 = \lambda$, $b_2 = 1 - \lambda$,

but not conversely.

Let $S =]0, \pi/2[$. Then S is invex w.r.t. η, where

$$\eta(x,u) = \begin{cases} (sinx - sinu)/cosu, & \text{if } x \geq u \\ 0, & \text{if } x \leq u, \end{cases}$$

Let $f : S \to R$ be defined by $f(x) = \sin x$. Then f is B-preinvex w.r.t. η, b_1, b_2, where

$$b_1(x,u,0) = 1,$$

$$b_1(x,u,\lambda) = \begin{cases} 1-\lambda, & \text{if } x \geq u, 0 < \lambda \leq 1 \\ 1, & \text{if } x < u, \end{cases}$$

$$b_2(x,u,\lambda) = \begin{cases} \lambda, & \text{if } x \geq u, 0 < \lambda \leq 1 \\ 0, & \text{if } x < u. \end{cases}$$

But f is not convex at $x = \pi/6$, $u = \pi/3$, $\lambda = 1/2$.

Theorem 35. Every preinvex function w.r.t. η is B-preinvex w.r.t. η, b_1, b_2, where $b_1 = \lambda$, $b_2 = 1 - \lambda$, but not conversely.

The function defined in Theorem-34, is B-preinvex w.r.t. η, b_1, b_2, defined there, but not preinvex w.r.t. η, at $x = \pi/6$, $u = \pi/3$, $\lambda = 1/2$.

Theorem 36. Every B-vex function w.r.t. b_1, b_2,is B-preinvex w.r.t. η, b_1, b_2, where $\eta(x,u) = x - u$; but not conversely.

The function defined in Theorem-34, is B-preinvex w.r.t.η, b_1, b_2, defined there, but f is not B-vex w.r.t. b_1, b_2, at $x = \pi/3$, $u = \pi/6$, $\lambda = 1/2$.

Theorem 37. Suppose that g is B-preinvex on an invex set w.r.t. η, b_1, b_2, and $g(x) > 0 \ \forall \ x \in X$. Then $1/g$ is B-preincave on X w.r.t. η, and some $\bar{b_1}$, $\bar{b_2}$.

But the similar result does not hold preinvex functions. Let $S =]0, \pi/2[$, and let

$$\eta(x,u) = \begin{cases} (sinx - sinu)/cosu, & \text{if } x \geq u \\ x - u, & \text{if } x \leq u, \end{cases}$$

Then S is invex w.r.t. η.

Define $f : S \to R$ by $f(x) = x$. Then $f(x) = x > 0 \ \forall \ x \in S$ and f is preinvex on S w.r.t. η, but $g = -1/f$ is not preinvex w.r.t. η, at $u = \pi/6$, $x = \pi/3$, $\lambda = 1/2$.

Theorem 38. If f is B-preinvex on an invex set $X \subseteq R^n$ w.r.t. η, b_1, b_2, and $b_1(x,u,\lambda) > 0 \ \forall \ x,u \in x$, $0 < \lambda < d < 1$, for some fixed $d \in R$, then every local minimum of f over X is a global minimum of f over X.

Theorem 39. There exist functions which are prequasiinvex w.r.t. η, but not B-preinvex w.r.t. η, b_1, b_2, where $b_1(x,u,\lambda) > 0$, for some d, $0 < \lambda < d < 1$.

Let $S = R \sim]-1/2, 1/2[$. Then S is invex w.r.t. η, where

$$\eta(x,u) = \begin{cases} x - u, & \text{if } x > 0,\ u > 0 \ or \ x < 0,\ u < 0 \\ u - x, & \text{if } x > 0,\ u < 0 \ or \ x < 0,\ u > 0, \end{cases}$$

Let $f : S \to R$ be defined by

$$f(x) = \begin{cases} x + 1, & \text{if } x < -1, \\ 0, & \text{if } -1 \leq x \leq 1, \\ 1 - x, & \text{if } x > 1. \end{cases}$$

f is prequasiinvex w.r.t. η, but not B-preinvex w.r.t. η, b_1, b_2, where $b_1(x,u,\lambda) > 0$, for $0 < \lambda < d < 1$, as $x = 3/4$ is a local minimum of f over S but not a global minimum of f over S.

Theorem 40. Every (α, λ)-convex function is B-preinvex w.r.t. η, b_1, b_2, where $\eta(x,u) = x - u$, $b_1 = \lambda$, $b_2 = 1 - \lambda$, but not conversely.

Let $S =]0, \pi/2[$. Then S is invex w.r.t. η, where

$$\eta(x,u) = \begin{cases} (sinx - sinu)/cosu, & \text{if } x \geq u \\ 0, & \text{if } x \leq u, \end{cases}$$

Let $f : S \to R$ be defined by $f(x) = \sin x$. Then f is B-preinvex w.r.t. η, b_1, b_2, where

$$b_1(x,u,0) = 1,$$

$$b_1(x,u,\lambda) = \begin{cases} 1-\lambda, & \text{if } x \geq u, 0 < \lambda \leq 1 \\ 1, & \text{if } x < u, \end{cases}$$

$$b_2(x,u,\lambda) = \begin{cases} \lambda, & \text{if } x \geq u, 0 < \lambda \leq 1 \\ 0, & \text{if } x < u. \end{cases}$$

f is B-preinvex w.r.t. η, b_1, b_2, but not (α,λ)-convex for $\alpha = \lambda$, $b_1 = \lambda$.

Theorem 41. Differentiable B-vex $\Rightarrow$ (ρ,θ)-pseudo-$\bar{B}$-vex

$$\not\Leftarrow$$

where

$$\bar{b}(x,y) = \lim_{\lambda \to 0+} b(x,y,\lambda)$$

Consider $f(x) = x + x^3$.

Define $\theta : D \times D \to \mathbb{R}$ by

$$\theta(x,y) = \begin{cases} \sqrt{x-y} & \text{if } x > y, \\ 0 & \text{if } x \leq y. \end{cases}$$

Taking $\rho = -1$. Define $b : D \times D \times [0,1] \to \mathbb{R}_+$ by

$$b(x,\, y,\, \lambda) = \begin{cases} 1-\lambda & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Then

$$\bar{b}(x,\, y) = \begin{cases} 1 & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

f is (ρ,θ)-pseudo-$\bar{B}$-vex but not B-vex at $x = -1/4$, $y = -1/2$, $\lambda = 1/2$.

Theorem 42. Pseudo-convex $\Rightarrow$ (ρ,θ)-pseudo-B-vex

$$\not\Leftarrow$$

when b(x,y)=0 for some $x, y \in D$.

Consider $f(x) = x^3$. Define $\theta : D \times D \to \mathbb{R}$ by

$$\theta(x, y) = \begin{cases} \sqrt{y-x} & \text{if } y > x, \\ 0 & \text{if } x \geq y. \end{cases}$$

Taking ρ = -1. Define $b : D \times D \to \mathbb{R}_+$ by

$$b(x, y) = \begin{cases} 1 & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

f is (ρ,θ)-pseudo-B-vex but not pseudo-convex at $x = -1/3,\ y = 0$.

Theorem 43. Every invex function f w.r.t. η is (ρ,θ)-η-B-invex w.r.t. the same η, where $b(x, y) = 1$ but not conversely when $b(x, y) \neq 1$.

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \sin x$.

Define $\theta : D \times D \to \mathbb{R}$ by

$$\theta(x, y) = \begin{cases} \sqrt{x-y} & \text{if } x > y, \\ 0 & \text{if } x \leq y. \end{cases}$$

Taking $\rho = -1$. Define $b : D \times D \to \mathbb{R}_+$ by

$$b(x, y) = \begin{cases} 2 & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Define $\eta : D \times D \to \mathbb{R}$ by

$$\eta(x, y) = x - y.$$

f is (ρ,η)- θ-B-invex, but not invex w.r. t. the same η at $x = \frac{\pi}{4}$ and $y = \frac{\pi}{6}$.

Theorem 44. Pseudo-invex w.r. t. $\eta \Rightarrow (\rho,\theta) - \eta$-pseudo-B-invex w.r. t. the same η

$$\not\Leftarrow$$

when b(x,y)=0 for some $x, y \in D$.

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \cos x$. Define $\theta : D \times D \to \mathbb{R}$ by

$$\theta(x, y) = \begin{cases} \sqrt{y-x} & \text{if } y > x, \\ 0 & \text{if } y \leq x. \end{cases}$$

Taking ρ = -1. Define $b : D \times D \to \mathbb{R}_+$ by

$$b(x, y) = \begin{cases} 0 & \text{if } x \geq y, \\ xy & \text{if } x < y. \end{cases}$$

Define $\eta : D \times D \to \mathbb{R}$ by

$$\eta(x, y) = y - x.$$

f is (ρ,θ)-η-pseudo-B-invex w.r. t. η but not invex w.r. t. the same η at $x = \frac{\pi}{3}$, $y = \frac{\pi}{6}$.

Theorem 45. Quasi-invex w.r. t. $\eta \Rightarrow (\rho,\theta) - \eta$-quasi-B-invex w.r. t. the same η

$$\not\Leftarrow$$

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \sin x$.

Define $\theta : D \times D \to \mathbb{R}$ by

$$\theta(x, y) = \begin{cases} \sqrt{y - x} & \text{if } y > x, \\ 0 & \text{if } y \leq x. \end{cases}$$

Taking $\rho = -1$. Define $b : D \times D \to \mathbb{R}_+$ by

$$b(x, y) = \begin{cases} 0 & \text{if } x \geq y, \\ xy & \text{if } x < y. \end{cases}$$

Define $\eta : D \times D \to \mathbb{R}$ by

$$\eta(x, y) = y - x.$$

f is (ρ,θ)-η-quasi-B-invex w. r. t. η but not quasi-invex w.r. t. the same η at $x = \frac{\pi}{6}$, $y = \frac{\pi}{3}$.

Theorem 46. B-vex $\Rightarrow (\rho,\theta) - \eta$-$\bar{B}$-invex w.r. t. the same η

$$\not\Leftarrow$$

where

$$\bar{b}(x, y) = \lim_{\lambda \to 0_+} b(x, y, \lambda).$$

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \sin x$. Define $\theta : D \times D \to \mathbb{R}$ by

$$\theta(x, y) = \begin{cases} \sqrt{x - y} & \text{if } x > y, \\ 0 & \text{if } x \leq y. \end{cases}$$

Taking ρ = -1. Define $b : D \times D \times [0,1] \to \mathbb{R}_+$ by

$$b(x, y, \lambda) = \begin{cases} 1 & \text{if } x \geq y, \\ \lambda & \text{if } x < y. \end{cases}$$

Then Define $\bar{b} : D \times D \to \mathbb{R}_+$ by

$$\bar{b}(x, y) = \begin{cases} 1 & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Define $\eta : D \times D \to \mathbb{R}$ by

$$\eta(x, y) = x - y.$$

f is $(\rho,\theta) - \eta$-$\bar{B}$-invex but not B-vex at $x = \frac{\pi}{3}$, $y = \frac{\pi}{6}$, and $\lambda_1 = \frac{1}{2}$

Theorem 47. B-invex w. r. t. $\eta \Rightarrow (\rho,\theta) - \eta$-quasi-B-invex w.r. t. the same η

$$\not\Leftarrow$$

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \sin x$.

Define $\theta : D \times D \to \mathbb{R}$ by

$$\theta(x, y) = \begin{cases} \sqrt{y - x} & \text{if } y > x, \\ 0 & \text{if } y \leq x. \end{cases}$$

Taking $\rho = -1$. Define $b : D \times D \to \mathbb{R}_+$ by

$$b(x, y) = \begin{cases} 0 & \text{if } x \geq y, \\ xy & \text{if } x < y. \end{cases}$$

Define $\eta : D \times D \to \mathbb{R}$ by

$$\eta(x, y) = y - x.$$

f is $(\rho,\theta) - \eta$-quasi-B-invex w.r. t. η but not B-invex w. r. t. the same η at $x = \frac{\pi}{6}$, $y = \frac{\pi}{3}$.

Theorem 48. Every Frechet differentiable invex function f is ρ-(η,θ)-invex for $\rho \leq 0$.

The converse is true $\rho \geq 0$, but for $\rho < 0$, this is false.

Let $f : (0, \frac{1}{2}) \to \mathbb{R}$ be a mapping defined by

$$f(x) = -x^3.$$

Let the maps η and θ be defined by

$$\eta(y,x) = \begin{cases} y-x & \text{if } y > x, \\ 0 & \text{if } y \leq x. \end{cases}$$

and

$$\theta(y,x) = \begin{cases} \sqrt{y-x} & \text{if } y > x, \\ 0 & \text{if } y \leq x. \end{cases}$$

Taking $\rho = -1$,

f is ρ-(η,θ)-invex but not invex at $x = \frac{1}{4}$, $y = \frac{1}{3}$

Theorem 49. Differentiable quasi-invex $\Rightarrow \rho - (\eta,\theta)$-quasi-invex

$$\not\Leftarrow$$

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \sin x$.

Let the maps η and θ be defined by

$$\eta(y,x) = x - y$$

and

$$\theta(y,x) = \begin{cases} \sqrt{x-y} & \text{if } y \leq x, \\ 0 & \text{if } y > x. \end{cases}$$

Taking $\rho = -1$,

f is $\rho - (\eta,\theta)$-quasi-invex but not quasi-invex at $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$,

Theorem 50. Differentiable pseudo-invex $\Rightarrow \rho - (\eta,\theta)$-pseudo-invex

$$\not\Leftarrow$$

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \sin x - 1$.

Let the maps η and θ be defined by

$$\eta(y,x) = \begin{cases} \sin y - \sin x & \text{if } y > x, \\ 0 & \text{if } y \leq x. \end{cases}$$

and

$$\theta(y,x) = \begin{cases} \sqrt{\sin x - \sin y} & \text{if } x > y, \\ 0 & \text{if } x \leq y. \end{cases}$$

Taking $\rho = -1$.

f is $\rho - (\eta, \theta)$-pseudo-invex but not pseudo-invex at $x = \frac{\pi}{4}$ and $y = \frac{\pi}{6}$,

Theorem 51. B-invex $\Rightarrow \rho - (\eta, \theta)$-B-invex

$\not\Leftarrow$

when $\bar{b}(x, u) \neq 0$

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \sin x$.

Let the functions η and θ be defined by

$$\eta(x, u) = x - u$$

and

$$\theta(x, u) = \begin{cases} \sqrt{x-u} & \text{if } x > u, \\ 0 & \text{if } x \leq u. \end{cases}$$

Let $\bar{b} : X \times X \to \mathbb{R}_+$ be a function defined as

$$\bar{b}(x, u) = \begin{cases} xu & \text{if } x \geq u, \\ 0 & \text{if } x < u. \end{cases}$$

Taking $\rho = -1$.

f is $\rho - (\eta, \theta)$-B-invex but not B-invex with respect to η at $x = \frac{\pi}{4}$ and $u = \frac{\pi}{6}$.

Theorem 52. Pseudo- B-invex $\Rightarrow \rho - (\eta, \theta)$-pseudo-B-invex

$\not\Leftarrow$

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ be a function defined by $f(x) = \sin x + x$.

Let the functions η and θ be defined by

$$\eta(x, u) = \begin{cases} \sin x - \sin u & \text{if } x > u, \\ 0 & \text{if } x \leq u, \end{cases}$$

and

$$\theta(x, u) = \begin{cases} \sqrt{\sin u - \sin x} & \text{if } u > x, \\ 0 & \text{if } u \leq x. \end{cases}$$

Define $\bar{b} : X \times X \to \mathbb{R}_+$ as

$$\bar{b}(x,u) = 2.$$

Taking $\rho = -1$.

f is $\rho - (\eta,\theta)$-pseudo-B-invex but not pseudo-B-invex with respect to η at $x = \frac{\pi}{6}$ and $u = \frac{\pi}{4}$.

Theorem 53. Quasi- B-invex $\Rightarrow \rho - (\eta,\theta)$-quasi-B-invex

$$\not\Leftarrow$$

Consider $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ by $f(x) = \cos x$.

Let the functions η and θ be defined by

$$\eta(x,u) = u - x$$

and

$$\theta(x,u) = \begin{cases} \sqrt{x-u} & \text{if } x > u, \\ 0 & \text{if } x \leq u. \end{cases}$$

Taking ρ = -1. Define $\bar{b} : D \times D \to R_+$ by

$$b(x,u) = \begin{cases} xu & \text{if } x \geq u, \\ 0 & \text{if } x < u. \end{cases}$$

f is $\rho - (\eta,\theta)$-quasi-B-invex but not pseudo-B-invex with respect to η at $x = \frac{\pi}{4}$ and $u = \frac{\pi}{6}$.

Theorem 54. There exist invex functins which are not quasi-convex and there exist quasi-convex functions which are not invex.

Consider $f(x) = x^3$ is quasi-convex is not invex for any η.

Similarly the functin $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = -x^2 + xy - e^y$$

is invex but not quasi-convex.

Theorem 55. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a monotone increasing differentiable convex function.

If f is invex, then the composite function $\phi \circ f$ is invex.

Theorem 56. If f is differentiable function and there exists a sequence $\{t_n\}$ of positive real numbers such that $t_n \to 0$ as $n \to \infty$ and

$$f(y + t_n\eta(x, y)) \leq t_n f(x) + (1 - t_n)f(y)$$

for all x, y on the domain of f, then f is invex.

Theorem 57. If f is differentiable and pre invex (with respect to η), then it is invex (with respect to same η).

Theorem 58. If f is differentiable and invex, it need not be pre-invex. Infact pre invexity is a stronger condition than invexity.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

f is invex with respect to η given by

$$\eta(x, y) = \begin{cases} \frac{x^2-y^2}{2y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0 \end{cases}$$

$$f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y)$$

holds if and only if

$$t^2(x^4 - 2x^2y^2 + y^2) \leq 0,$$

That is t=0 or x=y=0.

Theorem 59. Suppose that Λ is a pre-invex set with respect to η and $f : X \to \mathbb{R}$ is differentiable, where X is open and $\Lambda \subset X$. Further suppose that f is invex with respect to η on Λ and that η satisfies condition C. Then f is pre-invex with respect to η on Λ.

Theorem 60. Let $f : \mathbb{R}^n \to \mathbb{R}$ be semi-invex and let condition C be satisfied. Then f is pre-invex.

Theorem 61. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\eta : \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^n$ be such that

(i) f is pre-invex

(ii) $sup_t \frac{f(x+t\eta(y,x))-f(x)}{t} \geq (\xi, \eta(y,x))$

For some $\xi \in \mathbb{R}^n$ and $t \in [0,1]$. Then f is semi-invex.

Theorem 62. Let f be a positive function defined on a convex set $K \subset \mathbb{R}^n$.

(i) If f is H-convex on K, then it is L-convex and convex on K but not conversely.

(ii) If f is concave on K, then it is L-concave and H-concave on K but not conversely.

Theorem 63. (i) Any positive linear function on a convex set $K \subset \mathbb{R}^n$, except a positive constant, is not H-convex on K,

(ii) Each positive linear function on $K \subset \mathbb{R}^n$ is H-concave on K.

Theorem 64. The reciprocal f of a positive H-convex (concave) function ϕ on $K \subset \mathbb{R}^n$ is concave (convex) on K and conversely.

Theorem 65. Let f be H-concave on $K \subset \mathbb{R}^n$. Then f is strictly quasi-concave on K.

Theorem 66. If g is the identity map, then g-convexity of f reduces to convexity, and if $g(x) = -x$, g-convexity implies that f is concave.

Theorem 67. If g is the log function and f is g-convex, then

$$f(\lambda x + (1-\lambda)y) \leq (f(x))^{\lambda}(f(y))^{1-\lambda}$$

which is the L-convexity of f.

Theorem 68. $g(x) = -\frac{1}{x}$ and f is g-convex, if and only if f is H-convex (harmonic convex), given by

$$f(\lambda x + (1-\lambda)y) \leq \frac{1}{\frac{\lambda}{f(x)} + \frac{1-\lambda}{f(y)}}$$

for $x, y \in K$ and $\lambda \in [0,1]$.

Similarly $f(x) = \frac{1}{x}$ and f is g-convex if and only if f is H-concave.

Theorem 69. (i) If f is affine and g is convex, then f is g-convex.

(ii) If f and g are convex and g is nondecreasing, then f is g-convex.

Theorem 70. f is g-convex if and only if

$$G = \{(x,\alpha) : x \in K, \alpha \in \mathbb{R}, gf(x) \leq \alpha\}$$

is a convex set.

Theorem 71. Let f is g-convex. Then the set

$$A_\alpha = \{x \in K : gf(x) \leq \alpha\}$$

is convex for every real α.

Remark

1. If $H(x,y,t) = x + \xi(y,x,t)$, then

Arcwise convex $\Rightarrow$ g preinvex

2. If $\xi(y,x,t) = t\eta(y,x)$, then

g preinvex $\Rightarrow$ preinvex

Theorem 72. f is g-convex iff for each positive integer n, $x_1, x_2, ..., x_n \in K$, $p_1, p_2, ..., p_n \geq 0$, $p_1 + p_2 + ... + p_n = 1$, $gf(p_1x_1 + p_2x_2 + ... + p_nx_n) \leq p_1gf(x_1) + p_gf(x_2) + ... + p_ngf(x_n)$

Theorem 73. An affine function f is $g-\gamma qc$ iff g is $g-\gamma qc$.

Theorem 74. If f is preinvex and g is monotonically increasing and convex, then f is g-preinvex.

Theorem 75. H-preinvexity⇒L-preinvexity⇒preinvexity.

Theorem 76. Let $g : I \to \mathbb{R}$ be convex and strictly increasing. Let f be sspi with $rngf \subseteq I$. Then f is ssgpi.

Theorem 77. Let $f : \mathbb{R}^n \to \mathbb{R}$ be g-semi-invex with a given η satisfying the condition C and let g be additive. Then f is g-preinvex with respect to same η.

Theorem 78. Convex⇒ Invex

$\not\Leftarrow$

Consider $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be a function defined by $f(x) = x + \sin x$.

Invex w.r.t. $\eta(x_1, x_2) = \frac{\sin x_1 - \sin x_2}{cos x_2}$

Not convex for $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{6}$.

Theorem 79. St. Convex⇒St. Invex

$\not\Leftarrow$

Consider $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be a function defined by $f(x) = -x + \cos x$.

St. Invex w.r.t. $\eta(x_1, x_2) = \frac{\cos x_2 - \cos x_1}{\sin x_2}$

Not St. Convex for $x_1 = \frac{\pi}{6}$, $x_2 = \frac{\pi}{4}$.

Theorem 80. Let $f = -\mid x \mid \forall\ x \in K = [-2, 2]$

and let

$$\eta(x, y) = \begin{cases} x - y & \text{if } x \geq 0, y \geq 0, \\ x - y & \text{if } x < 0, y < 0, \\ -2 - y & \text{if } x > 0, y \leq 0, \\ 2 - y & \text{if } x \leq 0, y > 0 \end{cases}$$

It is observed that f is invex with respect to η on K and f and η satisfy

Conditions A and C but f is not convex.

Theorem 81. Let Γ of $\mathbb{R}^n$ be an invex set with respect to η, and let η satisfy Condition C.

Then a differentiable function f is prequasiinvex with respect to η on Γ

iff for every pair of points $x, y \in \Gamma$,

$f(y) \leq f(x) \Rightarrow \eta(x, y)^T \nabla f(x) \leq 0$.

Theorem 82. Let f and η satisfy Condition C. Assume that the differentiable function f is

pseudoinvex with respect to η on an invex set Γ of $\mathbb{R}^n$ and $\forall\ x, y \in \Gamma$

$f(y) \leq f(x) \Rightarrow f(y + \eta(x, y)) \leq f(x)$.

Theorem 83. The following two statements are equivalent:

(i) f is B-vex on $X \subseteq \mathbb{R}^n$ w.r.t. some function $b : X \times X \times [0, 1] \to \mathbb{R}_+$;

(ii) f is quasiconvex on X.

Theorem 84. f is locally lipschitz on X. If f is b-vex near $x \in X$, with $b(x, y, \lambda)$ continuous on

$X \times X \times [0, 1]$, then for any $y \in X$,

$b(y, x, 0)[f(y) - f(x)] \geq (y - x)^T \xi$, $\forall\ \xi \in \partial f(x)$.

Theorem 85. Let f be Lipschitz near $x \in X$ and regular at x. If f is quasiconvex at $x \in X$, then there exists $\bar{b}(y,x)[f(y) - f(x)] \geq (y-x)^T \xi$, $\forall\ \xi \in \partial f(x)$.

Theorem 86. f is quasiconvex on X iff

$x, y \in X$, $f(y) < f(x) \Rightarrow (y-x)^T \xi \leq 0$, $\forall\ \xi \in \partial f(x)$.

Theorem 87. Let f be locally Lipschitz on X. If there exists a function $\bar{b} : X \times X \to \mathbb{R}_+$

such that $\forall\ x, y \in X$,

$\bar{b}(y,x)[f(y) - f(x)] \geq (y-x)^T \xi$, $\forall\ \xi \in \partial f(x)$

then f is b-vex on X, where

$b(y,x,\lambda) = \bar{b}(y, \lambda y + (1-\lambda)x)[\lambda \bar{b}(y, \lambda y + (1-\lambda)x) + (1-\lambda)\bar{b}(x, \lambda y + (1-\lambda)x)]^{-1}$.

Theorem 88. On the set X, let f be b-vex and bounded below. If there exists a real number $K \geq 0$ such that for any $x, y \in X$ and $\lambda \in (0,1)$,

$1 - K(1-\lambda)\lambda^{-1} \leq b(y,x,\lambda) \leq 1 + K(1-\lambda)\lambda^{-1}$

Theorem 89. f is invex if and only if every stationary point is a global minimum.

Theorem 90. If f has no stationary points then f is invex.

Theorem 91. Pseudo-convex and pseudo-invex functions are both invex, this is not the case with quasi-convex and quasi-invex functions

(i) $f(x) = x^3$

f is not invex, since the stationary point x=0 is not a global minimum.

x^3 is quasi-convex and hence, also quasi-invex.

(ii) $f(x) = x_1^3 + x_1 - 10x_2^3 - x_2$.

Since there are no stationary points. f is invex.

Taking u=(0,0), $x_1 = 2$, $x_2 = 1$, gives $f(x) - f(u) < 0$ but

$(x-u)^t \nabla f(u) > 0$, so that f is not quasi-convex.

Theorem 92. If f is differentiable and there exists an n-dimensional vector function $\eta(x,u)$ such that

$f(u + \lambda\eta(x,u)) \leq \lambda f(x) + (1-\lambda) f(u), 0 \leq \lambda \leq 1.$

Theorem 93. Assume that for every $y \in \mathbb{R}^n$, the function $x \to \eta(x,y)$ is

differentiable at the point x=y, $\eta(y,y) = 0$ and $\eta_x(y,y) = 1$. If f is invex and

$f(x) < f(y) \Rightarrow \nabla f(y)\eta(x,y) \leq \nabla f(y)(x-y)$

and

$\nabla f(y)v = 0 \Rightarrow \nabla f(y)(v^T \eta_{xx}(y,y))v > 0.$

then f is strongly pseudo-convex.

Theorem 94. If f is pre-invex then f is p.p.i with respect to the same η.

Theorem 95. Let f be p.q.i. function with respect to η and assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function. Then $\phi \circ f$ is p.q.i. with respect to the same η

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