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Convexity of Spherical Bernstein-Bézier Patches and Circular Bernstein-Bézier Curves

T. X. He *and Ram Mohapatra†

Abstract

This paper discusses the criteria of convexity of spherical Bernstein-Bézier patches, circular Bernstein-Bézier curves, and homogeneous Bernstein-Bézier polynomials.

Keywords: Spherical Bernstein-Bézier patch, spherical Bernstein-Bézier polynomial, circular Bernstein-Bézier curve, homogeneous Bernstein-Bézier polynomial, Bézier coefficient, convexity.

1 Introduction

Let S be the unit sphere in R^3 with center at the origin. $T = \{v \in S : v = b_1v_1+b_2v_2+b_3v_3,\ b_i \geq 0\}$ is the spherical triangle generated by the three unit vectors v_1, v_2, $v_3 \in S$. Here the boundary

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of T, three circular arcs, lie on great circles. Let v be a point on S^1. The (spherical) barycentric coordinates of v relative to T are the unique real numbers b_1, b_2, and b_3 such that

$$v = b_1v_1 + b_2v_2 + b_3v_3 \tag{1.1}$$

It is clear from (1.1) that the spherical barycentric coordinates of a point v on the sphere S^1 are exactly the same as the trihedral coordinates of v with respect to the trihedron generated by $\{v_1, v_2, v_3\}$. This implies that they have the following properties (cf. [1]):

(i). At the vertices v_j, j=1, 2, 3, of T, $b_i(v_j) = \delta_{ij}$, i=1,2,3.

(ii). For all v in the interior of T, $b_i(v) > 0$.

(iii). In contrast to the usual barycentric coordinates on planar triangles (which always sum up to 1),

$b_1(v) + b_2(v) + b_3(v) > 1$ if v∈T and v$\neq v_1, v_2, v_3$.

For the set

$$f = \{f_i : i = (i_1, i_2, i_3),\ i_1, i_2, i_3 \geq 0,\ \ |i| = i_1 + i_2 + i_3 = n\}$$

, an n^{th} degree functional spherical Bernstein-Bézier (SBB) polynomial is defined on the spherical triangle T as follows (1.1).

$$p_n(v) = B_n[f; b] = \sum_{|i|=n} f_i\phi_i^n(b), \tag{1.2}$$

where v_1, v_2, v_3 are three vertices of T, b=(b_1, b_2, b_3), v=$b_1v_1 + b_2v_2 + b_3v_3$, and

$$\phi_i^n(b) = \frac{n!}{i!}b^i = \frac{n!}{i!i_2!i_3!}b_i^{i_1}b_2^{i_2}b_3^{i_3},\ \ |i| = i_1 + i_2 + i_3 = n. \tag{1.3}$$

f=$\{f_i\}$ is called the set of Bézier coefficients of the polynomial (1.2). If we do not restrict $\{v_1, v_2, v_3\}$ to be on the unit sphere S, then the $p_n(v)$ shown in (1.2) is called a homogeneous Bernstein-Bézier (HBB) polynomial of degree n on the trihedron

$$\hat{T} := \{v \in R^3 : v = b_1v_1 + b_2v_2 + b_3v_3,\ \ b_i \geq 0\}$$

generated by $\{v_1, v_2, v_3\}$ ([1.1]).

In many applications, the Bézier representation is used to form parametric surface patches by using vector-valued coefficients $\mathbf{f}=\{\mathbf{f}_i\}_{|i|=n}$. This will be indicated by using the boldface notation

$$\mathbf{p}_n(v) = \mathbf{B}_n[\mathbf{f}; b] = \sum_{|i|=n} \mathbf{f}_i \phi_i^n(b). \tag{1.4}$$

$\{(v^i, \mathbf{f}_i)\}_{|i|=n}$ is called the Bézier net of $\mathbf{p}_n(v)$. Here $v^i = \frac{1}{|i|}\sum_{\ell=1}^{3} i_\ell v_\ell$. In [1], the spherical Bernstein-Bézier (SBB) patch was defined as the surface $\{p_n(v)v : v \in T\}$. Using the notation $E_m^\ell c_i = c_{i+me^\ell}$, where e^ℓ is the ℓ^{th} coordinate vector in R^3, we can rewrite $p_n(v)v$ as

$$p_n(v)v = \sum_{|i|=n+1} \frac{1}{n+1}(i_1 v_1 E_{-1}^1 + i_2 v_2 E_{-1}^2 + i_3 v_3 E_{-1}^3) c_i \phi_i^{n+1}(b). \tag{1.5}$$

Clearly, from (1.5), $p_n(v)v$ is also a parametric surface patch $\mathbf{p}_{n+1}(v)$ with

$$\mathbf{f}_i = \frac{1}{|i|}(i_i v_i E_{-1}^1 + i_2 v_2 E_{-1}^2 + i_3 v_3 E_{-1}^3) c_i, \ |i| = n+1. \tag{1.6}$$

For this reason we also called (1.4) the SBB patch of degree n defined on the spherical triangle T. In (1.2), a theory of the circular Bernstein-Bézier (CBB) polynomials is developed. In addition to their intrinsic interest, the CBB polynomials are also useful for describing the behavior of SBB polynomials on the circular arcs making up the edges of spherical triangles.

Let C be the unit circle in R^2 with center at the origin, and let A be a circular arc on C with length less than π and vertices $v_1 \neq v_2$. Let v be a point on C. Then the (circular) barycentric coordinates of v relative to A are the unique pair of real numbers b_1, b_2 such that

$$v = b_1 v_1 + b_2 v_2. \tag{1.7}$$

Circular barycentric coordinates have a very simple form if we express points on C in polar coordinates. Suppose

$$v_1 = (\cos\theta_1, \sin\theta_1)^T, \qquad v_2 = (\cos\theta_2, \sin\theta_2)^T, \tag{1.8}$$

with $0 < \theta_2 - \theta_1 < \pi$. let $v \in C$ be expressed in polar coordinates as $v = (\cos\theta, \sin\theta)^T$. The circular barycentric coordinates of v relative to circular arc A are

$$b_1(v) = \frac{\sin(\theta_2 - \theta)}{\sin(\theta_2 - \theta_1)},$$
$$b_2(v) = \frac{\sin(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)}$$

Similarly, for a given integer $n > 0$, the Bernstein basis polynomial of degree n on the circular arc A is

$$\phi_i^n(\theta) := \binom{n}{i} b_1(\theta)^{n-i} b_2(\theta)^i, \quad i = 0, 1, \cdots, n.$$

We call

$$p(\theta) := \sum_{i=0}^{n} c_i \phi_i^n(\theta) \tag{1.9}$$

a circular Bernstein-Bézier (CBB) polynomial of degree n on the circular arc A. Given a CBB polynomial p defined on a circular arc A, we define an associated CBB curve by

$$P(\theta) = p(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{1.10}$$

The aim of this paper is to study the convexity properties of SBB patches and obtain some convexity criteria for HBB polynomials.

The paper is organized as follows. In Section 2, we will discuss the convexity of SBB patches $\mathbf{p}_n(v)$. Some properties of spherical barycentric coordinates will also be given in this section. In Section 3, we will discuss the convexity of CBB curves. Some convexity criteria for HBB polynomials will be shown in Section 4.

2 Convexity criteria of SBB patches

In order to discuss the convexity of SBB patches, we need the following lemmas about the relations among the spherical barycentric coordinates of v that are defined in equation (1.1).

Lemma 1. Let T be a spherical triangle with vertices $v_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$, let $v = (x, y, z)$ be a point on S, the unit sphere in R^3 with the center at the origin, and let the vector $b = (b_1, b_2, b_3)$ be the spherical barycentric coordinates of v relative to T. Then

$$\sum_{1\le i\ne j\le 3} b_i b_j \langle v_i, v_j\rangle = 1. \tag{2.11}$$

Proof. From equation (1.1), we have

$$\langle v, v\rangle = \langle b_1v_1 + b_2v_2 + b_3v_3, b_1v_1 + b_2v_2 + b_3v_3\rangle = \sum_{1\le i\ne j\le 3} b_i b_j \langle v_i, v_j\rangle.$$

Since $\langle v, v\rangle = 1$, we obtain equation (2.11).

Lemma 2. Let T be a spherical triangle with vertices $v_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$, let $v = (x, y, z)$ be a point on S, the unit sphere in R^3 with the center at the origin, and let the vector $b = (b_1, b_2, b_3)$ be the spherical barycentric coordinates of v relative to T. Then b_3 can be considered as a function of b_1 and b_2, and

$$\frac{\partial b_3}{\partial b_i} = -\frac{\beta_i(b)}{\beta_3(b)},\ i = 1, 2, \tag{2.12}$$

where $\beta_\ell = \sum_{k=1}^{3} b_k\langle v_\ell, v_k\rangle$, $\ell = 1, 2, 3$, and $\langle v_\ell, V_k\rangle$ is the inner product of v_ℓ and v_k.

Proof. It is sufficient to prove the expression of $\frac{\partial b_3}{\partial b_1}$. Taking derivative in terms of b_1 on both sides of equation (2.11), we have

$$\frac{\partial}{\partial b_1}\sum_{1\le i,j\le 3} b_i b_j \langle v_i, v_j\rangle = 0.$$

Expanding the left hand side of the above equation and transposing all terms with $\frac{\partial b_3}{\partial b_1}$ to the right hand side, we obtain

$$b_1\langle v_1, v_1\rangle + b_2\langle v_1, v_2\rangle + b_3\langle v_1, v_3\rangle = -\frac{\partial b_3}{\partial b_1}[b_1\langle v_3, v_1\rangle + b_2\langle v_3, v_2\rangle + b_3\langle v_3, v_3\rangle].$$

Thus the lemma is proved.

Lemma 3. Let the SBB patch of f be the $\mathbf{B}_n[f,b]$ defined in (1.4). Then for $\ell = 1,2$,

$$\frac{\partial}{\partial b_\ell}\mathbf{B}_n[f;b] = \frac{1}{\beta_3(b)}\sum_{|i|=n}\langle\psi(v,E), v_3E_1^\ell - v_\ell E_1^3\rangle \mathbf{f}_i\phi_i^n(b), \tag{2.13}$$

where $\beta_3 = \sum_{k=1}^3 b_k\langle v_3, v_k\rangle$, $\psi(v,E) = i_1v_1E_{-1}^1 + i_2v_2E_{-1}^2 + i_3v_3E_{-1}^3$, $\langle av_\alpha\ E_m^\ell,\ bv_\beta\ E_n^q\rangle = ab\langle v_\alpha,$ $v_\beta\rangle E_m^\ell E_n^q$, and the shift operator $E_m^\ell c_i = c_{i+me^\ell}$. Here e^ℓ is the ℓ^{th} coordinate vector in R^3.

Proof. It is sufficient to prove the expression $\frac{\partial}{\partial b_1}\mathbf{B}_n[f;b]$. Noting equation (2.12), we have

$$\begin{aligned}
\frac{\partial}{\partial b_1}\mathbf{B}_n[f;b] &= \sum_{|i|=n}\mathbf{f}_i\left[\frac{n!}{(i_1-1)!i_2!i_3!}b_1^{i_1-1}b_2^{i_2}b_3^{i_3} + \frac{n!}{i_1!i_2!(i_3-1)!}b_1^{i_1}b_2^{i_2}b_3^{i_3-1}\frac{\partial b_3}{\partial b_1}\right]\\
&= \frac{1}{\beta_3(b)}\left[\sum_{|i|=n}(b_1\langle v_3,v_1\rangle + b_2\langle v_3,v_2\rangle + b_3\langle v_3,v_3\rangle)\mathbf{f}_i\frac{n!}{(i_1-1)!i_2!i_3!}b_1^{i_1-1}b_2^{i_2}b_3^{i_3}\right.\\
&\quad \left. - \sum_{|i|=n}(b_1\langle v_1,v_1\rangle + b_2\langle v_1,v_2\rangle + b_1\langle v_3,v_3\rangle)\mathbf{f}_i\frac{n!}{i_1!i_2!(i_3-1)!}b_1^{i_1}b_2^{i_2}b_3^{i_3-1}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\beta_3(b)}\sum_{|i|=n}[i_1\langle v_3,v_1\rangle E_{-1}^1E_1^1 + i_2\langle v_3,v_2\rangle E_1^1E_{-1}^2 + i_3\langle v_3,v_3\rangle E_1^1E_{-1}^3\\
&- i_1\langle v_1,v_1\rangle E_{-1}^1E_1^3 - i_2\langle v_1,v_2\rangle E_{-1}^2E_1^3 - i_3\langle v_1,v_3\rangle E_1^3E_{-1}^3]\mathbf{f}_i\phi_i^n(b)\\
&= \frac{1}{\beta_3(b)}\sum_{|i|=n}(i_1v_1E_{-1}^1 + i_2v_2E_{-1}^2 + i_3v_3E_{-1}^3)\cdot(v_3E_1^1 - v_1E_1^3)\mathbf{f}_i\phi_i^n(b).
\end{aligned}$$

Thus equation (2.13) has been proved.

Similarly, we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial b_\ell^2}\mathbf{B}_n[f;b] &= \frac{1}{[\beta_3(b)]^2}\sum_{|i|=n}[\langle\psi(v,E), v_3E_1^\ell - v_\ell E_1^3\rangle]^2\mathbf{f}_i\phi_i^n(b)\\
&+ \beta_3(b)\frac{\partial}{\partial b_\ell}\mathbf{B}_n[f;b]\frac{\partial}{\partial b_\ell}\left(\frac{1}{\beta_3(b)}\right), \quad \ell = 1,2
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
\frac{\partial^2}{\partial b_1\partial b_2}\mathbf{B}_n[f;b] &= \frac{1}{[\beta_3(b)]^2}\sum_{|i|=n}\left[\prod_{\ell=1}^2\langle\psi(v,E), v_3E_1^\ell - v_\ell E_1^3\rangle\right]\mathbf{f}_i\phi_i^n(b)\\
&+ \beta_3(b)\frac{\partial}{\partial b_2}\mathbf{B}_n[f;b]\frac{\partial}{\partial b_1}\left(\frac{1}{\beta_3(b)}\right),
\end{aligned}$$

$$= \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \left[\prod_{\ell=1}^{2} \langle \psi(v,E), v_3 E_1^\ell - v_\ell E_1^3 \rangle \right] \mathbf{f}_i \phi_i^n(b) \tag{2.15}$$

$$+ \beta_3(b) \frac{\partial}{\partial b_1} \mathbf{B}_n[f;b] \frac{\partial}{\partial b_2} \left(\frac{1}{\beta_3(b)} \right). \tag{2.16}$$

We now denote

$$\mathbf{p}_i = \langle \psi(v,E), V_3 E_1^1 - V_1 E_1^3 \rangle \mathbf{f}_i, \tag{2.17}$$

$$\mathbf{q}_i = \langle \psi(v,E), v_3 E_1^2 - v_2 E_1^3 \rangle \mathbf{f}_i, \tag{2.18}$$

$$\mathbf{U}_i = \langle \psi(v,E), v_1 E_1^3 - v_3 E_1^1 \rangle \langle v_3 E_1^2 - v_3 E_1^1 + (v_1 - v_2) E_1^3, \psi(v,E) \rangle \mathbf{f}_i, \tag{2.19}$$

$$\mathbf{V}_i = \langle \psi(v,E), v_2 E_1^3 - v_3 E_1^2 \rangle \langle v_3 E_1^1 - v_3 E_1^2 + (v_2 - v_1) E_1^3, \psi(v,E) \rangle \mathbf{f}_i, \tag{2.20}$$

$$\mathbf{W}_i = \langle \psi(v,E), v_3 E_1^1 - v_1 E_1^3 \rangle \langle v_3 E_1^2 - v_2 E_1^3, \psi(v,E) \rangle \mathbf{f}_i. \tag{2.21}$$

Remark: By changing $\mathbf{f}_i$ to f_i in (2.13)-(2.21), we obtain the corresponding partial derivatives of $B_n[f;b]$ and the corresponding p_i, q_i, U_i, V_i, and W_i.

Obviously, we have

$$\frac{\partial}{\partial b_1} \mathbf{B}_n[f;b] = \frac{1}{\beta_3(b)} \sum_{|i|=n} \mathbf{p}_i \phi_i^n(b),$$

$$\frac{\partial}{\partial b_2} \mathbf{B}_n[f;b] = \frac{1}{\beta_3(b)} \sum_{|i|=n} \mathbf{q}_i \phi_i^n(b),$$

$$\frac{\partial^2}{\partial b_1^2} \mathbf{B}_n[f;b] = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} (\mathbf{U}_i + \mathbf{W}_i) \phi_i^n(b) + \beta_3(b) \frac{\partial}{\partial b_1} \mathbf{B}_n \frac{\partial}{\partial b_1} \left(\frac{1}{\beta_3(b)} \right),$$

$$\frac{\partial^2}{\partial b_2^2} \mathbf{B}_n[f;b] = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} (\mathbf{V}_i + \mathbf{W}_i) \phi_i^n(b) + \beta_3(b) \frac{\partial}{\partial b_2} \mathbf{B}_n \frac{\partial}{\partial b_2} \left(\frac{1}{\beta_3(b)} \right),$$

and

$$\frac{\partial^2}{\partial b_1^2 b_2} \mathbf{B}_n[f;b] = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{W}_i \phi_i^n(b) + \beta_3(b) \frac{\partial}{\partial b_1} \mathbf{B}_n \frac{\partial}{\partial b_2} \left(\frac{1}{\beta_3(b)} \right),$$

$$= \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{W}_i \phi_i^n(b) + \beta_3(b) \frac{\partial}{\partial b_2} \mathbf{B}_n \frac{\partial}{\partial b_1} \left(\frac{1}{\beta_3(b)} \right).$$

It is well known that if the Gaussian curvature of a compact surface π in R^3 is positive everywhere, then surface π is convex (lies on one side of each tangent plane)(1.9). In addition, if π is defined by $\mathbf{r} = \mathbf{r}(u,v)$, a parametric vector-valued function in C^2, then the Gaussian curvature of π is $k = (LN - M^2)/(EG - F^2)$, where E, G, F and L, M, N are respectively the first fundamental form and the second fundamental form of π. It is well known that $L = \frac{1}{D}(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uu})$, $M = \frac{1}{D}(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uv})$, $N = \frac{1}{D}(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{vv})$, and $D^2 = EG - F^2 > 0$. Here $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the scalar product of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and is defined by $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \langle \mathbf{a} \times \mathbf{b}, \mathbf{c}\rangle$. Therefore the Gaussian curvature K of $\mathbf{r} = \mathbf{r}(u,v)$ and $LN - M^2$, or $(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uu})$, $(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{vv}) - (\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uv})^2$ have the same sign. In particular, for the surface $\mathbf{r} = \mathbf{r}(b_1, b_2) = \mathbf{B}_n[f; b]$, we have

$$\mathbf{r}_{b_1} = \frac{\partial}{\partial b_1}\mathbf{B}_n[f;b], \quad \mathbf{r}_{b_2} = \frac{\partial}{\partial b_2}\mathbf{B}_n[f;b], \quad \mathbf{r}_{b_1 b_1} = \frac{\partial^2}{\partial b_1^2}\mathbf{B}_n[f;b],$$

etc. Thus, if $(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_1})$ $(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_2 b_2})$ $-(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_2})^2 > 0$, then $K > 0$. By using the notation in (2.17)-(2.21), from Lemma 3, we have

$$(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_1})(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_2 b_2}) - (\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_2})^2$$

$$= (\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^U)(\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^V) + (\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^U)(\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^W) + (\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^V)(\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^W), \qquad (2.22)$$

where $\mathbf{r}^p = \frac{1}{\beta_3(b)} \sum_{|i|=n} \mathbf{p}_i \phi_i^n(b)$, $\mathbf{r}^q = \frac{1}{\beta_3(b)} \sum_{|i|=n} \mathbf{q}_i \ \phi_i^n(b)$, $\mathbf{r}^U = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{U}_i \ \phi_i^n(b)$, $\mathbf{r}^V = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{V}_i \ \phi_i^n(b)$, and $\mathbf{r}^W = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{W}_i \ \phi_i^n(b)$.

Denoting $\nabla^{(1)}_{i,j,k} = (\mathbf{p}_i, \mathbf{q}_j, \mathbf{U}_k)$, $\nabla^{(2)}_{i,j,k} = (\mathbf{p}_i, \mathbf{q}_j, \mathbf{V}_k)$, and $\nabla^{(3)}_{i,j,k} = (\mathbf{p}_i, \mathbf{q}_j, \mathbf{W}_k)$, from (2.22) we have the following theorem.

Theorem 1. If

$$\sum_{1\le\alpha<\beta\le 3} \sum_{|j|=3n} \sum_{|s|=2n} \sum_{|r|=2n} \sum_{|t|=n} \sum_{|k|=n} \triangle^{(\alpha,\beta)}_{j,s,r,t,k}(i) > 0 \qquad (2.23)$$

holds for all $i \in Z^3_+$, $|i| = 6n$, where

$$\triangle^{(\alpha,\beta)}_{j,s,r,t,k}(i) = \nabla^{(\alpha)}_{t,s-t,j-s} \nabla^{(\beta)}_{k,r-k,i-j-r} \frac{\binom{s}{t}\binom{r}{k}\binom{j}{s}\binom{i-j}{r}\binom{i}{j}}{\binom{2n}{n}^2\binom{3n}{2n}^2\binom{6n}{3n}},$$

then $\mathbf{B}_n[f;b]$ is convex over the spherical triangle T.

Proof. To prove the theorem, we need the following two representations.

$$\sum_{|i|=n} \mathbf{f}_i \phi_i^n(b) \times \sum_{|i|=m} \mathbf{q}_i \phi_i^m(b) = \sum_{|i|=n+m} \mathbf{h}_i \phi_i^{n+m}(b), \tag{2.24}$$

where $\mathbf{h}_i = \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{q}_{i-j}) \binom{i}{j} / \binom{n+m}{n}$, and

$$\sum_{|i|=n} f_i \phi_i^n(b) \cdot \sum_{|i|=m} g_i \phi_i^m(b) = \sum_{|i|=n+m} h_i \phi_i^{n+m}(b) \tag{2.25}$$

where $h_i = \sum_{|j|=n} (f_j g_{i-j}) \binom{i}{j} / \binom{n+m}{n}$. In fact,

$$\begin{aligned}
& \sum_{|j|=n} \mathbf{f}_j \phi_j^n(b) \times \sum_{|i|=m} \mathbf{g}_i \phi_i^m(b) \\
= & \sum_{|i|=m} \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{g}_i)\, \phi_j^n(b) \phi_i^m(b) \\
= & \sum_{|i|=m} \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{g}_i) \binom{n}{j} \binom{m}{i} b^{i+j} \\
= & \sum_{|i'|=m} \sum_{|j|=n} \left(\mathbf{f}_j \times \mathbf{g}_{i'-j}\right) \binom{n}{j} \binom{m}{i'-j} b^{i'} \\
= & \sum_{|i'|=m} \sum_{|j|=n} \left(\mathbf{f}_j \times \mathbf{g}_{i'-j}\right) \frac{n!}{j!} \frac{m!}{(i'-j)!} b^{i'} \\
= & \sum_{|i'|=m} \sum_{|j|=n} \left(\mathbf{f}_j \times \mathbf{g}_{i'-j}\right) \frac{i!}{(i'-j)!j!} \frac{(n+m)!}{i'!} \frac{n!m!}{(n+m)!} b^{i'} \\
= & \sum_{|i'|=m} \sum_{|j|=n} \left(\mathbf{f}_j \times \mathbf{g}_{i'-j}\right) \binom{i'}{j} \phi_{i'}^{n+m}(b) / \binom{n+m}{n}.
\end{aligned}$$

Similarly, we may obtain equation 2.25 by using the same argument. Therefore,

$$\begin{aligned}
(\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^U) = & \ (\mathbf{r}^p \times \mathbf{r}^q) \cdot \mathbf{r}^U \\
= & \left(\frac{1}{\beta_3(b)} \sum_{|t|=n} \mathbf{p}_t \phi_t^n(b) \times \frac{1}{\beta_3(b)} \sum_{|s|=n} \mathbf{q}_s \phi_s^n(b) \right) \cdot \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{U}_i \phi_i^n(b) \\
= & \frac{1}{[\beta_3(b)]^4} \sum_{|s|=2n} \left(\sum_{|t|=n} \mathbf{p}_t \times \mathbf{q}_{s-t} \right) \frac{\binom{s}{t}}{\binom{2n}{n}} \phi_s^{2n}(b) \cdot \sum_{|i|=n} \mathbf{U}_i \phi_i^n(b) \\
= & \frac{1}{[\beta_3(b)]^4} \sum_{|i|=3n} \sum_{|s|=2n} \sum_{|t|=n} (\mathbf{p}_t \times \mathbf{q}_{s-t}) \cdot \mathbf{U}_{i-s} \binom{i}{s} \binom{s}{t} \phi_i^{3n}(b) / \binom{2n}{n} \binom{3n}{2n} \\
= & \frac{1}{[\beta_3(b)]^4} \sum_{|i|=3n} \sum_{|s|=2n} \sum_{|t|=n} \nabla^{(1)}_{t,s-t,i-s} \binom{i}{s} \binom{s}{t} \phi_i^{3n}(b) / \binom{2n}{n} \binom{3n}{2n}
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^V) = & \ (\mathbf{r}^p \times \mathbf{r}^q) \cdot \mathbf{r}^V \\
= & \left(\frac{1}{\beta_3(b)} \sum_{|k|=n} \mathbf{p}_k \phi_k^n(b) \times \frac{1}{\beta_3(b)} \sum_{|r|=n} \mathbf{q}_r \phi_r^n(b) \right) \cdot \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{V}_i \phi_i^n(b) \\
= & \frac{1}{[\beta_3(b)]^4} \sum_{|i|=3n} \sum_{|r|=2n} \sum_{|k|=n} \nabla^{(2)}_{k,r-k,i-r} \binom{i}{r} \binom{r}{k} \phi_i^{3n}(b) / \binom{2n}{n} \binom{3n}{2n}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& (\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^U) \cdot (\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^V) \\
= & \frac{1}{[\beta_3(b)]^4} \sum_{|i|=3n} \sum_{|s|=2n} \sum_{|t|=n} \nabla^{(1)}_{t,s-t,i-s} \binom{i}{s} \binom{s}{t} \phi_i^{3n}(b) \\
& \cdot \frac{1}{[\beta_3(b)]^4} \sum_{|i|=3n} \sum_{|r|=2n} \sum_{|k|=n} \nabla^{(2)}_{k,r-k,i-r} \binom{i}{r} \binom{r}{k} \phi_i^{3n}(b) / \binom{2n}{n}^2 \binom{3n}{2n}^2 \\
= & \frac{1}{[\beta_3(b)]^8} \sum_{|i|=6n} \sum_{|j|=3n} \sum_{|s|=2n} \sum_{|r|=2n} \sum_{|t|=n} \sum_{|k|=n} \\
& \nabla^{(1)}_{t,s-t,i-s} \nabla^{(2)}_{k,r-k,i-j-r} \frac{\binom{s}{t}\binom{r}{k}\binom{j}{s}\binom{i-j}{r}\binom{i}{j}}{\binom{2n}{n}^2 \binom{3n}{2n}^2 \binom{6n}{3n}} \phi_i^{6n}(b).
\end{aligned}$$

By using the above two representations and expression (2.18), we immediately have

$$\begin{aligned} &(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_1})(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_2 b_2}) - (\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_2})^2 \\ = \;& \frac{1}{[\beta_3(b)]^4} \sum_{|i|=6n} \sum_{|j|=3n} \sum_{|s|=2n} \sum_{|r|=2n} \sum_{|t|=n} \sum_{|k|=n} \sum_{1\le\alpha<\beta\le 3} \triangle^{(\alpha,\beta)}_{j,s,r,t,k}(i)\phi^{6n}_i(b). \end{aligned} \tag{2.26}$$

Thus Theorem 1 is proved.

From condition (2.26), we also have the following criterion for the convexity of $\mathbf{B}_n[f;b]$.

Theorem 2. If one of the following condition is satisfied for all $i, j, k \in Z^3_+$, $|i| = |j| = |k| = n$, then $\mathbf{B}_n[f;b]$ is convex over the spherical triangle T.

$$\begin{aligned} (i)\ \nabla^{(u)}_{i,j,k} &+ \nabla^{(w)}_{i,j,k} > 0, \quad \nabla^{(u)}_{i,j,k}\nabla^{(v)}_{i,j,k} + \nabla^{(v)}_{i,j,k}\nabla^{(w)}_{i,j,k} + \nabla^{(w)}_{i,j,k}\nabla^{(u)}_{i,j,k} > 0; \\ (ii)\nabla^{(1)}_{i,j,k} &+ \nabla^{(3)}_{i,j,k} > \left|\nabla^{(3)}_{i,j,k}\right| \ and\ \nabla^{(2)}_{i,j,k} + \nabla^{(3)}_{i,j,k} > \left|\nabla^{(3)}_{i,j,k}\right|; \\ (iii) & \qquad \sum_{1\le\alpha<\beta\le 3} \nabla^{(\alpha)}_{i,j,k}\nabla^{(\beta)}_{i,j,k} > 0; \\ (iv)\nabla^{(\alpha)}_{i,j,k} &> 0, \quad \alpha = 1,2,3, \end{aligned}$$

where (u, v, w) is a permutation of (1, 2, 3).

Proof. Denote $\nabla^{(\ell)}_{i,j,k}$, $\ell = 1,2,3$, by a_1, b_1, c_1, respectively and $\nabla^{(\ell)}_{i',j',k'}$, $\ell = 1,2,3$, by a_2, b_2, c_2, respectively. It is obvious that inequality (2.23) holds if

$$a_1b_2 + b_1c_2 + c_1a_2 + a_2b_1 + b_2c_1 + c_2a_1 > 0 \tag{2.27}$$

for all $i, j, k \in Z^3_+$, where $|i| = |j| = |k| = n$ and $|i'| = |j'| = |k'| = n$.

We can prove that inequality (2.27) holds if the following inequalities

$$a_\ell + c_\ell > 0, \ \ and\ \ a_\ell b_\ell + b_\ell c_\ell + c_\ell a_\ell > 0, \qquad \ell = 1,2,$$

hold. In fact, if the above inequalities hold, then we have

$$(a_1 + c_1)b_1 > -a_1c_1, \qquad (a_2 + c_2)b_2 > -a_2c_2,$$

and

$$
\begin{aligned}
& (a_1+c_1)(a_2+c_2)(a_1b_2+b_1c_2+c_1a_2+a_2b_1+b_2c_1+c_2a_1) \\
= \; & (a_1+c_1)(a_2+c_2)[b_1(a_2+c_2)+b_2(a_1+c_1)+c_1a_2+a_1c_2] \\
= \; & (a_2+c_2)^2(a_1+c_1)b_1+(a_1+c_1)^2(a_2+c_2)b_2+c_1a_2+a_1c_2 \\
> \; & -a_1c_1(a_2+c_2)^2-a_2c_2(a_1+c_1)^2+c_1a_2+a_1c_2 \\
= \; & (a_1c_2-a_2c_1)^2 \geq 0.
\end{aligned}
$$

Thus inequality (2.27) holds.

Consequently, inequality (2.23) holds if

$$\nabla^{(1)}_{i,j,k}+\nabla^{(3)}_{i,j,k}>0, \; and \; \nabla^{(1)}_{i,j,k}\nabla^{(2)}_{i,j,k}+\nabla^{(2)}_{i,j,k}\nabla^{(3)}_{i,j,k}+\nabla^{(3)}_{i,j,k}\nabla^{(1)}_{i,j,k}>0$$

which is equivalent to that the following matrix is positive definite.

$$\begin{pmatrix} \nabla^{(1)}_{i,j,k}+\nabla^{(3)}_{i,j,k} & \nabla^{(1)}_{i,j,k} \\ \nabla^{(1)}_{i,j,k} & \nabla^{(1)}_{i,j,k}+\nabla^{(2)}_{i,j,k} \end{pmatrix}$$

Obviously, the above matrix is positive definite if it is strictly strongly diagonally dominant, that is, $\sqrt[n]{}$

$$\nabla^{(1)}_{i,j,k}+\nabla^{(3)}_{i,j,k}>\left|\nabla^{(1)}_{i,j,k}\right| \; and \; \nabla^{(1)}_{i,j,k}+\nabla^{(2)}_{i,j,k}>\left|\nabla^{(1)}_{i,j,k}\right|.$$

Furthermore, this condition can be implied by

$$\nabla^{(\alpha)}_{i,j,k}>0, \quad \alpha=1,2,3.$$

Thus, Theorem 2 is proved.

Remark 5. Equation (2.24) and inequality (2.27) were given in [12] without any proof.

3 Convexity criteria of CBB curvess

In this section, we will give the convexity criteria for CBB curves. Obviously, a CBB curve is convex if and only if its curvature $k \geq 0$; i.e., the curve lies on only one side of each tangent line. The following lemma gives the curvature $k = k(\theta)$ of $P(\theta)$ at θ, which is defined as equation (10).

Lemma 4. Let the $P(\theta)$ defined as (1.10) be a CBB curve and $p(\theta)$ be the associated CBB polynomial defined in equation (1.9). The CBB curve is convex if and only if

$$(p(\theta))^2 + 2(p'(\theta))^2 - p(\theta)p''(\theta) \geq 0. \tag{3.28}$$

Proof. It is sufficient to prove that the sign of the curvature of the CBB curve $P(\theta)$ at any θ is

$$Sign[k(\theta)] = Sign[(p(\theta))^2 + 2(p'(\theta))^2 - p(\theta)p''(\theta)]. \tag{3.29}$$

Obviously, the curvature of the parametric curve $P(\theta) = (x(\theta), y(\theta))$ is

$$k(\theta) = \frac{y''(\theta)x'(\theta) - x''(\theta)y'(\theta)}{[(x'(\theta))^2 + (y'(\theta))^2]^{3/2}}.$$

Since $x(\theta) = p(\theta)\cos\theta$ and $y(\theta) = p(\theta)\sin\theta$, we obtain

$$\begin{aligned}
& y''(\theta)x'(\theta) - x''(\theta)y'(\theta) \\
= \; & [p''(\theta)\sin\theta + 2p'(\theta)\cos\theta - p(\theta)\sin\theta][p'(\theta)\cos\theta - p(\theta)\sin\theta] \\
- \; & [p''(\theta)\cos\theta - 2p'(\theta)\sin\theta - p(\theta)\cos\theta][p'(\theta)\sin\theta + p(\theta)\cos\theta] \\
= \; & (p(\theta))^2(\sin^2\theta + \cos^2\theta) + 2(p'(\theta))^2(\sin^2\theta + \cos^2\theta) - p(\theta)p''(\theta)(\sin^2\theta + \cos^2\theta) \\
= \; & (p(\theta))^2 + 2(p'(\theta))^2 - p(\theta)p''(\theta).
\end{aligned}$$

Thus the lemma is proved.

We now derive $p'(\theta)$ and $p''(\theta)$.

Since $p(\theta) = \sum_{i=0}^{n} c_i\binom{n}{i}b_1(\theta)^{n-i}b_2(\theta)^i$, where $b_1(\theta) = \sin(\theta_2-\theta)/\sin(\theta_2-\theta_1)$ and $b_2(\theta) = \sin(\theta-\theta_1)/\sin(\theta_2-\theta_1)$, we have

$$\begin{aligned}
p'(\theta) &= -\sum_{i=0}^{n-1} c_i \frac{n!}{(n-i-1)!i!} b_1(\theta)^{n-i-1} b_2(\theta)^i \frac{\cos(\theta_2-\theta)}{\sin(\theta_2-\theta_1)} \\
&+ \sum_{i=1}^{n} c_i \frac{n!}{(n-i)!(i-1)!} b_1(\theta)^{n-i} b_2(\theta)^{i-1} \frac{\cos(\theta-\theta_1)}{\sin(\theta_2-\theta_1)} \\
&= -n\sum_{i=0}^{n-1} c_i \binom{n-1}{i} b_1(\theta)^{n-i-1} b_2(\theta)^i \frac{\cos(\theta_2-\theta)}{\sin(\theta_2-\theta_1)} \\
&+ n\sum_{i=0}^{n-1} c_{i+1} \binom{n-1}{i} b_1(\theta)^{n-i-1} b_2(\theta)^i \frac{\cos(\theta-\theta_1)}{\sin(\theta_2-\theta_1)} \\
&= \frac{n}{\sin(\theta_2-\theta_1)} \sum_{i=0}^{n-1} \left[\cos(\theta-\theta_1)c_{i+1} - \cos(\theta_2-\theta)c_i\right] \phi_i^{n-1}(\theta).
\end{aligned}$$

Thus

$$\begin{aligned}
p''(\theta) &= \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{n-2} \left[(\cos(\theta-\theta_1)c_{i+2} - \cos(\theta_2-\theta)c_{i+1})\cos(\theta-\theta_1)\right. \\
&\left. -(\cos(\theta-\theta_1)c_{i+1} - \cos(\theta_2-\theta)c_i)\cos(\theta_2-\theta)\right] \phi_i^{n-2}(\theta) \\
&- \frac{n}{\sin(\theta_2-\theta_1)} \sum_{i=0}^{n-1} \left[\sin(\theta-\theta_1)c_{i+1} + \sin(\theta_2-\theta)c_i\right] \phi_i^{n-1}(\theta) \\
&= \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{n-2} \left[\cos^2(\theta-\theta_1)c_{i+2} - 2\cos(\theta_2-\theta)\cos(\theta-\theta_1)c_{i+1}\right. \\
&\left. +\cos^2(\theta_2-\theta)c_i\right] \phi_i^{n-2}(\theta)
\end{aligned}$$

$$
\begin{aligned}
&- \quad n\sum_{i=0}^{n-1}\left[b_2(\theta)c_{i+1}+b_1(\theta)c_i\right]\frac{(n-1)!}{i!(n-i-1)!}b_1(\theta)^{n-i-1}b_2(\theta)^i \\
&= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[\cos^2(\theta-\theta_1)c_{i+2}-2\cos(\theta_2-\theta)\cos(\theta-\theta_1)c_{i+1}\right. \\
&\qquad \left.+\cos^2(\theta_2-\theta)c_i\right]\phi_i^{n-2}(\theta) \\
&- \quad \sum_{i=0}^{n-1}(i+1)c_{i+1}\phi_{i+1}^n(\theta)-\sum_{i=0}^{n-1}(n-i)c_i\phi_i^n(\theta) \\
&= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[\cos^2(\theta-\theta_1)c_{i+2}-2\cos(\theta_2-\theta)\cos(\theta-\theta_1)c_{i+1}\right. \\
&\qquad \left.+\cos^2(\theta_2-\theta)c_i\right]\phi_i^{n-2}(\theta)-\sum_{i=1}^{n}ic_i\phi_i^n(\theta)-\sum_{i=0}^{n-1}(n-i)c_i\phi_i^n(\theta) \\
&= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[\cos^2(\theta-\theta_1)c_{i+2}-2\cos(\theta_2-\theta)\cos(\theta-\theta_1)c_{i+1}\right. \\
&\qquad \left.+\cos^2(\theta_2-\theta)c_i\right]\phi_i^{n-2}(\theta)-\sum_{i=0}^{n}ic_i\phi_i^n(\theta)-\sum_{i=0}^{n}(n-i)c_i\phi_i^n(\theta) \\
&= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[\cos^2(\theta-\theta_1)c_{i+2}-2\cos(\theta_2-\theta)\cos(\theta-\theta_1)c_{i+1}\right. \\
&\qquad \left.+\cos^2(\theta_2-\theta)c_i\right]\phi_i^{n-2}(\theta)-n\sum_{i=0}^{n}c_i\phi_i^n(\theta)
\end{aligned}
$$

Noting the trigonometric identities $\cos^2(\theta_2-\theta)=1-\sin^2(\theta_2-\theta)$, $\cos^2(\theta-\theta_1)=1-\sin^2(\theta-\theta_1)$, and $\cos(\theta_2-\theta)\cos(\theta-\theta_1)=\cos(\theta_2-\theta_1)+\sin(\theta_2-\theta)\sin(\theta-\theta_1)$, we may re-write $p''(\theta)$ into

$$
\begin{aligned}
p''(\theta) \quad &= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[c_{i+2}-2\cos(\theta_2-\theta_1)c_{i+1}+c_i\right]\phi_i^{n-2}(\theta) \\
&- \quad n(n-1)\sum_{i=0}^{n-2}b_2(\theta)^2c_{i+2}\phi_i^{n-2}(\theta)-2n(n-1)\sum_{i=0}^{n-2}b_2(\theta)b_1(\theta)c_{i+1}\phi_i^{n-2}(\theta) \\
&- \quad n(n-1)\sum_{i=0}^{n-2}b_1(\theta)^2c_i\phi_i^{n-2}(\theta)-n\sum_{i=0}^{n}c_i\phi_i^n(\theta) \\
&= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[c_{i+2}-2\cos(\theta_2-\theta_1)c_{i+1}+c_i\right]\phi_i^{n-2}(\theta)
\end{aligned}
$$

$$
\begin{aligned}
&- \quad n(n-1)\sum_{i=0}^{n-2} c_{i+2}\frac{(n-2)!}{(n-i-2)!i!}b_1(\theta)^{n-i-2}b_2(\theta)^{i+2}\\
&- \quad 2n(n-1)\sum_{i=0}^{n-2} c_{i+1}\frac{(n-2)!}{(n-i-2)!i!}b_1(\theta)^{n-i-1}b_2(\theta)^{i+1}\\
&- \quad n(n-1)\sum_{i=0}^{n-2} c_{i}\frac{(n-2)!}{(n-i-2)!i!}b_1(\theta)^{n-i}b_2(\theta)^{i} - n\sum_{i=0}^{n} c_i\phi_i^n(\theta)\\
&= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[c_{i+2}-2\cos(\theta_2-\theta_1)c_{i+1}+c_i\right]\phi_i^{n-2}(\theta)\\
&- \quad \sum_{i=0}^{n-2}(i+2)(i+1)c_{i+2}\frac{n!}{(n-i-2)!(i+2)!}b_1(\theta)^{n-i-2}b_2(\theta)^{i+2}\\
&- \quad 2\sum_{i=0}^{n-2}(i+1)(n-i-1)c_{i+1}\frac{n!}{(n-i-1)!(i+1)!}b_1(\theta)^{n-i-1}b_2(\theta)^{i+1}\\
&- \quad \sum_{i=0}^{n-2}(n-i)(n-i-1)c_{i}\frac{n!}{(n-i)!i!}b_1(\theta)^{n-i}b_2(\theta)^{i} - n\sum_{i=0}^{n} c_i\phi_i^n(\theta)\\
&= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[c_{i+2}-2\cos(\theta_2-\theta_1)c_{i+1}+c_i\right]\phi_i^{n-2}(\theta)\\
&- \quad \sum_{i=0}^{n} i(i-1)c_i\phi_i^n(\theta) - 2\sum_{i=0}^{n} i(n-i)c_i\phi_i^n(\theta)\\
&- \quad \sum_{i=0}^{n}(n-i)(n-i-1)c_i\phi_i^n(\theta) - n\sum_{i=0}^{n} c_i\phi_i^n(\theta)\\
&= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[c_{i+2}-2\cos(\theta_2-\theta_1)c_{i+1}+c_i\right]\phi_i^{n-2}(\theta)\\
&- \quad \sum_{i=0}^{n}\left[i(i-1)+2i(n-i)+(n-i)(n-i-1)+n\right]c_i\phi_i^n(\theta)
\end{aligned}
$$

$$
= \quad \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[c_{i+2}-2\cos(\theta_2-\theta_1)c_{i+1}+c_i\right]\phi_i^{n-2}(\theta) \tag{3.30}
$$

$$
- \quad n^2\sum_{i=0}^{n} c_i\phi_i^n(\theta) \tag{3.31}
$$

By using equation (2.25) for the case of two dimension:

$$
\sum_{i=0}^{n} f_i\phi_i^n(\theta)\cdot\sum_{j=0}^{m} g_j\phi_j^m(\theta)
$$

$$= \sum_{i=0}^{n+m} \sum_{j=0}^{m} f_{i-j} g_j \binom{n+m-i}{m-j} \phi_i^{n+m}(\theta) / \binom{n+m}{m}, \tag{3.32}$$

we obtain

$$[p(\theta)]^2 = \frac{1}{\binom{2n}{n}} \sum_{i=0}^{2n} d_i \phi_i^{2n}(\theta),$$

where

$$d_i = \sum_{j=0}^{n} c_j c_{i-j} \binom{i}{j} \binom{2n-i}{n-j}$$

and

$$\begin{aligned} p(\theta)p''(\theta) &= \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{2n-2} \sum_{j=0}^{n} c_j \left[c_{i-j+2} - 2\cos(\theta_2-\theta_1) c_{i-j+1}\right. \\ & \left. + c_{i-j}\right] \binom{i}{j} \binom{2n-i-2}{n-j} \phi_i^{2n-2}(\theta) / \binom{2n-2}{n} \\ &- n^2 \sum_{i=0}^{2n} d_i \phi_i^{2n}(\theta) / \binom{2n}{n} \\ &= \frac{2n(2n-1)}{\binom{2n}{n} \sin^2(\theta_2-\theta_1)} \sum_{i=0}^{2n-2} \sum_{j=0}^{n} c_j \left[c_{i-j+2} - 2\cos(\theta_2-\theta_1) c_{i-j+1}\right. \\ & \left. + c_{i-j}\right] \binom{i}{j} \binom{2n-i-2}{n-j} \phi_i^{2n-2}(\theta) \\ &- \frac{n^2}{\binom{2n}{n}} \sum_{i=0}^{2n} \sum_{j=0}^{n} c_j c_{i-j} \binom{i}{j} \binom{2n-i}{n-j} \phi_i^{2n}(\theta). \end{aligned}$$

Here, $c_{i-j} = 0$ if $i < j$ or $i > j+n$.

For the sake of convenience, we will use the relation $[(p(\theta))^2]'' = 2(p'(\theta))^2 + 2p(\theta)p''(\theta)$ to derive the expression for $(p'(\theta))^2$. Similar to the process of deriving equation (3.30), we can obtain

$$\begin{aligned}
[(p(\theta))^2]'' &= \frac{2n(2n-1)}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{2n-2}\left[d_{i+2}-2\cos(\theta_2-\theta_1)d_{i+1}+d_i\right]\phi_i^{2n-2}(\theta)\\
&- \frac{(2n)^2}{\binom{2n}{n}}\sum_{i=0}^{2n} d_i\phi_i^n(\theta)\\
&= \frac{2n(2n-1)}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{2n-2}\sum_{j=0}^{n} c_j\left[c_{i-j+2}\binom{i+2}{j}\binom{2n-i-2}{n-j}\right.\\
&- 2\cos(\theta_2-\theta_1)c_{i-j+1}\binom{i+1}{j}\binom{2n-i-1}{n-j}\\
&\left.+c_{i-j}\binom{i}{j}\binom{2n-i}{n-j}\right]\phi_i^{2n-2}(\theta)\\
&- \frac{4n^2}{\binom{2n}{n}}\sum_{i=0}^{2n}\sum_{j=0}^{n} c_jc_{i-j}\binom{i}{j}\binom{2n-i}{n-j}\phi_i^{2n}(\theta).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(p(\theta))^2+2(p'(\theta))^2-p(\theta)p''(\theta)\\
&= (p(\theta))^2+[(p(\theta))^2]''-3p(\theta)p''(\theta)
\end{aligned}$$

$$= \frac{2n(2n-1)}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{2n-2}\sum_{j=0}^{n} c_j\left[c_{i-j+2}\left(\binom{i+2}{j}\binom{2n-i-2}{n-j}-3\binom{i}{j}\binom{2n-i-2}{n-j}\right)\right. \tag{3.33}$$

$$- 2\cos(\theta_2-\theta_1)c_{i-j+1}\left(\binom{i+1}{j}\binom{2n-i-1}{n-j}-3\binom{i}{j}\binom{2n-i-2}{n-j}\right) \tag{3.34}$$

$$\left.+c_{i-j}\left(\binom{i}{j}\binom{2n-i}{n-j}-3\binom{i}{j}\binom{2n-i-2}{n-j}\right)\right]\phi_i^{2n-2}(\theta) \tag{3.35}$$

$$+ \frac{1-n^2}{\binom{2n}{n}}\sum_{i=0}^{2n}\sum_{j=0}^{n} c_jc_{i-j}\binom{i}{j}\binom{2n-i}{n-j}\phi_i^{2n}(\theta) \tag{3.36}$$

We will use the following degree-raising formula which was given in [2], to raise the degree of the first summation in the above expression of $(p(\theta))^2+2(p'(\theta))^2-p(\theta)p''(\theta)$ to 2n.

$$p(\theta)=\sum_{i=0}^{d} e_i\phi_i^d(\theta)=\sum_{i=0}^{d+2}\bar{e}_i\phi_i^{d+2}(\theta),$$

where

$$\begin{aligned}\bar{e}_i(\theta) &= \frac{1}{(d+2)(d+1)}\left[i(i-1)e_{i-2} + 2\cos(\theta_2-\theta_1)i(d-i+2)e_{i-1}\right.\\ &\quad \left. +(d-i+2)(d-i+1)e_i\right],\end{aligned}$$

for $i = 0, 1, \cdots, d+2$.

Thus, from equation (3.33) and the degree-raising formula, we obtain

$$\begin{aligned}
&(p(\theta))^2 + 2(p'(\theta))^2 - p(\theta)p''(\theta)\\
= &\ \frac{1}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{2n}\sum_{j=0}^{n} c_j\left[i(i-1)\left[c_{i-j}\binom{2n-i}{n-j}\left(\binom{i}{j} - 3\binom{i-2}{j}\right)\right.\right.\\
- &\ 2\cos(\theta_2-\theta_1)c_{i-j-1}\left(\binom{i-1}{j}\binom{2n-i+1}{n-j} - 3\binom{i-2}{j}\binom{2n-i}{n-j}\right)\\
&\ \left. +c_{i-j-2}\binom{i-2}{j}\left(\binom{2n-i+2}{n-j} - 3\binom{2n-i}{n-j}\right)\right]\\
+ &\ 2\cos(\theta_2-\theta_1)i(2n-i)\left[c_{i-j+1}\binom{2n-i-1}{n-j}\left(\binom{i+1}{j} - 3\binom{i-1}{j}\right)\right.\\
- &\ 2\cos(\theta_2-\theta_1)c_{i-j}\left(\binom{i}{j}\binom{2n-i}{n-j} - 3\binom{i-1}{j}\binom{2n-i-1}{n-j}\right)\\
&\ \left. +c_{i-j-1}\binom{i-1}{j}\left(\binom{2n-i+1}{n-j} - 3\binom{2n-i-1}{n-j}\right)\right]\\
+ &\ (2n-i)(2n-i-1)\left[c_{i-j+2}\binom{2n-i-2}{n-j}\left(\binom{i+2}{j} - 3\binom{i}{j}\right)\right.\\
- &\ 2\cos(\theta_2-\theta_1)c_{i-j+1}\left(\binom{i+1}{j}\binom{2n-i-1}{n-j} - 3\binom{i}{j}\binom{2n-i-2}{n-j}\right)\\
&\ \left.\left. +c_{i-j}\binom{i}{j}\left(\binom{2n-i}{n-j} - 3\binom{2n-i-2}{n-j}\right)\right]\phi_i^{2n}(\theta)\\
+ &\ \frac{1-n^2}{\binom{2n}{n}}\sum_{i=0}^{2n}\sum_{j=0}^{n} c_j c_{i-j}\binom{i}{j}\binom{2n-i}{n-j}\phi_i^{2n}(\theta)
\end{aligned}$$

$$= \frac{1}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{2n} a_i\phi_i^{2n}(\theta), \qquad (3.37)$$

where

$$
\begin{aligned}
a_i &= \sum_{j=0}^{n} \binom{i}{j}\binom{2n-i}{n-j} c_j \Big[b^{(2)}(i,j)c_{i-j+2} + b^{(1)}(i,j)c_{i-j+1} + b^{(0)}(i,j)c_{i-j} && (3.38)\\
&\quad + b^{(-1)}(i,j)c_{i-j-1} + b^{(-2)}(i,j)c_{i-j-2} \Big], && (3.39)
\end{aligned}
$$

$c_k = 0$ for $k = -1, -2, \cdots, -n-2$ or $k = n+1, n+2, \cdots, 2n+2$, and $b^{(\ell)}_{i-j+\ell}$ can be found from equation (32) by using the combination formulas:

$$
\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k} \quad or \quad \binom{n-1}{k} = \frac{n-k}{n}\binom{n}{k}.
$$

In fact, for $i < j$ or $j < i-n$, we have $b^{(\ell)}(i,j) = 0$; for $i - n \le j \le i$, we obtain

$$
\begin{aligned}
b^{(2)}(i,j) &= (2n-i)(2n-i-1)\binom{2n-i-2}{n-j}\left(\binom{i+2}{j} - 3\binom{i}{j}\right) \Big/ \binom{i}{j}\binom{2n-i}{n-j}\\
&= (n-i+j)(n-i+j-1)\left(\frac{(i+2)(i+1)}{(i-j+2)(i-j+1)} - 3\right), && (3.40)\\
b^{(1)}(i,j) &= 2\cos(\theta_2-\theta_1)\left[i(2n-i)\binom{2n-i-1}{n-j}\left(\binom{i+1}{j} - 3\binom{i-1}{j}\right)\right.\\
&- (2n-i)(2n-i-1)\left(\binom{i+1}{j}\binom{2n-i-1}{n-j}\right.\\
&\quad \binom{-3\binom{i}{j}\binom{2n-i-2}{n-j}\Big)\Big]\{i}{j\}\binom{2n-i}{n-j}}\\
&= 2(n-i+j)\cos(\theta_2-\theta_1)\left[\frac{(2i-2n+1)(i+1)}{i-j+1}\right. && (3.41)\\
&\quad \left. +3(n-2i+2j-1)\right],
\end{aligned}
$$

$$\begin{aligned}
b^{(0)}(i,j) &= \left[i(i-1)\binom{2n-i}{n-j}\left(\binom{i}{j} - 3\binom{i-2}{j}\right)\right. \\
&- 4i(2n-i)\cos^2(\theta_2-\theta_1)\left(\binom{i}{j}\binom{2n-i}{n-j} - 3\binom{i-1}{j}\binom{2n-i-1}{n-j}\right) \\
&+ (2n-i)(2n-i-1)\binom{i}{j}\left(\binom{2n-i}{n-j} - 3\binom{2n-i-2}{n-j}\right) \\
&\quad \left. +(1-n^2)\sin^2(\theta_2-\theta_1)\binom{i}{j}\binom{2n-i}{n-j}\right] / \binom{i}{j}\binom{2n-i}{n-j} \\
&= i(i-1) - 3(i-j)(i-j-1) - 4i(2n-i)\cos^2(\theta_2-\theta_1) \\
&+ 12(i-j)(n-i+j)\cos^2(\theta_2-\theta_1) + (2n-i)(2n-i-1) \\
&- 3(n-i+j)(n-i+j-1) + (1-n^2)\sin^2(\theta_2-\theta_1) \\
&= i(i-1) + (2n-i)(2n-i-1) + 1 - n^2 \qquad (3.42) \\
&- 3\left[(i-j)(i-j-1) + (n-i+j)(n-i+j-1)\right] \qquad (3.43) \\
&+ \cos^2(\theta_2-\theta_1)\left[n^2 - 1 + 4i(2n-i) - 12(i-j)(n-i+j)\right], \qquad (3.44)
\end{aligned}$$

$$\begin{aligned}
b^{(-1)}(i,j) &= 2\cos(\theta_2-\theta_1)\left[-i(i-1)\left(\binom{i-1}{j}\binom{2n-i+1}{n-j} - 3\binom{i-2}{j}\binom{2n-i}{n-j}\right)\right. \\
&\quad \left. +i(2n-i)\binom{i-1}{j}\left(\binom{2n-i+1}{n-j} - 3\binom{2n-i-1}{n-j}\right)\right] / \binom{i}{j}\binom{2n-i}{n-j}
\end{aligned}$$

$$\begin{aligned}
&= 2(i-j)\cos(\theta_2-\theta_1)\left[\frac{(2n-2i+1)(2n-i+1)}{n-i+j+1}\right. \qquad (3.45) \\
&\quad \left. -3(n-2i+2j+1)\right], \qquad (3.46) \\
b^{(-2)}(i,j) &= i(i-1)\binom{i-2}{j}\left(\binom{2n-i+2}{n-j} - 3\binom{2n-i}{n-j}\right)\binom{i}{j}\binom{2n-i}{n-j} \qquad (3.47) \\
&= (i-j)(i-j-1)\left[\frac{(2n-i+2)(2n-i+1)}{(n-i+j+2)(n-i+j+1)} - 3\right]. \qquad (3.48)
\end{aligned}$$

From [7], we have the following positivity criterion for CBB polynomial

$p(\theta) = \sum_{i=0}^{n} a_i \phi_i^n(\theta)$.

If

$$a_0 + (n-1)!\left(\frac{2}{n}\right)^{n-1}\sum_{\substack{i=1\\a_i<0}}^{n-1}\frac{i^n}{i!(n-i)!}a_i \geq 0$$

$$a_n + (n-1)!\left(\frac{2}{n}\right)^{n-1}\sum_{\substack{i=1\\a_i<0}}^{n-1}\frac{(n-i)^n}{i!(n-i)!}a_i \geq 0,$$

then $p(\theta) \geq 0$. Therefore, from Lemma 4 and equation (3.37) we obtain the following convexity criterion for $p(\theta)$.

Theorem 3. Let the $P(\theta)$ defined in (1.10) be a CBB curve and $p(\theta)$ be the associated CBB polynomial defined in equation (1.9). If

$$a_0 + (2n-1)!\left(\frac{2}{2n}\right)^{2n-1}\sum_{\substack{i=1\\a_i<0}}^{2n-1}\frac{i^{2n}}{i!(2n-i)!}a_i \geq 0$$

$$a_{2n} + (2n-1)!\left(\frac{2}{2n}\right)^{2n-1}\sum_{\substack{i=1\\a_i<0}}^{2n-1}\frac{(2n-i)^{2n}}{i!(2n-i)!}a_i \geq 0,$$

where a_i is defined by (3.38)-(3.47), then the CBB curve $P(\theta)$ is convex.

If we use the positivity criterion given in [11] (if

$$a_0 + (n-1)!\sum_{\substack{i=1\\a_i<0}}^{n-1}\frac{i}{i!(n-i)!}a_i \geq 0$$

and

$$a_n + (n-1)!\sum_{\substack{i=1\\a_i<0}}^{n-1}\frac{n-i}{i!(n-i)!}a_i \geq 0,$$

then $p(\theta) \geq 0$), we have another convexity criterion for $p(\theta)$, which is as follows.

Theorem 4. Let the $P(\theta)$ defined in (1.10) be a CBB curve and $p(\theta)$ be the associated CBB polynomial defined in equation (1.9). If

$$a_0 + (2n-1)!\sum_{\substack{i=1\\a_i<0}}^{2n-1}\frac{i}{i!(2n-i)!}a_i \geq 0$$

and

$$a_{2n} + (2n-1)! \sum_{\substack{i=1 \\ a_i<0}}^{2n-1} \frac{2n-i}{i!(2n-i)!} a_i \geq 0,$$

where a_i is defined by (3.38)-(3.47), then the CBB curve $P(\theta)$ is convex.

4 Convexity criteria of HBB polynomials

In Section 1, we have shown that if $\hat{T}$ is a trihedron generated by $\{v_1, v_2, v_3\}$ and if $b_1(v)$, $b_2(v)$, $b_3(v)$ denote the trihedron coordinates, i.e.,

$$\hat{T} = \{v \in R^3 : v = b_1v_1 + b_2v_2 + b_3v_3,\ b_i \geq 0\},$$

then the HBB polynomials of degree n can be written as

$$p_n(v) = \sum_{|i|=n} a_i \phi_i^n(b). \tag{4.49}$$

Here ϕ_i^n was defined in equation (1.3) in Section 1. In this section, we will discuss the convexity criteria of HBB polynomial (ref39). Since a positively homogeneous convex function is called a gauge function, these convexity criteria can be also considered as conditions for making a HBB polynomial a gauge function (cf. [4]).

In the following, we will use the notation $D_\gamma = \gamma_1\frac{\partial}{\partial x} + \gamma_2\frac{\partial}{\partial y} + \gamma_3\frac{\partial}{\partial z}$, where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. For $\gamma = v_\ell$, $\ell = 1, 2, 3$, we denote $D_\ell = D_\gamma = D_{v_\ell}$. If we define that $E_\ell a_i = a_{i+\mathbf{e}^\ell}$, where $\mathbf{e}^\ell$ denotes the ℓ^{th} coordinate vector in $\mathbf{R}^3$, we have

$$D_\ell p_n = n \sum_{|i|=n-1} E_\ell a_i \phi_i^{n-1}(b). \tag{4.50}$$

For any direction V, there exists a vector $\mathbf{c}_V = (c_1, c_2, c_3)$ such that

$$V = \sum_{\ell=1}^{s} c_\ell v_\ell. \tag{4.51}$$

Thus, from (4.51), we have

$$D_V^2 p_n = n(n-1) \sum_{|i|=n-2} \mathbf{c}_V^T Q_{i,a} \mathbf{c}_V \phi_i^{n-2}(b), \tag{4.52}$$

where $c_V = (c_1, c_2, c_3)^T$ and

$$Q_{i,a} := (E_u E_w a_i)_{u,w=1}^{3,3} \tag{4.53}$$

for $|i| = n-2$.

Obviously, $p_n(v)$ is convex on $\hat{T}$ if and only if $D_V^2 p_n(v) \geq 0$ for any directional vector V and at any point $v \in \hat{T}$. Denoting $q_{i,a}(\mathbf{c}_V) = \mathbf{c}_V^T Q_{i,a} \mathbf{c}_V$, we have

$$D_V^2 p_n(v) = n(n-1) \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) \phi_i^{n-2}(b). \tag{4.54}$$

We define a function $\mathbf{c}_V$ associated with $q_{i,f}(\mathbf{c})$ as follows:

$$w_{i,a}(\mathbf{c}_V) = \begin{cases} 0 & \text{if } q_{i,a}\ (\mathbf{c}_V) \geq 0, \\ 1 & \text{if } q_{i,a}\ (\mathbf{c}_V) < 0 \end{cases}$$

where $|i| = n-2$ and $i_\ell \neq n-2$, $\ell = 1,2,3$. If $|i| = n-2$ and $i_\ell = n-2$ for $\ell = 1,2,3$, then $w_{i,a}(\mathbf{c}_V) = 1$.

The following two inequalities about $\phi_i^n(b)$, (46) and (47), were obtained in [7] and [11], respectively, by using inequalities from [6, p. 17].

$$0 \leq \phi_i^n(b) \leq \frac{(n-1)!}{i! n^{n-1}} (\sum_{\ell=1}^{3} i_\ell b_\ell)^n, \tag{4.55}$$

$$0 \leq \phi_i^n(b) \leq \frac{(n-1)!}{i!} \sum_{\ell=1}^{3} i_\ell b_\ell^n. \tag{4.56}$$

We now give some convexity criteria for the homogeneous Bernstein-Bézier polynomials over triangle $\hat{T}$.

Theorem 5. Let $r_i \in \{0,1\}$ for $|i| = n-2$ and $i \neq (n-2)\mathbf{e}^\ell$, $\ell = 1,2,3$, and $r_i = 1$ for $i = (n-2)\mathbf{e}^\ell$, $\ell = 1,2,3$. The Bernstein-Bézier polynomial $p_n(v)$ shown in (39) is convex on $\hat{T}$

if for all $u \in \{1,2,3\}$ its Bézier coefficients satisfy either

$$\sum_{|i|=n-2}\left(\sum_{\ell=0}^{s} i_\ell^2\right)^{\frac{n-2}{2}} \frac{r_i}{i!} E_u E_u a_i \geq \sum_{\substack{w=1,2,3\\ w\neq u}} \left| \sum_{|i|=n-2}\left(\sum_{\ell=0}^{s} i_\ell^2\right)^{\frac{n-2}{2}} \frac{r_i}{i!} E_u E_w a_i \right| \text{ and } (48) \tag{4.57}$$

$$\sum_{|i|=n-2}\left(\sum_{\ell=0}^{s} i_\ell^2\right)^{\frac{n-2}{2}} \frac{r_i}{i!} \left(E_u E_u a_i - \sum_{\substack{w=1,2,3\\ w\neq u}} |E_u E_w a_i| \right) \geq 0. \tag{4.58}$$

Proof. It is sufficient to prove inequality (4.57), since inequality (4.58) is implied by inequality (4.57). Noting inequality (4.55), we have

$$\begin{aligned}
&\frac{1}{n(n-1)} D_V^2 p_n(v) \\
= & \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V)\phi_i^{n-2}(b) \\
&\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V)\phi_i^{n-2}(b) \\
&\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V)\frac{(n-3)!}{i!(n-2)^{n-3}}\left(\sum_{\ell=1}^{3} i_\ell b_\ell\right)^{n-2} \\
&\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V)\frac{(n-3)!}{i!(n-2)^{n-3}}\left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}}\left(\sum_{\ell=1}^{3} b_\ell^2\right)^{\frac{n-2}{2}} \\
= & \frac{(n-3)!}{(n-2)^{n-3}}\left(\sum_{\ell=1}^{3} b_\ell^2\right)^{\frac{n-2}{2}} \mathbf{c}_V^T \left[\sum_{|i|=n-2}\left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V)}{i!} Q_{i,a}\right]\mathbf{c}_V \\
= & \frac{(n-3)!}{(n-2)^{n-3}}\left(\sum_{\ell=1}^{3} b_\ell^2\right)^{\frac{n-2}{2}} \mathbf{c}_V^T \left[\sum_{|i|=n-2}\left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V)}{i!} E_u E_w a_i\right]_{u,w=1}^{3,3}\mathbf{c}_V.
\end{aligned}$$

Obviously, if the last symmetric matrix is strongly diagonally dominant, i.e., for all $u = 1,2,3$,

$$\sum_{|i|=n-2}\left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V)}{i!} E_u E_u a_i \geq \sum_{\substack{w=1,2,3\\ w\neq u}} \left| \sum_{|i|=n-2}\left(\sum_{\ell=0}^{s} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V}{i!} E_u E_w a_i \right|,$$

then the matrix

$$\left[\sum_{|i|=n-2}\left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V)}{i!} E_u E_w a_i\right]_{u,w=1}^{3,3}$$

is semi-positive definite. Thus $D_V^2 p_n(v) \geq 0$ and $p_n(v)$ is convex. Obviously, the above condition is implied by inequality (4.57). Thus, Theorem 5 is proved.

Similarly, we may use inequality (3.32) to obtain the following result.

Theorem 6. Let $r_i \in \{0,1\}$ for $|i| = n-2$ and $i \neq (n-2)\mathbf{e}^\ell$, $\ell = 1,2,3$, and $r_i = 1$ for $i = (n-2)\mathbf{e}^\ell$, $\ell = 1,2,3$. The Bernstein-Bézier polynomial $p_n(v)$ shown in (3.41) is convex on $\hat{T}$ if for $\ell = 1,2,3$ and $u = 1,2,3$, its Bézier coefficients satisfy either

$$\sum_{|i|=n-2} \frac{i_\ell}{i!} r_i \quad E_u E_u a_i \geq \sum_{\substack{w=1,2,3\\ w\neq u}} \left| \sum_{|i|=n-2} \frac{i_\ell}{i!} r_i E_u E_w a_i \right| \tag{4.59}$$

or

$$\sum_{|i|=n-2} \frac{i_\ell}{i!} r_i \left(E_u E_u a_i - \sum_{\substack{w=1,2,3\\ w\neq u}} |E_u E_w a_i| \right) \geq 0. \tag{4.60}$$

Proof. It is sufficient to prove inequality (4.59), since inequality (4.60) is implied by inequality (4.59). Noting inequality (4.56), we have

$$\begin{aligned}
&\frac{1}{n(n-1)} D_V^2 p_n(v) \\
= &\sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) \phi_i^{n-2}(b) \\
&\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V) \phi_i^{n-2}(b)
\end{aligned}$$

$$\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V) \frac{(n-3)!}{i!} \sum_{\ell=1}^{3} i_\ell b_\ell^{n-2}$$

$$= (n-3)! \sum_{\ell=1}^{3} b_\ell^{n-2} \mathbf{c}_V^T \left[\sum_{|i|=n-2} \frac{i_\ell}{i!} w_{i,a}(\mathbf{c}_V) Q_{i,a} \right] \mathbf{c}_V$$

$$= (n-3)! \sum_{\ell=1}^{3} b_\ell^{n-2} \mathbf{c}_V^T \left[\sum_{|i|=n-2} \frac{i_\ell}{i!} w_{i,a}(\mathbf{c}_V) E_u E_w a_i \right]_{u,w=1}^{3,3} \mathbf{c}_V$$

Obviously, if the last symmetric matrix is strongly diagonally dominant, i.e., for all $u = 1, 2, 3$,

$$\sum_{|i|=n-2} \frac{i_\ell}{i!} \quad w_{i,a}(\mathbf{c}_V) E_u E_u a_i \geq \sum_{\substack{w=1,2,3 \\ w \neq u}} \left| \sum_{|i|=n-2} \frac{i!}{i!} w_{i,a}(\mathbf{c}_V) E_u E_w a_i \right|,$$

then the matrix

$$\left[\sum_{|i|=n-2} \frac{i!}{i!} w_{i,a}(\mathbf{c}_V) E_u E_w a_i \right]_{u,w=1}^{3,3}$$

is semi-positive definite. Thus $D_V^2 p_n(v) \geq 0$ and $p_n(v)$ is convex. Obviously, the above condition is implied by inequality (4.59). Thus, Theorem 6 is proved.

Remark 4. A stronger convexity condition is implied by inequalities (4.58) and (4.60) as follows: $E_u E_u a_i \geq \sum_{\substack{w=1,2,3 \\ w \neq u}} \mid E_u E_w a_i \mid$. Here, $u = 1, 2, 3$.

Remark 5. The conditions given in Theorem 5 and Theorem 6 are independent (see [7] and [8]).

Remark 6. There is another approach for finding convexity criteria from the positivity criteria and is shown in [8] for plane BB polynomials. This approach can also be applied here for HBB polynomials.

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Role of Higher Order Invexity in Parametric Optimality Conditions for Discrete Minmax Fractional Programming

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Abstract

A new class of second order (ϕ, η, ρ, θ, m)-invexities is introduced, and then a set of higher-order parametric necessary optimality conditions and several sets of higher order sufficient optimality conditions for a discrete minmax fractional programming problem applying various second-order (ϕ, η, ρ, θ, m)-invexity constraints are established. This class of the generalized invexities not only generalizes/unifies existing concepts for generalized invexities in the literature, but also is more application-oriented to further research developments.

Keywords: Discrete minmax fractional programming, (ϕ, η, ρ, θ, m)-invex functions, necessary optimality conditions, sufficient optimality conditions.

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1 Introduction

This communication primarily deals with a new class of generalized second-order $(\phi, \eta, \rho, \theta, m)$-invex functions, which is applied to establish a set of second-order necessary optimality conditions leading to several sets of second-order sufficient optimality conditions and theorems for the following discrete minmax fractional programming problem:

$$(P) \qquad \text{Minimize} \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)}$$

subject to $G_j(x) \leq 0,\ j \in \underline{q},\quad H_k(x) = 0,\ k \in \underline{r},\quad x \in X$, where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), $f_i,\ g_i,\ i \in \underline{p} = \{1, 2, \ldots, p\}$,
$G_j,\ j \in \underline{q}$, and $H_k,\ k \in \underline{r}$, are real-valued functions defined on X, and for each $i \in \underline{p},\ g_i(x) > 0$ for all x satisfying the constraints of (P).

The first part of this presentation deals with a new notion of the generalized $(\phi, \eta, \rho, \theta, m)$-invexities, which generalizes/unifies most of the existing generalized invexities and variants in the literature, while the second part deals with investigating the second-order optimality and duality of our principal problem (P) as well as its semi infinite counterpart in a series of papers. We begin our investigation here by establishing a set of second-order parametric necessary optimality conditions and several sets of sufficient optimality conditions for (P). The results thus obtained here in this communication seem to be new to context of results available in the literature.

2 Preliminaries

Verma and Zalmai [27] introduced the notion of the generalized $(\phi, \eta, \rho, \theta, m)$-invexities, and applied to establish a class of second order parametric necessary optimality conditions as well as sufficient optimality conditions for a discrete minmax fractional programming problem using the general frameworks for the $(\phi, \eta, \rho, \theta, m)$-invexities. In this section, we first generalize the notion of the generalized $(\phi, \eta, \rho, \theta, m)$-invexities, and then recall some important auxiliary results for the problem (P).

Definition 2.1 *Let f be a differentiable real-valued function defined on $\mathbb{R}^n$. Then f is said to be η-invex (invex with respect to η) at y if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,*

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y) \rangle,$$

where $\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \ldots, \partial f(y)/\partial y_n)$ is the gradient of f at y, and $\langle a, b \rangle$ denotes the inner product of the vectors a and b; f is said to be η-invex on $\mathbb{R}^n$ if the above inequality holds for all $x, y \in \mathbb{R}^n$,

From this definition it is clear that every differentiable real-valued convex function is invex with $\eta(x, y) = x - y$. This generalization of the concept of convexity was originally proposed by Hanson [6] who showed that for a nonlinear programming problem of the form

Minimize $f(x)$ subject to $g_i(x) \leq 0, \ i \in \underline{m}, \ \ x \in \mathbb{R}^n$,

where the differentiable functions $f, \ g_i : \mathbb{R}^n \to \mathbb{R}, i \in \underline{m}$, are invex with respect to the same function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient.

Let f be a twice differentiable real-valued function defined on $\mathbb{R}^n$. Now we introduce the new classes of generalized second-order hybrid invex functions which seem to be application-oriented to developing a new optimality-duality theory for nonlinear programming based on second-order necessary and sufficient optimality conditions. We shall abbreviate "second-order invex" as **sonvex**. Let $f : X \to \mathbb{R}$ be a twice differentiable function.

Definition 2.2 *The function f is said to be* (strictly) $(\phi, \eta, \rho, \theta, m)$-sonvex *at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}, \ \rho : X \times X \to \mathbb{R}$, and $\eta, \theta : X \times X \to \mathbb{R}^n$, and a positive integer m such* *that for each* $x \in X$ $(x \neq x^*)$ *and* $z \in \mathbb{R}^n$,

$$\phi\big(f(x) - f(x^*) + \frac{1}{2}\langle \nabla f(x^*), z\rangle\big)(>) \geq \langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^m.$$

Definition 2.3 *The function f is said to be* (strictly) $(\phi, \eta, \rho, \theta, m)$-pseudosonvex *at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, and $\eta, \theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,*

$$\begin{aligned} &\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle \geq -\rho(x, x^*)\|\theta(x, x^*)\|^m \\ \Rightarrow\quad &\phi\big(f(x) - f(x^*) + \frac{1}{2}\langle \nabla f(x^*), z\rangle\big)(>) \geq 0. \end{aligned}$$

Definition 2.4 *The function f is said to be* (prestrictly) $(\phi, \eta, \rho, \theta, m)$-quasisonvex *at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}, \rho : X \times X \to \mathbb{R}$, and $\eta, \theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ and $z \in \mathbb{R}^n$,*

$$\begin{aligned} &\phi\big(f(x) - f(x^*) + \frac{1}{2}\langle \nabla f(x^*), z\rangle\big)(<) \leq 0 \Rightarrow \\ &\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle \leq -\rho(x, x^*)\|\theta(x, x^*)\|^m, \end{aligned}$$

equivalently

$$\begin{aligned} &\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle > -\rho(x, x^*)\|\theta(x, x^*)\|^m \\ \Rightarrow\quad &\phi\big(f(x) - f(x^*) + \frac{1}{2}\langle \nabla f(x^*), z\rangle\big)(\geq) > 0. \end{aligned}$$

We need to recall the following auxiliary results which are needed for establishing our main results based on generalized invexity to the context of minmax fractional programming.

Lemma 2.1 [27] *Let λ^* be the optimal value of (P), and let $v(\lambda)$ be the optimal value of $(P\lambda)$ for any fixed $\lambda \in \mathbb{R}$ such that $(P\lambda)$ has an optimal solution. Then the following statements are valid:*

(a) If x^* is an optimal solution of (P), then it is an optimal solution of $(P\lambda^*)$ and $v(\lambda^*) = 0$.

(b) If $(P\bar{\lambda})$ has an optimal solution $\bar{x}$ for some $\bar{\lambda} \in \mathbb{R}$ with $v(\bar{\lambda}) = 0$, then $\bar{x}$ is an optimal solution of (P) and $\bar{\lambda} = \lambda^*$.

It is clear that $(P\lambda)$ is equivalent to the following problem:

$(EP\lambda)$ Minimize μ

subject to $x \in \mathbb{F}$ and $f_i(x) - \lambda g_i(x) - \mu \leq 0, \ i \in \underline{p}$.

Theorem 2.1 [27] *Let x^* be an optimal solution of (P), let $\lambda^* = \max_{1\leq i \leq p} f_i(x^*)/g_i(x^*)$, and assume that the functions $f_i, \ g_i, \ i \in \underline{p}, \ G_j, \ j \in \underline{q}$, and $H_k, \ k \in \underline{r}$, are twice continuously differentiable at x^*, and that the second-order Guignard constraint qualification holds at x^*. Then for each critical direction z^*, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that*

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \quad (2.1)$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*) \Big\} z^* \Big\rangle \geq 0, \quad (2.2)$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \ \ i \in \underline{p}, \quad (2.3)$$

$$v_j^* G_j(x^*) = 0, \ \ j \in \underline{q}. \quad (2.4)$$

3 Sufficient Optimality Conditions

In this section, we present several second-order sufficiency results in which various generalized hybrid $(\phi, z, \rho, \theta, m)$-sonvexity assumptions are imposed on the individual as well as certain combinations of the problem functions.

For the sake of the compactness, we shall use the following notations during the statements as well as the proofs of sufficiency theorems:

$$\begin{aligned}
\mathcal{C}(x,v) &= \sum_{j=1}^{q} v_j G_j(x),\\
\mathcal{D}_k(x,w) &= w_k H_k(x),\\
\mathcal{D}(x,w) &= \sum_{k=1}^{r} w_k H_k(x),\\
\mathcal{E}_i(x,\lambda) &= f_i(x) - \lambda g_i(x),\\
\mathcal{E}(x,u,\lambda) &= \sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)],\\
\mathcal{G}(x,v,w) &= \sum_{j=1}^{q} v_j G_j(x) + \sum_{k=1}^{r} w_k H_k(x),
\end{aligned}$$

$$I_+(u) = \{i \in \underline{p} : u_i > 0\}, \quad J_+(v) = \{j \in \underline{q} : v_j > 0\}, \quad K_*(w) = \{k \in \underline{r} : w_k \neq 0\}.$$

During the course of proofs for our sufficiency theorems, we shall use the following auxiliary result which provides an alternative expression for the objective function of (P).

Lemma 3.1 [27] *For each* $x \in X$,

$$\varphi(x) = \max_{1 \le i \le p} \frac{f_i(x)}{g_i(x)} = \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

Theorem 3.1 *Let* $x^* \in \mathbb{F}$, $\lambda^* = \varphi(x^*) \ge 0$, *the functions* f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, *and* H_k, $k \in \underline{r}$, *be twice differentiable at* x^*. *Assume that for each critical direction* z^*, *there exist* $u^* \in U$, $v^* \in \mathbb{R}^q_+$, *and* $w^* \in \mathbb{R}^r$ *such that*

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \tag{3.5}$$

$$\begin{aligned}
&\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)]\\
&+ \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*) \Big\} z^* \Big\rangle \ge 0,
\end{aligned} \tag{3.6}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] - \frac{1}{2}\langle z, \nabla f_i(x^*) - \lambda \nabla g_i(x^*)\rangle \geq 0, \ \ i \in \underline{p}, \tag{3.7}$$

$$v_j^* G_j(x^*) - \frac{1}{2}\langle z^*, v_j^* \nabla G_j(x^*)\rangle \geq 0, \ \ j \in \underline{q}, \tag{3.8}$$

$$w_k^* H_k(x^*) - \frac{1}{2}\langle z^*, w_k^* \nabla H_k(x^*)\rangle \geq 0, \ \ k \in \underline{r}. \tag{3.9}$$

In addition, assume that any one of the following six sets of conditions holds:

(a) *(i) for each* $i \in I_+ \equiv I_+(u^*)$, f_i *is* $(\phi, \eta, \bar{\rho}_i, \theta, m)$*-sonvex and* $-g_i$ *is* $(\phi, \eta, \tilde{\rho}_i, \theta, m)$*-sonvex at* x^*, ϕ *is superlinear, and* $\phi(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) for each $j \in J_+ \equiv J_+(v^*)$, G_j *is* $(\hat{\phi}_j, \eta, \hat{\rho}_j, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$*;*

(iii) for each $k \in K_* \equiv K_*(w^*)$, $\xi \to \mathcal{D}_k(\xi, w^*)$ *is* $(\breve{\phi}_k, \eta, \breve{\rho}_k, \theta, m)$*-quasisonvex at* x^* *and* $\breve{\phi}_k(0) = 0$*;*

(iv) $\rho^*(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*, where* $\rho^*(x, x^*) = \sum_{i \in I_+} u_i^*[\bar{\rho}_i(x, x^*) + \lambda^* \tilde{\rho}_i(x, x^*)]$*;*

(b) *(i) for each* $i \in I_+$, f_i *is* $(\phi, \eta, \bar{\rho}_i, \theta, m)$*-sonvex and* $-g_i$ *is* $(\phi, \eta, \tilde{\rho}_i, \theta, m)$*-sonvex at* x^*, ϕ *is superlinear, and* $\phi(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) $\mathcal{C}(\cdot, v^*)$ *is* $(\hat{\phi}, \eta, \hat{\rho}, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) for each $k \in K_*$, $\xi \to \mathcal{D}_k(\xi, w^*)$ *is* $(\breve{\phi}_k, \eta, \breve{\rho}_k, \theta, m)$*-quasisonvex at* x^* *and* $\breve{\phi}_k(0) = 0$*;*

(iv) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(c) *(i) for each* $i \in I_+$, f_i *is* $(\phi, \eta, \bar{\rho}_i, \theta, m)$*-sonvex and* $-g_i$ *is* $(\phi, \eta, \tilde{\rho}_i, \theta, m)$*-sonvex at* x^*, ϕ *is superlinear, and* $\phi(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) for each $j \in J_+$, G_j *is* $(\hat{\phi}_j, \eta, \hat{\rho}_j, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$*;*

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ *is* $(\breve{\phi}, \eta, \breve{\rho}, \theta, m)$*-quasisonvex at* x^* *and* $\breve{\phi}(0) = 0$*;*

(iv) $\rho^*(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(d) *(i) for each* $i \in I_+$*,* f_i *is* $(\phi, \eta, \bar{\rho}_i, \theta, m)$*-sonvex and* $-g_i$ *is hybrid* $(\phi, \eta, \tilde{\rho}_i, \theta, m)$*-sonvex at* x^**,* ϕ *is superlinear, and* $\phi(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ *is* $(\hat{\phi}, \eta, \hat{\rho}, \theta, m)$*-quasisonvex at* x^**,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ *is* $(\breve{\phi}, \eta, \breve{\rho}, \theta, m)$*-quasisonvex at* x^* *and* $\breve{\phi}(0) = 0$*;*

(iv) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(e) *(i) for each* $i \in I_+$*,* f_i *is* $(\phi, \eta, \bar{\rho}_i, \theta, m)$*-sonvex and* $-g_i$ *is* $(\phi, \eta, \tilde{\rho}_i, \theta, m)$*-sonvex at* x^**,* ϕ *is superlinear, and* $\phi(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ *is* $(\hat{\phi}, \eta, \hat{\rho}, \theta, m)$*-quasisonvex at* x^**,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(f) the Lagrangian-type function

$$\xi \to L(\xi, u^*, v^*, w^*, \lambda^*) = \sum_{i=1}^{p} u_i^*[f_i(\xi) - \lambda^* g_i(\xi)] + \sum_{j=1}^{q} v_j^* G_j(\xi) + \sum_{k=1}^{r} w_k^* H_k(\xi)$$

is $(\phi, \eta, \rho, \theta, m)$*-pseudosonvex at* x^**,* $\rho(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*,* $\phi(a) \geq 0 \Rightarrow a \geq 0$*, and*

$$\Big(L(x^*, u^*, v^*, w^*, \lambda^*) - \frac{1}{2}\langle \nabla L(x^*, u^*, v^*, w^*, \lambda^*), z^* \rangle\Big) \geq 0.$$

Then x^* *is an optimal solution of (P).*

Proof 3.1 *Let* x *be an arbitrary feasible solution of* (P)*.*

(a): Using the hypotheses specified in (i), we have for each $i \in I_+$,

$$\begin{aligned}&\phi\big(f_i(x) - f_i(x^*) + \frac{1}{2}\langle \nabla f_i(x^*), z^*\rangle\big)\\ \geq\ &\langle \nabla f_i(x^*) + \frac{1}{2}\nabla^2 f_i(x^*)z^*, \eta(x,x^*)\rangle + \bar{\rho}_i(x,x^*)\|\theta(x,x^*)\|^m\end{aligned}$$

and

$$\begin{aligned}&\phi\big(-g_i(x) + g_i(x^*) - \frac{1}{2}\langle \nabla g_i(x^*), z^*\rangle\big)\\ \geq\ &-\langle \nabla g_i(x^*) + \frac{1}{2}\nabla^2 g_i(x^*)z^*, \eta(x,x^*)\rangle + \tilde{\rho}_i(x,x^*)\|\theta(x,x^*)\|^m.\end{aligned}$$

As $\lambda^* \geq 0$, $u^* \geq 0$, $\sum_{i=1}^p u_i^* = 1$, and ϕ is superlinear, we deduce from the above inequalities that

$$\begin{aligned}\phi\Big(\sum_{i=1}^p u_i^*[f_i(x) - \lambda^* g_i(x)] &- \sum_{i=1}^p u_i^*[f_i(x^*) - \lambda^* g_i(x^*)]\\ &+ \frac{1}{2}\Big\langle \sum_{i=1}^p u_i^*[\nabla f_i(x^*) - \lambda^*\nabla g_i(x^*)], z\Big\rangle\Big)\\ &\geq \frac{1}{2}\Big\langle \sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - \lambda^*\nabla^2 g_i(x^*)]z^*, \eta(x,x^*)\Big\rangle\\ &+ \Big\langle \sum_{i=1}^p u_i^*[\nabla f_i(x^*) - \lambda^*\nabla g_i(x^*)], \eta(x,x^*)\Big\rangle\\ &+ \sum_{i\in I_+} u_i^*[\bar{\rho}_i(x,x^*) + \lambda^*\tilde{\rho}_i(x,x^*)]\|\theta(x,x^*)\|^m. \quad (3.10)\end{aligned}$$

Since $x \in \mathbb{F}$ and (3.4) holds, it follows from the properties of the functions $\hat{\phi}_j$ that for each $j \in J_+$, $\big(v_j^*G_j(x) \leq 0 \leq v_j^*G_j(x^*) - \frac{1}{2}\langle z^*, v_j^*\nabla G_j(x^*)\rangle$, which implies

$$\hat{\phi}_j\big(v_j^*G_j(x) - v_j^*G_j(x^*) + \frac{1}{2}\langle z^*, v_j^*\nabla G_j(x^*)\rangle\big) \leq 0$$

which in view of (ii) implies that

$$\langle \nabla G_j(x^*) + \frac{1}{2}\nabla^2 G_j(x^*)z^*, \eta(x,x^*)\rangle \leq -\hat{\rho}_j(x,x^*)\|\theta(x,x^*)\|^m.$$

As $v_j^* \geq 0$ *for each* $j \in \underline{q}$ *and* $v_j^* = 0$ *for each* $j \in \underline{q}\backslash J_+$ *(complement of* J_+ *relative to* $\underline{q}$*), the above inequalities yield*

$$\Big\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \frac{1}{2}\sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) z^*, \eta(x,x^*)\Big\rangle \leq -\sum_{j\in J_+} v_j^* \hat{\rho}_j(x,x^*)\|\theta(x,x^*)\|^m. \tag{3.11}$$

In a similar manner, we can show that (iii) leads to the following inequality:

$$\Big\langle \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) + \frac{1}{2}\sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*) z^*, \eta(x,x^*)\Big\rangle \leq -\sum_{k\in K_*} \breve{\rho}_k(x,x^*)\|\theta(x,x^*)\|^m. \tag{3.12}$$

Now, using (3.5), (3.6), and (3.10) - (3.12), we find that

$$\begin{aligned}
&\phi\Big(\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] - \big(\sum_{i=1}^{p} u_i^*[f_i(x^*) - \lambda^* g_i(x^*) \\
&- \frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)], z^*\Big\rangle]\big)\Big) \\
&\geq -\Big[\Big\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \frac{1}{2}\sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) z^*, \eta(x,x^*)\Big\rangle \\
&+ \Big\langle \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) + \frac{1}{2}\sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*) z^*, \eta(x,x^*)\Big\rangle\Big] \\
&\quad + \sum_{i\in I_+} u_i^*[\bar{\rho}_i(x,x^*) + \lambda^* \tilde{\rho}_i(x,x^*)]\|\theta(x,x^*)\|^m \ \ \textit{(by (3.5), (3.6), and (3.9))}
\end{aligned}$$

$$\geq \Big\{ \sum_{i\in I_+} u_i^*[\bar{\rho}_i(x,x^*) + \lambda^* \tilde{\rho}_i(x,x^*)] + \sum_{j\in J_+} v_j^* \hat{\rho}_j(x,x^*) + \sum_{k\in K_*} \breve{\rho}_k(x,x^*)\Big\}\|\theta(x,x^*)\|^m$$

(by (3.11) and (3.12))

≥ 0 *(by (iv)).*

But $\phi(a) \geq 0 \Rightarrow a \geq 0$*, and hence using (3.7), we have*

$$\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] \geq 0, \tag{3.13}$$

which using (3.7) implies that

$$\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] \geq 0. \tag{3.14}$$

Now using this inequality and Lemma 3.1, we have

$$\varphi(x^*) = \lambda^* \leq \frac{\sum_{i=1}^{p} u_i^* f_i(x)}{\sum_{i=1}^{p} u_i^* g_i(x)} \leq \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} = \varphi(x).$$

Since $x \in \mathbb{F}$ *is arbitrary, we conclude from this inequality that* x^* *is an optimal solution to* (P).

(b): Proceeding as in part (a), for each $j \in J_+$, *we have* $\big(v_j^* G_j(x) \leq 0 \leq v_j^* G_j(x^*) - \frac{1}{2}\langle z^*, v_j^* \nabla G_j(x^*)\rangle$, *which implies*

$$\hat{\phi}_j\big(v_j^* G_j(x) - v_j^* G_j(x^*) + \frac{1}{2}\langle z^*, v_j^* \nabla G_j(x^*)\rangle\big) \leq 0, \tag{3.15}$$

which in view of (ii) implies that

$$\Big\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \frac{1}{2}\sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) z^*, \eta(x, x^*) \Big\rangle \leq -\hat{\rho}(x, x^*)\|\theta(x, x^*)\|^m.$$

Now proceeding as in the proof of part (a) and using this inequality instead of (3.10), we arrive at (3.12), which leads to the desired conclusion that x^* *is an optimal solution of* (P).

(c) - (e) : The proofs are similar to those of parts (a) and (b).

(f) : Since $\rho(x, x^*) \geq 0$, *(3.5) and (3.6) imply*

$$\langle \eta(x, x^*), \nabla L(x^*, u^*, v^*, w^*, \lambda^*) + \frac{1}{2}\nabla^2 L(x^*, u^*, v^*, w^*, \lambda^*) z^*\rangle$$
$$\geq -\rho(x, x^*)\|\theta(x, x^*)\|^m,$$

which in view of our $(\phi, \eta, \rho, \theta, m)$*-pseudosonvexity assumption implies that*

$$\phi\big(L(x, u^*, v^*, w^*, \lambda^*) - \Big[L(x^*, u^*, v^*, w^*, \lambda^*) - \frac{1}{2}\langle \nabla L(x^*, u^*, v^*, w^*, \lambda^*), z^*\rangle\Big]\big) \geq 0.$$

But $\phi(a) \geq 0 \Rightarrow a \geq 0$ *and hence we have*

$$L(x, u^*, v^*, w^*, \lambda^*) \geq 0.$$

Because $x, x^* \in \mathbb{F}$, $v^* \geq 0$, *and (3.3), (3.4) and (3.5) hold, we get*

$$\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] \geq 0.$$

As seen in the proof of part (a), this inequality leads to the desired conclusion that x^* *is an optimal solution to* (P).

Theorem 3.2 *Let* $x^* \in \mathbb{F}$, $\lambda^* = \varphi(x^*)$, *the functions* f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, *and* H_k, $k \in \underline{r}$, *be twice differentiable at* x^*. *Then there exist* $u^* \in U$, $v^* \in \mathbb{R}^q_+$, *and* $w^* \in \mathbb{R}^r$ *such that (3.1) - (3.5) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:*

(a) *(i)* $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ *is* $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$*-pseudosonvex at* x^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $j \in J_+ \equiv J(v^*)$, G_j *is* $(\hat{\phi}_j, \eta, \hat{\rho}_j, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) *for each* $k \in K_* \equiv K(w^*)$, $\xi \to \mathcal{D}_k(\xi, w^*)$ *is* $(\breve{\phi}_k, \eta, \breve{\rho}_k, \theta, m)$*-quasisonvex at* x^*, *and* $\breve{\phi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$;

(b) *(i)* $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ *is* $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$*-pseudosonvex at* x^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ *is* $(\hat{\phi}, \eta, \hat{\rho}, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) *for each* $k \in K_*$, $\xi \to \mathcal{D}_k(\xi, w^*)$ *is* $(\breve{\phi}_k, \eta, \breve{\rho}_k, \theta, m)$*-quasisonvex at* x^*, *and* $\breve{\phi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$;

(c) *(i)* $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ *is* $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$*-pseudosonvex at* x^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+$, G_j *is* $(\hat{\phi}_m, \eta, \hat{\rho}_j, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$*;*

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ *is* $(\breve{\phi}, \eta, \breve{\rho}, \theta, m)$*-quasisonvex at* x^**, and* $\breve{\phi}(0) = 0$*;*

(iv) $\bar{\rho}(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(d) *(i)* $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ *is* $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$*-pseudosonvex at* x^**, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ *is* $(\hat{\phi}, \eta, \hat{\rho}, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ *is* $(\breve{\phi}, \eta, \breve{\rho}, \theta, m)$*-quasisonvex at* x^**, and* $\breve{\phi}(0) = 0$*;*

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(e) *(i)* $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ *is* $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$*-pseudosonvex at* x^**, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ *is* $(\hat{\phi}, \eta, \hat{\rho}, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$.

Then x^* *is an optimal solution of (P).*

Proof 3.2 *Let* x *be an arbitrary feasible solution of* (P).

(a): Based on assumptions specified in (ii) and (iii), (3.10) - (3.12) still hold for this case. From

(3.5), (3.6),(3.10), (3.11), (3.12) and (iv) we deduce that

$$\Big\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^*\nabla g_i(x^*)] + \frac{1}{2}\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^*\nabla^2 g_i(x^*)]z^*, \eta(x,x^*)\Big\rangle$$
$$\geq -\Big[\Big\langle \sum_{j=1}^{q} v_j^*\nabla G_j(x^*) + \frac{1}{2}\sum_{j=1}^{q} v_j^*\nabla^2 G_j(x^*)z^*, \eta(x,x^*)\Big\rangle$$
$$+ \Big\langle \sum_{k=1}^{r} w_k^*\nabla H_k(x^*) + \frac{1}{2}\sum_{k=1}^{r} w_k^*\nabla^2 H_k(x^*)z^*, \eta(x,x^*)\Big\rangle$$
$$\geq \Big[\sum_{j\in J_+} v_j^*\hat{\rho}_j(x,x^*) + \sum_{k\in K_*} \breve{\rho}_k(x,x^*)\Big]\|\theta(x,x^*)\|^m \quad \textit{(by (3.6) and (3.7))}$$
$$\geq -\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^m \quad \textit{(by (iv))},$$

which in view of (i) implies that

$$\bar{\phi}\big(\mathcal{E}(x,u^*,\lambda^*) - \big[\mathcal{E}(x^*,u^*,\lambda^*) - \frac{1}{2}\langle\nabla\mathcal{E}(x^*,u^*,\lambda^*), z^*\rangle\big]\big) \geq 0.$$

Based on the properties of the function $\bar{\phi}$, the last inequality yields

$$\mathcal{E}(x,u^*,\lambda^*) \geqq 0.$$

As shown in the proof of Theorem 3.1, this inequality leads to the conclusion that x^* is an optimal solution to (P).

(b) - (e) : The proofs are similar to that of part (a).

Theorem 3.3 *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, and assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are twice differentiable at x^*, and that there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that (3.1) - (3.4) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:*

(a) *(i) $\xi \to \mathcal{E}(\xi,u^*,\lambda^*)$ is prestrictly $(\bar{\phi},\eta,\bar{\rho},\theta,m)$-quasisonvex at x^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) for each $j \in J_+ \equiv J_+(v^*)$, G_j is $(\hat{\phi}_j,\eta,\hat{\rho}_j,\theta,m)$-quasisonvex at x^*, $\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0) = 0$;

(iii) *for each* $k \in K_* \equiv K(w^*)$, $\xi \to \mathcal{D}_k(\xi, w^*)$ *is* $(\breve{\phi}_k, \eta, \breve{\rho}_k, \theta, m)$*-quasisonvex at* x^*, *and* $\breve{\phi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j\in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k\in K_*} \breve{\rho}_k(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(b) *(i)* $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ *is prestrictly* $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$*-quasisonvex at* x^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ *is* $(\hat{\phi}, \eta, \hat{\rho}, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) *for each* $k \in K_*$, $\xi \to \mathcal{D}_k(\xi, w^*)$ *is* $(\breve{\phi}_k, \eta, \breve{\rho}_k, \theta, m)$*-quasisonvex at* x^*, *and* $\breve{\phi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k\in K_*} \breve{\rho}_k(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(c) *(i)* $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ *is prestrictly* $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$*-quasisonvex at* x^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $j \in J_+$, G_j *is* $(\hat{\phi}_j, \eta, \hat{\rho}_j, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ *is* $(\breve{\phi}, \eta, \breve{\rho}, \theta, m)$*-quasisonvex at* x^*, *and* $\breve{\phi}(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j\in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(d) *(i)* $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ *is prestrictly* $(\bar{\phi}, \eta^*, \bar{\rho}, \theta, m)$*-quasisonvex at* x^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, w^*)$ *is* $(\hat{\phi}, \eta, \hat{\rho}, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ *is* $(\breve{\phi}, \eta, \breve{\rho}, \theta, m)$*-quasisonvex at* x^*, *and* $\breve{\phi}(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(e) *(i)* $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ *is prestrictly* $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$*-quasisonvex at* x^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ *is* $(\hat{\phi}, \eta, \hat{\rho}, \theta, m)$*-quasisonvex at* x^*, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) > 0$ *for all* $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof 3.3 *The proof is similar to that of Theorem 3.2.*

4 Concluding Remarks

We established several results applying the new notion of higher order invexity to the context of discrete minmax fractional programming, which offer further applications to other fields for research endeavors relating to discrete fractional programming problems.

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Imbedding Theorems and Domains

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Abstract

The aim of this survey is to describe various types of domains which appear in the theory of function spaces and to show they role in some imbedding theorems.

Keywords: Sobolev Spaces, Sobolev imbedding theorem.

1 The Sobolev space and an imbedding

For Ω a bounded domain in the N-dimensional Euclidean space $\mathbb{R}^N$ and for a parameter $p > 1$, let us define the *Sobolev space* $W^{1,p}(\Omega)$ (roughly speaking) as the set of functions $f = f(x)$ defined a.e. in Ω which, together with their first order derivatives $\frac{\partial f}{\partial x_i}$ $(i = 1, 2, \ldots, N)$, belong to the Lebesgue space $L^p(\Omega)$. The famous *Sobolev imbedding theorem* claims that, for $1 < p < N$, a function $f \in W^{1,p}(\Omega)$ belongs to the space

$$L^q(\Omega) \quad \text{with} \quad 1 < p < q \leq \frac{Np}{N-p}.$$

We denote this result as

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \tag{1.1}$$

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and call $q_0 = \frac{Np}{N-p}$ the *critical value* of the imbedding (1.1). For the validity of an imbedding of type (1.1), the domain Ω plays an important role, as follows from the following (counter)example:

Example 1. Consider the plane domain Ω $(N = 2)$

$$\Omega = \{(x_1, x_2) : 0 < x_1 < 1,\ |x_2| < x_1^{2p}\exp(-p/x_1)\},\ 1 < p \le 2.$$

The function $f(x) = \exp(1/x_1)$ is an element of $W^{1,p}(\Omega)$, but $f \notin L^q(\Omega)$ for any $q > p$ since

$$\int_\Omega |f(x)|^q dx = 2\int_0^1 \exp(q/x_1)\exp(-p/x_1)x_1^{2p}dx_1 = \infty.$$

2 Domains

A *domain* Ω is an *open* and *connected* set in the space $\mathbb{R}^N$. Now, let us describe various types of domains, from various points of view. — The *boundary* of the domain Ω will be denoted by $\partial\Omega$. Sometimes the properties of $\partial\Omega$ are more important than that of Ω.

A. Boundedness

A.1 The domain $\Omega \subset \mathbb{R}^N$ is *bounded*, if there exists a ball $B(0, R)$ with center at origin and with radius $R > 0$ such that $\Omega \subset B(0, R)$.

A.2 In the opposite case, the domain Ω is *unbounded*. We can differ the following cases:

A.2.1 $\mu(\Omega) = \infty$ [μ denotes the Lebesgue measure];

A.2.2 $\mu(\Omega) < \infty$;

A.2.3 Ω is *quasibounded*, i.e.

$$\lim_{|x|\to\infty} \operatorname{dist}(x, \partial\Omega) = 0$$

($\operatorname{dist}(x, M)$ denotes the distance between the point x and the set M).

B. A plausible criterion

Usually, we consider domains Ω in $\mathbb{R}^N$ whose boundaries consist only of $(N-1)$-dimensional surfaces. Furthermore, we claim that

B.1 to every point $x \in \partial\Omega$ there exists a neighbourhood U_x in which it is possible to describe – in terms of an appropriate system of coordinates $y = (y', y_N) = (y_1, y_2, \ldots, y_{N-1}, y_N)$ – the boundary $\partial\Omega$ [more precisely: the set $\partial\Omega \cap U_x$] by a function a of $(N-1)$ variables as $y_N = a(y')$;

B.2 the domain Ω "lies on only one side of the boundary $\partial\Omega$".

Example 2. Let $\Omega \subset \mathbb{R}^2$ be the plane domain obtained by deleting from the unit ball the segment $S = \{(x_1, x_2) : 0 < x_1 < 1,\ x_2 = 0\}$. The boundary $\partial\Omega$ consists of the unit circle and of the segment S. - For this domain, both conditions B.1 and B.2 are violated: In the neighbourhood of the point $(1, 0) \in \partial\Omega$, the domain cannot be described by a function, and the domain Ω lies "on both sides of S". Domains of type B are called domains *with continuous*

boundary (if the function a from B.1 is continuous) or *with Hölderian boundary* (shortly Hölder domains) (if a satisfies the Hölder condition), or *with Lipschitzian boundary* (shortly Lipschitz domains) (if a satisfies the Lipschitz condition).

C. Convexity

A domain Ω is called *convex*, if it contains with any two points $x, y \in \Omega$ the whole segment $\langle x, y\rangle$ with end-points x, y.

D. Starshapedness

A domain Ω is called *starshaped* (with respect to $x_0 \in \Omega$) if there exists a point $x_0 \in \Omega$ such that, with every point $x \in \Omega$, the whole segment $\langle x_0, x \rangle$ belongs to Ω. [Alternatively: A bounded domain containing the origin 0 is called starshaped with respect to the origin, if there exists a positive function h defined on the unit sphere such that $\Omega = \{x \in \mathbb{R}^N;\ |x| < h(x/|x|) \text{ for } x \neq 0\}$. In this case $\partial\Omega = \{x \in \mathbb{R}^N;\ |x| = h(x/(|x|)$.]

Remark 1. (i) When S. L. SOBOLEV started in 1938 to introduce and investigate the function spaces bearing now his name, he used starshaped domains. (ii) Convex domains are starshaped with respect to any of their points.

E. John domain

Roughly speaking, the bounded domain $\Omega \subset \mathbb{R}^N$ is a *John domain*, if it is possible to move from one point of Ω to another without passing too close to the boundary $\partial\Omega$, or, more precisely, if there is a constant C_J and a distinguished point $x_0 \in \Omega$ so that each point $x \in \Omega$ can be joined to x_0 by a curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and $\operatorname{dist}(\gamma(t), \partial\Omega) \geq (1/C_J)|x - \gamma(t)|$ for every $t \in [0,1]$.

The class of John domains was considered first by F. JOHN in 1961 and named in 1978 by O. Martio and J. Sarvas.

H. Domains with property (S)

These domains have been introduced in [KJF] and used for the investigation of spaces of smooth functions. A domain Ω has the property (S), if there exists a constant $M > 0$ such that for every two points $x, y \in \Omega$ there exists a finite set of points $x_0 = x$, $x_1, x_2, \ldots, x_m = y$ such that the segments $\langle x_i, x_{i+1}\rangle$ $(i = 0, 1, \ldots, m-1)$ lie in Ω and $\sum_{i=1}^{m-1} |x_i - x_{i+1}| \le M|x-y|$.

Remark 2. Convex and starshaped domains have the property (S). The plane domain Ω lying – roughly speaking – "between two infinite disjoint spirals which converge to the same point" (called "pole") does *not* satisfy the condition (S). [The two spirals forming the boundary $\partial\Omega$ can be described in polar coordinates by $r\varphi = a$ and $r\varphi = b$, $0 < a < b$, with the origin being the pole.]

I. Domains with the cone condition

A bounded domain $\Omega \subset \mathbb{R}^N$ is called a *domain with the* (interior) *cone condition* if there exists a finite cone

$$C = \{x \in \mathbb{R}^N : x_1^2 + x_2^2 + \ldots x_{N-1}^2 \le a x_N^2,\ 0 \le x_N \le b\},$$

$a, b > 0$, such that any point of Ω is a vertex of a cone that is congruent to C and is entirely contained in Ω.

Example 3. (i) Every domain with the cone condition is a John domain. Also the plane domain from Example 2 satisfies the cone condition.
(ii) Domains with the cone property are "user-friendly" and appear frequently in applications.

J. Domains with the segment condition

We say that a domain $\Omega \subset \mathbb{R}^N$ satisfies the *segment condition* if every point $x \in \partial\Omega$ has a neighbourhood U_x and a nonzero vector y_x such that if $z \in \overline{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $0 < t < 1$.

K. Domains with cusps

Such domains are investigated in detail in [AF]. We say – roughly speaking – that Ω has a *cusp* at point $x_0 \in \partial\Omega$ if no open cone of positive volume contained in Ω has its vertex at x_0.

K.1 For $1 \leq k \leq N-1$ and $\lambda > 1$, the *standard cusp* $Q_{k,\lambda}$ is defined as the set of points $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ that satisfy the inequalities

$$x_1^2 + \cdots + x_k^2 < x_{k+1}^{2\lambda}, \;\; x_{k+1} > 0, \dots, x_N > 0,$$
$$(x_1^2 + \cdots + x_k^2)^{1/\lambda} + x_{k+1}^2 + \cdots + x_N^2 < a^2$$

where a is the radius of the ball of unit volume in $\mathbb{R}^N$.

The cusp $Q_{k,\lambda}$ has an axial plane spanned by the axes $x_k, x_{k+1}, \dots, x_N$, and a cusp plane spanned by $x_{k+2}, \dots, x_N$. If $k = N-1$, then the origin is the only vertex point of $Q_{k,\lambda}$.

Example 4. In the case $N = 2$ and $k = 1$ (i.e., $k = N-1$), the cusp in the neighbourhood of the vertex (= the origin) can be illustrated as $\{(x_1, x_2), x_1 \in (0, \varepsilon), x_2 = |x_1|^{1/\lambda}\}$.

K.2 Due to Example 4, we can also say that $Q_{k,\lambda}$ is a *cusp with power sharpness* (the sharpness being characterized by the parameter λ [more precisely: by $1/\lambda$]). We say that Ω has at $x_0 \in \partial\Omega$ a *cusp of exponential sharpness* if – for $\Omega_r = B(x_0, r)$, $S_r = \partial B_r(x_0, r) \cap \Omega$ and $A(r, \Omega)$ the surface area of S_r – it is

$$\lim_{r \to 0+} \frac{A(r, \Omega)}{r^k} = 0 \text{ for every real } k.$$

L. Domains with the flexible cone property

In the Russian literature (see e.g. [Be]), domains with the *flexible λ-horn* or *flexible λ-cone property* appear. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_i \geq 1$, a domain $\Omega \subset \mathbb{R}^N$ will be said to have the flexible λ-horn property [λ-cone property, if $\lambda_1 = \lambda_2 = \cdots = \lambda_N = \lambda$], if for some $\delta \in (0,1]$, $T > 0$ and for any $x \in \Omega$, there exists a curve $\rho(t^\lambda) = \rho(t^\lambda, x) = (\rho_1(t^{\lambda_1}), \ldots, \rho_N(t^{\lambda_N}))$, $0 \le t \le T$, such that

$$\rho_i(t) \text{ are absolutely continuous on } [0, T^{\lambda_i}],\ |\rho_i'(t)| \le 1 \text{ for a.e. } t \in [0, T^{\lambda_i}]$$

$$\rho(0,x) = x,\ x + \bigcup_{0<t\le T} [\rho(t^\lambda, x) + t^\lambda \delta^\lambda (-1,1)^N] \subset \Omega.$$

Remark 3. Domains with the flexible cone property are more general than the John domains.

M. Domains of type $\mathfrak{N}$

These domains have been introduced and described by J. NEČAS in 1962 and have been used in his book [N] and, e.g., in the book [KJF], but they appear, even in a particular or modified form, on several places in the literature (see, e.g., [AF] or [Bu]). Let us give at least a (raw) description: We assume that (a) the domain Ω is bounded;

(b) there exists a finite covering of the boundary $\partial\Omega$ by open sets U_i, $i = 1, 2, \ldots, m$, and an open set $U_0 \subset \Omega$ such that $\bigcup_{i=1}^{m} U_i \supset \partial\Omega$, $\bigcup_{i=0}^{m} U_i \supset \overline{\Omega}$;

(c) there exist m systems of coordinates $y_i = (y_i', y_{iN})$, $y_i' = (y_{i1}, y_{i2}, \ldots, y_{i,N-1})$, and m functions $a_i = a_i(y_i')$ defined on some appropriate sets $\Delta_i \subset \mathbb{R}^{N-1}$ such that a_i describes the set $\partial\Omega \cap U_i : y_{iN} = a_i(y_i')$, $[a_i(y_i'), y_{iN}] \in \partial\Omega$ $(i = 1, 2, \ldots, m)$;

(d) there exists $\varepsilon > 0$ such that, for $t \in (0, \varepsilon)$, the points $[y_i', a_i(y_i') - t]$ $(y_i' \in \Delta_i)$ lie in Ω and the points $[y_i', a_i(y_i') + t]$ lie in $\mathbb{R}^N \setminus \overline{\Omega}$ $(i = 1, 2, \ldots, m)$.

We shall say that Ω is of type $\mathfrak{N}^0$ or $\mathfrak{N}^{0,\lambda}$ or $\mathfrak{N}^{0,1}$ $(0 < \lambda < 1)$ if all functions a_i are continuous $[a_i \in C^0(\Delta_i)]$ or satisfy the Hölder condition $[a_i \in C^{0,\lambda}(\Delta_i)]$ or satisfy the Lipschitz condition $[a_i \in C^{0,1}(\Delta_i)]$, respectively.

Remark 4. (i) The description above allows to reduce the investigation of a function of N variables [in U_i, i.e. in the "cylinder" $\Delta_i \times (0, \infty)$] to the investigation of a function of a single variable y_{iN} with the parameter $y_i' \in \Delta_i$.

(ii) It is easy to show that $\Omega \in \mathfrak{N}^0$ is a domain with "continuous boundary" (see Section B), $\Omega \in \mathfrak{N}^{0,1}$ a domain with the cone condition (see Section I) and $\Omega \in \mathfrak{N}^{0,\lambda}$ a domain with cusps (Section K) with cusps of power sharpness.

N. Some special domains

In the literature, one can find many examples of domains constructed in terms of some limiting procedure. Here, let us consider two special cases. **N.1 von Koch's snowflake**. Let us consider a segment of length l; divide it into three equal parts and replace the middle part by an equilateral triangle. We obtain a broken line of four segments of length $l/3$, with total length $4l/3$. We proceed similarly with every of this four segments and obtain a broken line consisting of 16 segments of length $l/9$ each and of total length $16l/9 = (\frac{4}{3})^2 l$. We continue similarly with every of these 16 segments and obtain a broken line of total length $(\frac{4}{3})^3 l$ etc. Finally, tending to ∞, we obtain a line of infinite length. – If we use this procedure for every side of a equilateral triangle, we obtain a domain with boundary of infinite length, which was described in 1904 by the Swedish mathematician Helge von Koch. – This "snowflake" is a typical example of a so-called *fractal domain*, i.e. a domain with fractal boundary. **N.2** The "spiny urchin" is a domain Ω in $\mathbb{R}^2$, obtained by deleting from the plane the union of the sets S_k ($k = 1, 2, \dots$) specified in polar coordinates by

$$S_k = \Big\{(r, \varphi) : r \geq k,\ \varphi = \frac{n\pi}{2^k},\ n = 1, 2, \dots, 2^{k+1}\Big\}.$$

Hence, the set S_k is a collection of 2^{k+1} rays; the domain Ω is unbounded, but quasibounded (see A.2.3); the union of the sets S_k forms its boundary $\partial\Omega$; and $\overline{\Omega} = \mathbb{R}^2$.

Remark 5. There are also many other classes of domains in $\mathbb{R}^N$, e.g. MAZ'JA's $J_{(N-1)/N}$ and $I_{p,1/p-1/N}$-domains from the 1980-ies (see [M]); the (ε, δ)-domains of JONES (see [Jo]); the d-sets (see [JW]), domains of the type "rooms and passages" (see [EE]), but we will not go into details here and refer to the literature.

3 Back to the imbedding theorem

In Part 1, we have mentioned the classical result of SOBOLEV: If $1 < p < N$, then

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for} \quad 1 < p < q \leq q_0 \tag{3.2}$$

with $q_0 = \frac{Np}{N-p}$. Moreover, the value q_0 *cannot be improved.* **3.1** This result, derived for bounded and starshaped domains, holds for many of the domains mentioned above, in particular

- for domains with the cone condition (Section I)
- for John domains (Section E)
- for domains of type $\mathfrak{N}^{0,1}$ (Section M)
- for the von Koch's snowflake (being a John domain)
- for unbounded, but quasibounded domains (Section A) with "reasonable boundaries" (like the "spiny urchin", Section N.2).

3.2 The imbedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ holds due to the definition of the Sobolev space $W^{1,p}(\Omega)$, but in some cases, the imbedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *does not hold* for any $q > p$. Example 1 can serve as an illustration; in general, it is the case if

(i) Ω is unbounded, but $\mu(\Omega) < \infty$ ([AF, Theorem 4.46]) or

(ii) Ω has a cusp of exponential sharpness with vertex at $x_0 \in \partial\Omega$ ([AF, Theorem 4.48], Example 1).

3.3 In some cases, the imbedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ holds for $q > p$, but $q < q_0 = Np/(N-p)$. More precisely: (i) Let Ω have a cusp with power sharpness $1/\lambda$ (Section K.2) and let $\nu \geq \lambda - 1$. Then the imbedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ holds for

$$1 < p < q \leq \frac{(\nu + N)p}{\nu + N - p} := q_1.$$

(ii) Let Ω have the flexible λ-cone property (Section L, $\lambda \geq 1$). Then the imbedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ holds for

$$1 < p < q \leq \frac{Np}{\lambda(N-1) + 1 - p} := q_2.$$

Obviously, it is $q_1 \leq q_0$ and $q_2 \leq q_0$; hence these imbeddings are worse than that in the classical case. [The estimates for upper bounds q_1 and q_2 can be found in [AF] for (i) and in [Be] for (ii).]

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Generalization of Popoviciu Type Inequalities Via Abel-Gontscharoff Interpolating Polynomial

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Abstract

The inequality of Popoviciu for convex functions is generalized via Abel-Gontscharoff interpolating polynomial for higher order convex functions. Using Čebyšev functional, Grüss and Ostrowski type inequalities are obtained with best possible bounds. At last exponential convexity and Cauchy means are presented for linear functional coming from the general inequality with applications.

Keywords: Convex function, divided difference, Čebyšev functional, Grüss inequality, Ostrowski inequality, exponential convexity.

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1 Introduction

The theory developed in the study of convex functions, arising from intuitive geometrical observations, may be readily applied to topics in real analysis and economics. In the modern era the theory of convex functions has experienced a rapid development. This can be attributed to several causes: firstly, so many areas in modern analysis directly or indirectly involve the application of convex functions; secondly, convex functions are closely related to the theory of inequalities and many important inequalities are consequences of the applications of convex functions (see [16]).

A characterization of convex function established by T. Popoviciu [17] is studied by many people (see [18, 16] and references with in). For recent work, we refer [8, 11, 12, 13, 14]. The following form of Popoviciu's inequality is by Vasić and Stanković in [18] (see also page 173 [16]):

Theorem 1.1 *Let $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_m) \in [\alpha, \beta]^m$, $\mathbf{p} = (p_1, ..., p_m)$ be a positive m-tuple such that $\sum_{i=1}^m p_i = 1$. Also let $f : [\alpha, \beta] \to \mathbb{R}$ be a convex function. Then*

$$p_{k,m}(\mathbf{x}, \mathbf{p}; f) \leq \frac{m-k}{m-1} p_{1,m}(\mathbf{x}, \mathbf{p}; f) + \frac{k-1}{m-1} p_{m,m}(\mathbf{x}, \mathbf{p}; f), \tag{1.1}$$

where

$$p_{k,m}(\mathbf{x}, \mathbf{p}; f) = p_{k,m}(\mathbf{x}, \mathbf{p}; f(x)) := \frac{1}{C_{k-1}^{m-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \left(\sum_{j=1}^{k} p_{i_j} \right) f \left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \right)$$

is the linear functional with respect to f.

By inequality (1.1), we write

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; f) := \frac{m-k}{m-1} p_{1,m}(\mathbf{x}, \mathbf{p}; f) + \frac{k-1}{m-1} p_{m,m}(\mathbf{x}, \mathbf{p}; f) - p_{k,m}(\mathbf{x}, \mathbf{p}; f). \tag{1.2}$$

Remark 1.1 *It is important to note that under the assumptions of Theorem 1.1, if the function f is convex then $\mathbf{P}(\mathbf{x},\mathbf{p};f) \geq 0$ and $\mathbf{P}(\mathbf{x},\mathbf{p};f) = 0$ for $f(x) = x$ or f is constant function.*

The mean value theorems and exponential convexity of the linear functional $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ are given in [11] for a positive m-tuple $\mathbf{p}$. Some special classes of convex functions are considered to construct the exponential convexity of $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ in [11]. In [12] (see also [8]), the results related to $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ are generalized with help of Green function and $n-$exponential convexity is proved instead of exponential convexity.

In the present paper, we use Abel-Gontscharoff interpolating polynomial and prove many interesting results. The Abel-Gontscharoff interpolation problem in the real case was introduced in 1935 by Whittaker [19] and subsequently by Gontscharoff [7] and Davis [6]. The Abel-Gontscharoff interpolating polynomial for two points with integral remainder is given in [1]:

Theorem 1.2 *Let $n, l \in \mathbb{N}$, $n \geq 2$, $0 \leq l \leq n-1$ and $\lambda \in C^n([\alpha,\beta])$. Then we have*

$$\lambda(s) = T_{n-1}(\alpha,\beta,s;\lambda) + R(s;\lambda), \tag{1.3}$$

where $T_{n-1}(\alpha,\beta,s;\lambda)$ is the Abel-Gontscharoff interpolating polynomial of degree $n-1$ for two points, i.e.

$$T_{n-1}(\alpha,\beta,s;\lambda) = \sum_{v=0}^{l} \frac{(s-\alpha)^v}{v!}\lambda^{(v)}(\alpha) + \sum_{w=0}^{n-l-2}\left[\sum_{v=0}^{w} \frac{(s-\alpha)^{l+1+v}(\alpha-\beta)^{w-v}}{(l+1+v)!(w-v)!}\right]\lambda^{(l+1+w)}(\beta)$$

and the remainder is given by

$$R(s;\lambda) = \int_\alpha^\beta G_n(s,t)\lambda^{(n)}(t)dt,$$

where as $G_n(s,t)$ be Green's function [1, p. 177]

$$G_n(s,t) = \frac{1}{(n-1)!}\begin{cases} \displaystyle\sum_{v=0}^{l}\binom{n-1}{v}(s-\alpha)^v(\alpha-t)^{n-v-1}, & \alpha \leq t \leq s, \\ \displaystyle -\sum_{v=l+1}^{n-l}\binom{n-1}{v}(s-\alpha)^v(\alpha-t)^{n-v-1}, & s \leq t \leq \beta. \end{cases} \tag{1.4}$$

Further, for $\alpha \leq s, \quad t \leq \beta$ the following inequalities hold

$$(-1)^{n-l-1} \frac{\partial^v G_n(s,t)}{\partial s^v} \geq 0, \qquad 0 \leq v \leq l, \tag{1.5}$$

$$(-1)^{n-l} \frac{\partial^v G_n(s,t)}{\partial s^v} \geq 0, \qquad l+1 \leq v \leq n-1. \tag{1.6}$$

The presentation of the paper follows the following pattern: we start our main results from Section 2, in which we present generalization of the Popoviciu's inequality by using Abel-Gontscharoff interpolating polynomial combine together with the $n-$convexity of the function λ. Further in Section 3, we present some interesting results by using Čebyšev functional and Grüss-type inequalities along with some results relating to the Ostrowski-type inequality. In Section 4, we study the functional defined as the difference between the R.H.S. and the L.H.S. of the generalized inequality and our aim is to investigate the properties of functional, such as $n-$exponential and logarithmic convexity. Furthermore, we prove monotonicity property of the generalized Cauchy means obtained via this functional. Finally, in Section 5 we give several examples of the families of functions for which the obtained results can be applied.

2 Generalization of Popoviciu's Inequality for $n-$convex Functions Via Abel-Gontscharoff Interpolating Polynomial

Motivated by identity (1.2), we construct the following identity with help of Abel-Gontscharoff interpolating polynomial.

Theorem 2.1 *Let $n, l \in \mathbb{N}, \ n \geq 2, \ 0 \leq l \leq n-1$ and $\lambda \in C^n([\alpha,\beta])$ and let $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1, ..., p_m)$ be a real m-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < ... < i_k \leq m$ and $\sum_{i=1}^{m} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for* *any* $1 \leq i_1 < ... < i_k \leq m$ *with* G_n *defined in* (1.4)*. Then we have the following identity:*

$$\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) = \sum_{v=2}^{l} \frac{\lambda^{(v)}(\alpha)}{v!}\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^v) + \sum_{w=0}^{n-l-2}\left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\lambda^{(l+1+w)}(\beta)}{(l+1+v)!(w-v)!}\right]\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+1+v}) + \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G_n(x,t))\lambda^{(n)}(t)dt. \quad (2.7)$$

Proof 2.1 *Using Theorem 1.2, we have*

$$\lambda(x) = \sum_{v=0}^{l} \frac{(x-\alpha)^v}{v!}\lambda^{(v)}(\alpha) + \sum_{w=0}^{n-l-2}\left[\sum_{v=0}^{w} \frac{(x-\alpha)^{l+1+v}(\alpha-\beta)^{w-v}}{(l+1+v)!(w-v)!}\right]\lambda^{(l+1+w)}(\beta) + \int_\alpha^\beta G_n(x,t)\lambda^{(n)}(t)dt. \quad (2.8)$$

Substituting this value of λ *in* (1.2) *and following Remark 1.1, we get* (2.7)*.*

In the following theorem we obtain generalizations of Popoviciu's inequality for $n-$convex functions.

Theorem 2.2 *Let all the assumptions of Theorem 2.1 be satisfied and let for* $n \geq 2$

$$\mathbf{P}(\mathbf{x},\mathbf{p};G_n(x,t)) \geq 0, \quad t \in [\alpha,\beta]. \quad (2.9)$$

If λ *is* $n-$*convex, then we have*

$$\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) \geq \sum_{v=2}^{l} \frac{\lambda^{(v)}(\alpha)}{v!}\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^v) + \sum_{w=0}^{n-l-2}\left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\lambda^{(l+1+w)}(\beta)}{(l+1+v)!(w-v)!}\right]\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+1+v}). \quad (2.10)$$

Proof 2.2 *Since the function* λ *is* $n-$*convex, therefore without loss of generality we can assume that* λ *is n-times differentiable and* $\lambda^{(n)}(x) \geq 0$ *for all* $x \in [\alpha,\beta]$ *(see [16], p. 16). Hence we can apply Theorem* (2.1) *to obtain* (2.10)*.*

Now, we give generalization of Popoviciu's inequality for m-tuples.

Theorem 2.3 *Let all the assumptions of Theorem* (2.1) *be satisfied in addition with the condition that* $\mathbf{p} = (p_1, ..., p_m)$ *be a positive* m*-tuple such that* $\sum_{i=1}^{m} p_i = 1$ *and consider* $\lambda : [\alpha, \beta] \to R$ *is* $n-$*convex function.*

(i) If $(n = even, \; l = odd)$ or $(l = even, \; n = odd)$, then (2.10) holds.

(ii) Let the inequality (2.10) be satisfied. If the function

$$F(x) := \sum_{v=2}^{l} \frac{(x-\alpha)^v}{v!} \lambda^{(v)}(\alpha) + \sum_{w=0}^{n-l-2} \left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}(x-\alpha)^{l+1+v}}{(l+1+v)!(w-v)!} \right] \lambda^{(l+1+w)}(\beta). \tag{2.11}$$

is convex, the R.H.S. of (2.10) is non negative and we have inequality

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; \lambda(x)) \geq 0. \tag{2.12}$$

Proof 2.3 (*i*) *By using* (1.5), *for* $\alpha \leq x, \;\; t \leq \beta$ *the following inequality holds*

$$(-1)^{n-l-1} \frac{\partial^2 G_n(x,t)}{\partial x^2} \geq 0, \tag{2.13}$$

therefore it is easy to conclude that if $(n = even, \; l = odd)$ *or* $(l = even, \; n = odd)$ *then* $\frac{\partial^2 G_n(x,t)}{\partial x^2} \geq 0$ *and if* $(n = odd, \; l = odd)$ *or* $(l = even, \; n = even)$ *then* $\frac{\partial^2 G_n(x,t)}{\partial x^2} \leq 0$. *So for the cases* $(n = even, \; l = odd)$ *or* $(l = even, \; n = odd)$, G_n *is convex with respect to the first variable therefore by following Remark 1.1, the inequality* (2.9) *holds for* m*-tuples. Hence by Theorem 2.2, the inequality* (2.10) *holds.*

(*ii*) *Since* $\mathbf{P}(\cdot)$ *is a linear functional, so we can rewrite the R.H.S. of* (2.10) *in the form* $\mathbf{P}(\mathbf{x}, \mathbf{p}; F(x))$ *where* F *is defined in* (2.11). *Since* F *is (assumed to be) convex, therefore by Remark 1.1 the non negativity of the R.H.S. of* (2.10) *is immediate and we have* (2.12).

3 Bounds for Identities Related to Generalization of Popoviciu's Inequality

In this section we present some interesting results by using Čebyšev functional and Grüss type inequalities. For two Lebesgue integrable functions $f, h : [\alpha, \beta] \to R$, we consider the Čebyšev functional

$$\Delta(f,h) = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)h(t)dt - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)dt.\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} h(t)dt.$$

The following Grüss type inequalities are given in [5].

Theorem 3.1 *Let $f : [\alpha, \beta] \to R$ be a Lebesgue integrable function and $h : [\alpha, \beta] \to R$ be an absolutely continuous function with $(. - \alpha)(\beta - .)[h']^2 \in L[\alpha, \beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leq \frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_{\alpha}^{\beta}(x-\alpha)(\beta-x)[h'(x)]^2dx\right)^{\frac{1}{2}}. \tag{3.14}$$

The constant $\frac{1}{\sqrt{2}}$ in (3.14) is the best possible.

Theorem 3.2 *Assume that $h : [\alpha, \beta] \to R$ is monotonic nondecreasing on $[\alpha, \beta]$ and $f : [\alpha, \beta] \to R$ be an absolutely continuous with $f' \in L_\infty[\alpha, \beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leq \frac{1}{2(\beta-\alpha)}||f'||_\infty \int_{\alpha}^{\beta}(x-\alpha)(\beta-x)[h'(x)]^2dh(x). \tag{3.15}$$

The constant $\frac{1}{2}$ in (3.15) is the best possible.

In the sequel, we consider above theorems to derive generalizations of the results proved in the previous section. In order to avoid many notions let us denote

$$\mathfrak{F}(t) = \mathbf{P}(\mathbf{x}, \mathbf{p}; G_n(x,t)), \;\; t \in [\alpha, \beta], \tag{3.16}$$

Consider the Čebyšev functional $\Delta(\mathfrak{F}, \mathfrak{F})$ given as:

$$\Delta(\mathfrak{F},\mathfrak{F}) = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{F}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{F}(t)dt\right)^2, \tag{3.17}$$

Theorem 3.3 *Let* $n, l \in \mathbb{N}$, $n \geq 2$, $0 \leq l \leq n-1$ *and* $\lambda \in C^n([\alpha,\beta])$ *with* $(.-\alpha)(\beta-.)[\lambda^{(n+1)}]^2 \in L[\alpha,\beta]$. *Let* $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1, ..., p_m)$ *be a real m-tuple such that* $\sum_{j=1}^{k} p_{i_j} \neq 0$ *for any* $1 \leq i_1 < ... < i_k \leq m$ *and* $\sum_{i=1}^{m} p_i = 1$. *Also let* $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ *for any* $1 \leq i_1 < ... < i_k \leq m$ *with* $\mathfrak{F}$ *defined in* (3.16). *Then*

$$\mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) = \sum_{v=2}^{l} \frac{\lambda^{(v)}(\alpha)}{v!}\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^v)$$
$$+ \sum_{w=0}^{n-l-2}\left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\lambda^{(l+1+w)}(\beta)}{(l+1+v)!(w-v)!}\right]\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+1+v})$$
$$+ \frac{\lambda^{(n-1)}(\beta)-\lambda^{(n-1)}(\alpha)}{(\beta-\alpha)}\int_\alpha^\beta \mathfrak{F}(t)dt + \mathfrak{K}_n(\alpha,\beta;\lambda), \quad (3.18)$$

where the remainder $\mathfrak{K}_n(\alpha,\beta;\lambda)$ *satisfies the bound*

$$|\mathfrak{K}_n(\alpha,\beta;\lambda)| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}[\Delta(\mathfrak{F},\mathfrak{F})]^{\frac{1}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\lambda^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}. \quad (3.19)$$

Proof 3.1 (*i*) *If we apply Theorem 3.1 for* $f \to \mathfrak{F}$ *and* $h \to \lambda^{(n)}$, *we get*

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{F}(t)\lambda^{(n)}(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{F}(t)dt.\frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda^{(n)}(t)dt\right|$$
$$\leq \frac{1}{\sqrt{2}}[\Delta(\mathfrak{F},\mathfrak{F})]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\lambda^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}. \quad (3.20)$$

By denoting

$$\mathfrak{K}_n(\alpha,\beta;\lambda) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{F}(t)\lambda^{(n)}(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{F}(t)dt.\frac{\lambda^{(n-1)}(\beta)-\lambda^{(n-1)}(\alpha)}{(\beta-\alpha)}. \quad (3.21)$$

in (3.20), *we have* (3.19). *Hence, we have*

$$\int_\alpha^\beta \mathfrak{F}(t)\lambda^{(n)}(t)dt = \frac{\lambda^{(n-1)}(\beta)-\lambda^{(n-1)}(\alpha)}{(\beta-\alpha)}\int_\alpha^\beta \mathfrak{F}(t)dt + \mathfrak{K}_n(\alpha,\beta;\lambda),$$

where the remainder $\mathfrak{K}_n(\alpha, \beta; \lambda)$ *satisfies the estimation* (3.19). *Now from identity* (2.7) , *we obtain* (3.18).

The following Grüss type inequalities can be obtained by using Theorem 3.2

Theorem 3.4 *Let* $n, l \in \mathbb{N},\ n \geq 2,\ 0 \leq l \leq n-1$ *and* $\lambda \in C^n([\alpha, \beta])$ *such that* $\lambda^{(n+1)} \geq 0$ *on* $[\alpha, \beta]$ *with* $\mathfrak{F}$ *defined in* (3.16) *respectively. Then the representation* (3.18) *and the remainder* $\mathfrak{K}_n(\alpha, \beta; \lambda)$ *satisfies the estimation*

$$|\mathfrak{K}_n(\alpha, \beta; \lambda)| \leq ||\mathfrak{F}'||_\infty \left[\frac{\lambda^{(n-1)}(\beta) + \lambda^{(n-1)}(\alpha)}{2} - \frac{\lambda^{(n-2)}(\beta) - \lambda^{(n-2)}(\alpha)}{\beta - \alpha}\right]. \tag{3.22}$$

Proof 3.2 *Applying Theorem 3.2 for* $f \to \mathfrak{F}$ *and* $h \to \lambda^{(n)}$, *we get*

$$\left|\frac{1}{\beta - \alpha}\int_\alpha^\beta \mathfrak{F}(t)\lambda^{(n)}(t)dt - \frac{1}{\beta - \alpha}\int_\alpha^\beta \mathfrak{F}(t)dt.\frac{1}{\beta - \alpha}\int_\alpha^\beta \lambda^{(n)}(t)dt\right|$$
$$\leq \frac{1}{2(\beta - \alpha)}||\mathfrak{F}'||_\infty \int_\alpha^\beta (t - \alpha)(\beta - t)\lambda^{(n+1)}(t)dt. \tag{3.23}$$

Since

$$\int_\alpha^\beta (t - \alpha)(\beta - t)\lambda^{(n+1)}(t)dt = \int_\alpha^\beta [2t - (\alpha + \beta)]\lambda^{(n)}(t)dt$$
$$= (\beta - \alpha)\big[\lambda^{(n-1)}(\beta) + \lambda^{(n-1)}(\alpha)\big] - 2\big(\lambda^{(n-2)}(\beta) - \lambda^{(n-2)}(\alpha)\big).$$

Therefore, using identity (2.7) *and the inequality* (3.23), *we deduce* (3.22).

Now we intend to give the Ostrowski type inequalities related to generalizations of Popoviciu's inequality.

Theorem 3.5 *Suppose all the assumptions of Theorem 2.1 be satisfied. Moreover, assume* (p, q) *is a pair of conjugate exponents, that is* $p, q \in [1, \infty]$ *such that* $1/p + 1/q = 1$. *Let*

$|\lambda^{(n)}|^p : [\alpha,\beta] \to R$ be a R-integrable function for some $n \geq 2$. Then, we have

$$\left| \begin{array}{c} \mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) - \sum_{v=2}^{l} \frac{\lambda^{(v)}(\alpha)}{v!}\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^v) \\ -\sum_{w=0}^{n-l-2}\left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\lambda^{(l+1+w)}(\beta)}{(l+1+v)!(w-v)!}\right]\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+1+v}) \end{array} \right|$$
$$\leq ||\lambda^{(n)}||_p \left(\int_\alpha^\beta \left|\mathbf{P}(\mathbf{x},\mathbf{p};G_n(x,t))\right|^q dt\right)^{1/q}. \tag{3.24}$$

The constant on the R.H.S. of (3.24) *is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.*

Proof 3.3 *Using identity* (2.7)*, we obtain*

$$\left| \begin{array}{c} \mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) - \sum_{v=2}^{l} \frac{\lambda^{(v)}(\alpha)}{v!}\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^v) \\ -\sum_{w=0}^{n-l-2}\left[\sum_{v=0}^{w} \frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\lambda^{(l+1+w)}(\beta)}{(l+1+v)!(w-v)!}\right]\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+1+v}) \end{array} \right|$$
$$= \left| \int_\alpha^\beta \mathfrak{F}(t)\lambda^{(n)}(t)dt \right|. \tag{3.25}$$

Apply Hölder's inequality for integrals on the right hand side of (3.25)*, we have*

$$\left| \int_\alpha^\beta \mathfrak{F}(t)\lambda^{(n)}(t)dt \right| \leq \left(\int_\alpha^\beta \left|\lambda^{(n)}(t)\right|^p dt \right)^{\frac{1}{p}} \left(\int_\alpha^\beta |\mathfrak{F}(t)|^q dt \right)^{\frac{1}{q}},$$

which combine together with (3.25) *gives* (3.24)*.*

For the proof of the sharpness of the constant $\left(\int_\alpha^\beta |\mathfrak{F}(t)|^q dt\right)^{1/q}$, let us define the function λ for which the equality in (3.24) *is obtained.*

For $1 < p \leq \infty$ take λ to be such that

$$\lambda^{(n)}(t) = sgn\mathfrak{F}(t)|\mathfrak{F}(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $\lambda^{(n)}(t) = sgn\mathfrak{F}(t)$.

For $p = 1$, we prove that

$$\left| \int_\alpha^\beta \mathfrak{F}(t)\lambda^{(n)}(t)dt \right| \leq \max_{t\in[\alpha,\beta]} |\mathfrak{F}(t)| \left(\int_\alpha^\beta \lambda^{(n)}(t)dt \right) \tag{3.26}$$

is the best possible inequality. Suppose that $|\mathfrak{F}(t)|$ *attains its maximum at* $t_0 \in [\alpha, \beta]$. *To start with first we assume that* $\mathfrak{F}(t_0) > 0$. *For* δ *small enough we define* $\lambda_\delta(t)$ *by*

$$\lambda_\delta(t) = \begin{cases} 0 , & \alpha \leq t \leq t_0 , \\ \frac{1}{\delta n!}(t-t_0)^n , & t_o \leq t \leq t_0 + \delta , \\ \frac{1}{n!}(t-t_0)^{n-1} , & t_0 + \delta \leq t \leq \beta . \end{cases}$$

Then for δ *small enough*

$$\left| \int_\alpha^\beta \mathfrak{F}(t)\lambda^{(n)}(t)dt \right| = \left| \int_{t_0}^{t_0+\delta} \mathfrak{F}(t)\frac{1}{\delta}dt \right| = \frac{1}{\delta}\int_{t_0}^{t_0+\delta} \mathfrak{F}(t)dt.$$

Now from inequality (3.26), *we have*

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta} \mathfrak{F}(t)dt \leq \mathfrak{F}(t_0)\int_{t_0}^{t_0+\delta} \frac{1}{\delta}dt = \mathfrak{F}(t_0).$$

Since

$$\lim_{\delta\to 0} \frac{1}{\delta}\int_{t_0}^{t_0+\delta} \mathfrak{F}(t)dt = \mathfrak{F}(t_0),$$

the statement follows. The case when $\mathfrak{F}(t_0) < 0$, *we define* $\lambda_\delta(t)$ *by*

$$\lambda_\delta(t) = \begin{cases} \frac{1}{n!}(t-t_0-\delta)^{n-1} , & \alpha \leq t \leq t_0 , \\ \frac{-1}{\delta n!}(t-t_0-\delta)^n , & t_o \leq t \leq t_0 + \delta , \\ 0 , & t_0 + \delta \leq t \leq \beta , \end{cases}$$

and rest of the proof is the same as above.

4 Mean Value Theorems and $n-$exponential convexity

We recall some definitions and basic results from [2], [9] and [15] which are required in sequel.

Definition 4.1 *A function $\lambda : I \to \mathbb{R}$ is $n-$exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \lambda\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

hold for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$. A function $\lambda : I \to \mathbb{R}$ is $n-$exponentially convex if it is $n-$exponentially convex in the Jensen sense and continuous on I.

Definition 4.2 *A function $\lambda : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is $n-$exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*
A function $\lambda : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 4.1 *If $\lambda : I \to \mathbb{R}$ is an $n-$exponentially convex in the Jensen sense, then the matrix $\left[\lambda\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \leq n$. Particularly,*

$$\det\left[\lambda\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, ..., n$.

Remark 4.1 It is known that $\lambda : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2 \lambda(x) + 2\alpha\beta\lambda\left(\frac{x+y}{2}\right) + \beta^2 \lambda(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is $2-$exponentially convex in the Jensen sense.
A positive function is log-convex if and only if it is $2-$exponentially convex.

Remark 4.2 *By the virtue of Theorem 2.2, we define the positive linear functional with respect* *to $n-$convex function λ as follows*

$$\Lambda(\lambda) := \mathbf{P}(\mathbf{x},\mathbf{p};\lambda(x)) - \sum_{v=2}^{l}\frac{\lambda^{(v)}(\alpha)}{v!}\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{v}) - \sum_{w=0}^{n-l-2}\left[\sum_{v=0}^{w}\frac{(-1)^{w-v}(\beta-\alpha)^{w-v}\lambda^{(l+1+w)}(\beta)}{(l+1+v)!(w-v)!}\right]\mathbf{P}(\mathbf{x},\mathbf{p};(x-\alpha)^{l+1+v}) \geq 0. \quad (4.27)$$

Lagrange and Cauchy type mean value theorems related to defined functional is given in the following theorems.

Theorem 4.1 *Let $\lambda : [\alpha,\beta] \to R$ be such that $\lambda \in C^{n}[\alpha,\beta]$. If the inequality in (2.9) holds, then there exist $\xi \in [\alpha,\beta]$ such that*

$$\Lambda(\lambda) = \lambda^{(n)}(\xi)\Lambda(\varphi), \quad (4.28)$$

where $\varphi(x) = \frac{x^{n}}{n!}$ and $\Lambda(\cdot)$ is defined by (4.27).

Proof 4.1 *Similar to the proof of Theorem* 4.1 *in [10] (see also [3]).*

Theorem 4.2 *Let $\lambda,\psi : [\alpha,\beta] \to R$ be such that $\lambda,\psi \in C^{n}[\alpha,\beta]$. If the inequality in (2.9) holds, then there exist $\xi \in [\alpha,\beta]$ such that*

$$\frac{\Lambda(\lambda)}{\Lambda(\psi)} = \frac{\lambda^{(n)}(\xi)}{\psi^{(n)}(\xi)}, \quad (4.29)$$

provided that the denominators are non-zero and $\Lambda(\cdot)$ is defined by (4.27).

Proof 4.2 *Similar to the proof of Corollary* 4.2 *in [10] (see also [3]).*

Theorem 4.2 enables us to define Cauchy means, because if

$$\xi = \left(\frac{\lambda^{(n)}}{\psi^{(n)}}\right)^{-1}\left(\frac{\Lambda(\lambda)}{\Lambda(\psi)}\right),$$

which means that ξ is mean of α, β for given functions λ and ψ.

Next we construct the non trivial examples of $n-$exponentially and exponentially convex functions from positive linear functional $\Lambda(\cdot)$. We use the idea given in [15]. In the sequel I and J are intervals in R.

Theorem 4.3 *Let $\Gamma = \{\lambda_t : t \in J\}$, where J is an interval in R, be a family of functions defined on an interval I in R such that the function $t \mapsto [x_0, \dots, x_n; \lambda_t]$ is $n-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in I$. Then for the linear functional $\Lambda(\lambda_t)$ as defined by (4.27), the following statements are valid:*

(i) The function $t \to \Lambda(\lambda_t)$ is $n-$exponentially convex in the Jensen sense on J and the matrix $[\Lambda(\lambda_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, .., t_m \in J$. Particularly,

$$\det[\Lambda(\lambda_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, ..., n.$$

(ii) If the function $t \to \Lambda(\lambda_t)$ is continuous on J, then it is $n-$exponentially convex on J.

Proof 4.3 *The proof is similar to Theorem 4.6 in [4].*

The following corollary is an immediate consequence of the above theorem

Corollary 4.1 *Let $\Gamma = \{\lambda_t : t \in J\}$, where J is an interval in R, be a family of functions defined on an interval I in R, such that the function $t \mapsto [x_0, \dots, x_n; \lambda_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in I$. Then for the linear functional $\Lambda(\lambda_t)$ as defined by (4.27), the following statements hold:*

(i) The function $t \to \Lambda(\lambda_t)$ is exponentially convex in the Jensen sense on J and the matrix $\left[\Lambda\left(\lambda_{\frac{t_j+t_l}{2}}\right)\right]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, .., t_m \in J$. Particularly,

$$\det\left[\Lambda\left(\lambda_{\frac{t_j+t_l}{2}}\right)\right]_{j,l=1}^m \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, ..., n.$$

(ii) If the function $t \to \Lambda(\lambda_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 4.2 *Let $\Gamma = \{\lambda_t : t \in J\}$, where J is an interval in R, be a family of functions defined on an interval I in R, such that the function $t \mapsto [x_0, \dots, x_n; \lambda_t]$ is $2-$exponentially* *convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Let $\Lambda(\cdot)$ be linear functional defined by* (4.27)*. Then the following statements hold:*

(i) If the function $t \mapsto \Lambda(\lambda_t)$ is continuous on J, then it is $2-$exponentially convex function on J. If $t \mapsto \Lambda(\lambda_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[\Lambda(\lambda_s)]^{t-r} \leq [\Lambda(\lambda_r)]^{t-s} [\Lambda(\lambda_t)]^{s-r},$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto \Lambda(\lambda_t)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(\Lambda, \Gamma) \leq \mu_{u,v}(\Lambda, \Gamma), \tag{4.30}$$

where

$$\mu_{p,q}(\Lambda, \Gamma) = \begin{cases} \left(\frac{\Lambda(\lambda_p)}{\Lambda(\lambda_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\Lambda(\lambda_p)}{\Lambda(\lambda_p)}\right), & p = q, \end{cases} \tag{4.31}$$

for $\lambda_p, \lambda_q \in \Gamma$.

Proof 4.4 *The proof is similar to Corollary 4.8 in [4].*

5 Applications to Cauchy means

In this section, we present some families of functions which fulfil the conditions of Theorem 4.3, Corollary 4.1 and Corollary 4.2. This enables us to construct a large families of functions which are exponentially convex. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

Example 5.1 Let us consider a family of functions

$$\Gamma_1 = \{\lambda_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\lambda_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & t \neq 0, \\ \frac{x^n}{n!}, & t = 0. \end{cases}$$

Since $\frac{d^n \lambda_t}{dx^n}(x) = e^{tx} > 0$, the function λ_t is $n-$convex on R for every $t \in R$ and $t \mapsto \frac{d^n \lambda_t}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.3 we also have that $t \mapsto [x_0, \ldots, x_n; \lambda_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 4.1 we conclude that $t \mapsto \Lambda(\lambda_t)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $t \mapsto \lambda_t$ is not continuous for $t = 0$), so it is exponentially convex. For this family of functions, $\mu_{t,q}(\Lambda, \Gamma_1)$, from (4.31), becomes

$$\mu_{t,q}(\Lambda, \Gamma_1) = \begin{cases} \left(\frac{\Lambda(\lambda_t)}{\Lambda(\lambda_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left(\frac{\Lambda(id \cdot \lambda_t)}{\Lambda(\lambda_t)} - \frac{n}{t}\right), & t = q \neq 0, \\ \exp\left(\frac{1}{n+1} \frac{\Lambda(id \cdot \lambda_0)}{\Lambda(\lambda_0)}\right), & t = q = 0, \end{cases}$$

where "*id*" is the identity function. By Corollary 4.2 $\mu_{t,q}(\Lambda, \Gamma_1)$ is a monotone function in parameters t and q.

Since

$$\left(\frac{\frac{d^n f_t}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{t-q}} (\log x) = x,$$

using Theorem 4.2 it follows that:

$$M_{t,q}(\Lambda, \Gamma_1) = \log \mu_{t,q}(\Lambda, \Gamma_1),$$

satisfies

$$\alpha \leq M_{t,q}(\Lambda, \Gamma_1) \leq \beta.$$

Hence $M_{t,q}(\Lambda, \Gamma_1)$ is a monotonic mean.

Example 5.2 *Let us consider a family of functions*

$$\Gamma_2 = \{g_t : (0,\infty) \to R : t \in R\}$$

defined by

$$g_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0,1,\ldots,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Since $\frac{d^n g_t}{dx^n}(x) = x^{t-n} > 0$, *the function* g_t *is* $n-$*convex for* $x > 0$ *and* $t \mapsto \frac{d^n g_t}{dx^n}(x)$ *is exponentially convex by definition. Arguing as in Example 5.1 we get that the mappings* $t \mapsto \Lambda(g_t)$ *is exponentially convex. Hence, for this family of functions* $\mu_{p,q}(\Lambda, \Gamma_2)$ *, from* (4.31), *are equal to*

$$\mu_{t,q}(\Lambda, \Gamma_2) = \begin{cases} \left(\frac{\Lambda(g_t)}{\Lambda(g_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Lambda(g_0 g_t)}{\Lambda(g_t)} + \sum\limits_{k=0}^{n-1} \frac{1}{k-t}\right), & t = q \notin \{0,1,\ldots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Lambda(g_0 g_t)}{2\Lambda(g_t)} + \sum\limits_{\substack{k=0 \\ k\neq t}}^{n-1} \frac{1}{k-t}\right), & t = q \in \{0,1,\ldots,n-1\}. \end{cases}$$

Again, using Theorem 4.2 we conclude that

$$\alpha \leq \left(\frac{\Lambda(g_t)}{\Lambda(g_q)}\right)^{\frac{1}{t-q}} \leq \beta. \tag{5.32}$$

Hence $\mu_{t,q}(\Lambda, \Gamma_2)$ *is a mean and its monotonicity is followed by (4.30).*

Example 5.3 *Let*

$$\Gamma_3 = \{\zeta_t : (0,\infty) \to R : t \in (0,\infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(-\log t)^n}, & t \neq 1; \\ \frac{x^n}{(n)!}, & t = 1. \end{cases}$$

Since $\frac{d^n \zeta_t}{dx^n}(x) = t^{-x}$ *is the Laplace transform of a non-negative function (see [**?**]) it is exponentially convex. Obviously* ζ_t *are* $n-$*convex functions for every* $t > 0$.

For this family of functions, $\mu_{t,q}(\Lambda, \Gamma_3)$ *, in this case for* $[\alpha, \beta] \subset \mathbb{R}^+$*, from (4.31) becomes*

$$\mu_{t,q}(\Lambda, \Gamma_3) = \begin{cases} \left(\frac{\Lambda(\zeta_t)}{\Lambda(\zeta_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Lambda(id.\zeta_t)}{t\Lambda(\zeta_t)} - \frac{n}{t\log t}\right), & t = q \neq 1; \\ exp\left(-\frac{1}{n+1}\frac{\Lambda(id.\zeta_1)}{\Lambda(\zeta_1)}\right), & t = q = 1, \end{cases}$$

where id is the identity function. By Corollary 4.2 $\mu_{p,q}(\Lambda, \Gamma_3)$ *is a monotone function in parameters* t *and* q*.*

Using Theorem 4.2 it follows that

$$M_{t,q}(\Lambda, \Gamma_3) = -L(t, q) log \mu_{t,q}(\Lambda, \Gamma_3),$$

satisfy

$$\alpha \leq M_{t,q}(\Lambda, \Gamma_3) \leq \beta.$$

This shows that $M_{t,q}(\Lambda, \Gamma_3)$ *is a mean. Because of the inequality (4.30), this mean is monotonic. Furthermore,* $L(t, q)$ *is logarithmic mean defined by*

$$L(t, q) = \begin{cases} \frac{t-q}{\log t - \log q}, & t \neq q; \\ t, & t = q. \end{cases}$$

Example 5.4 *Let*

$$\Gamma_4 = \{\Lambda_t : (0, \infty) \to R : t \in (0, \infty)\}$$

be a family of functions defined by

$$\Lambda_t(x) = \frac{e^{-x\sqrt{t}}}{(-\sqrt{t})^n}.$$

Since $\frac{d^n\Lambda_t}{dx^n}(x) = e^{-x\sqrt{t}}$ *is the Laplace transform of a non-negative function (see [**?**]) it is exponentially convex. Obviously* Λ_t *are* $n-$*convex function for every* $t > 0$*.*

For this family of functions, $\mu_{t,q}(\Lambda, \Gamma_4)$, *in this case for* $[\alpha, \beta] \subset \mathbb{R}^+$, *from (4.31) becomes*

$$\mu_{t,q}(\Lambda, \Gamma_4) = \begin{cases} \left(\frac{\Lambda(\Lambda_t)}{\Lambda(\Lambda_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Lambda(id.\Lambda_t)}{2\sqrt{t}\Lambda(\Lambda_t)} - \frac{n}{2t}\right), & t = q; \end{cases} \quad i = 1, 2.$$

By Corollary 4.2, it is a monotone function in parameters t *and* q.

Using Theorem 4.2 it follows that

$$M_{t,q}(\Lambda, \Gamma_4) = -\left(\sqrt{t} + \sqrt{q}\right) ln\mu_{t,q}(\Lambda, \Gamma_4),$$

satisfy

$$\alpha \leq M_{t,q}(\Lambda, \Gamma_4) \leq \beta.$$

This shows that $M_{t,q}(\Lambda, \Gamma_4)$ *is a mean. Because of the above inequality (4.30), this mean is monotonic.*

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Conflict of Interests

The authors declare that there is no conflict of interests.

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Matrix Transformations in Sequence Spaces

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Abstract

The purpose of this paper is to present a survey of results on matrix transformations in sequence spaces which contains some open problems for further study.

Keywords: Sequence spaces, matrix transformations, bounded sequences, convergence sequences, almost convergence.

AMS Mathematics Subject Classification: 46A45, 40C05

1 Introduction

Let X and Y be any two non empty subsets of the space of all complex sequences. Let $A = (a_{nk}), (n, k = 1, 2, 3, ...)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$, if $A_n(x) = \sum_k a_{nk}x_k$ converges for each n. If $x = (x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, then we say that A defines a matrix transformation from X into Y and we denote it by $A : X \to Y$. The sequence Ax is called the A-transform of x. By (X, Y) we mean the class of all matrixes A such that $A : X \to Y$. If in X and Y there is some notion of limit or sum, then we write (X, Y,

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P) to denote the subset of (X, Y) which preserves the limit or sum. In this paper, we present a table which is a survey of results, contains some open problems on matrix transformations. In the table the number, say 48, stands for the result on the transformation from the space occurring in the row containing 48 into the space occurring in the column containing 48. In the table, where no number is mentioned is an open problem, the result corresponding to that has not been solved yet. This is an extension of the table presented in Stieglitz and Tietz (35). In the list of results N, K and K* denote arbitrary finite subsets of the set of all positive integers. We first present some important definitions, notations and conventions which will be used in describing results on matrix transformations.

If $\{p_k\}$ is a bounded sequence of strictly positive real numbers, then we define :

$$l_\infty(p) = m(p) = \{x : sup_k |x_k|^{p_k} < \infty\},$$

$$c_o(p) = \{x : |x_k|^{p_k} \to 0\},$$

$$c(p) = \{x : |x_k - l|^{p_k} \to 0, \text{ for some } l\}$$

$$l(p) = \{x : \sum_k |x_k|^{p_k} < \infty\},$$

$$w(p) = \{x : \frac{1}{n}\sum_{k=1}^{n} |x_k - l|^{p_k} \to 0, \text{ for some } l\}$$

Write

$$S_x = \sum_{k=0}^{n} x_k$$

Then,

$$m_s(p) = \{x : S_x \in l_\infty(p)\}$$

$$c_s(p) = \{x : S_x \in c(p)\}$$

$$(c_o)_s(p) = \{x : S_x \in c_o(p)\}$$

If $p_k = p$ for all k, then we have

$$l_\infty(p) = m(p) = l_\infty = m, c_o(p) = c_o, c(p) = c,$$

$$l(p) = l_p, w(p) = w_p, m_s(p) = m_s$$

$$c_s(p) = c_s, (c_o)_s(p) = (c_o)_s$$

$$\text{If } p = 1 \text{ then } l_p = l_1 = l \text{ and } w_p = w_1 = w$$

$$\text{If } p_k = \frac{1}{k} \text{ then } c_o(p) = \Gamma$$

We have

$$bv = \{x : \sum_n |x_n - x_{n-1}| < \infty\},$$

$$bv_o = bv \bigcap c_o$$

Write

$$\triangle^\alpha x^n = \sum_{k=0}^{\infty} (-1)^k (\alpha_k) x_{n+k}$$

$$q^\alpha = c \bigcap \{x : \sum_n (n + \alpha_n - 1) |\triangle^\alpha x^n < \infty\}$$

Let $\{p_n\}$ be a bounded sequence of positive real numbers and $\{v_n\}$ any fixed sequence of non zero complex numbers satisfying

$$\lim_n |v_n|^{\frac{1}{n}} = r (0 \leq r \leq \infty)$$

Define a function

$$\Lambda \ : \mathbb{C} \rightarrow \ \mathbb{C} \ by$$

$$\Lambda(z) = \sum \frac{z^n}{v_n}$$

We define

$$D_o^\Lambda(p) = \{f : f(z) = a_n z^n,\ |a_n v_n|^{p_n} \to 0\},$$

$$D_\infty^\Lambda(p) = \{f : f(z) = a_n z^n,\ sup|a_n v_n|^{p_n} < \infty\},$$

$$D^\Lambda(p) = \{f : f(z) = a_n z^n,\ \sum |a_n v_n|^{p_n} < \infty\},$$

if $p_n = 1$ for all n,

$$D_0^\Lambda(p) = D_0^\Lambda, D_\infty^\Lambda(p) = D_\infty^\Lambda \text{ and } D^\Lambda(p) = D_1^\Lambda$$

$$\text{Put } t_{mn}(x) = \frac{1}{m}\sum_{i=1}^{m} x_{n+i}$$

$$x_n = a_1 + a_2 + ... + a_n$$

$$\Phi_{m,n}(a) = \frac{1}{m(m+1)}\sum_{i=1}^{m} i\ a_{n+i},\ m \geq 1$$

$$\Phi_{o,n}(a) = a_n$$

Then for a bounded sequence $\{p_m\}, p_m > 0$ we have

$$\hat{c}(p) = \{x : |t_{m,n}(x) - l|^{p_m} \to 0, \text{ for some} l \text{uniformly in n}\}$$

$$\hat{l}(p) = \{a : \sum_m |\Phi_{m,n}(a)|^{p_m} \text{ converges uniformly in n}\},$$

if $p_m = p$ for all m, $\hat{c}(p) = \hat{c},\ \hat{l}(p) = \hat{l}$ and if $p = 1,\ \hat{l}(p) = \hat{l}_1 = \hat{l}$

We define

$$E_r(c) = \{x = \{x_k\} : k^r x \in c\}$$

$$E_r(c_o) = \{x : k^r x \in c_o\}$$

$$E_r(m) = \{x : k^r x \in m\}$$

We now state all the results mentioned in Table-1, which appears at the end.

$$A \in (m,\ m) = (c,\ m) = (c_0,\ m) \Leftrightarrow 1.1 \tag{1}$$

$$\sup_n \sum_k |a_{nk}| < \infty, \tag{1.1}$$

$$A \in (m_s,\ m) \Leftrightarrow (2.1), (2.2), \tag{2}$$

$$\lim_k a_{nk} = 0 \text{ for all n}, \tag{2.1}$$

$$\sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty. \tag{2.2}$$

$$A \in (c_s,\ m) \Leftrightarrow (2.2), (3.1) \Leftrightarrow (3.2) \tag{3}$$

$$\sup_n |\lim_k a_{nk}| < \infty \tag{3.1}$$

$$\sup_n \sum_k |a_{nk} - a_{n,k-1}| < \infty \tag{3.2}$$

$$A \in ((c_0)_s,\ m) \Leftrightarrow (2.2). \tag{4}$$

$$A \in (l_p),\ m) \Leftrightarrow (5.1) \tag{5}$$

$$\sup_n \sum_k |a_{nk}|^q < \infty, 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1. \tag{5.1}$$

$$A \in (l,\ m) \Leftrightarrow (6.1) \tag{6}$$

$$\sup_{n,\ k} |a_{nk}| < \infty. \tag{6.1}$$

$$A \in (q^\alpha, m) \Leftrightarrow (7.1), (7.2) \tag{7}$$

$$\sup_n |\sum_k a_{nk}| < \infty. \tag{7.1}$$

$$\sup_{n,l} |\binom{l+\alpha-1}{l}^{-1} \sum_{k=0}^{l} a_{nk} \binom{l+k+\alpha-1}{l-k}| < \infty. \tag{7.2}$$

$$A \in (b\nu,\ m) \Leftrightarrow (7.1), (8.1) \Leftrightarrow (8.2) \tag{8}$$

$$\sup_{n,\ l} |\sum_{k=0}^{l} a_{nk}| < \infty, \tag{8.1}$$

$$\sup_{n,\ l} |\sum_{k=l}^{\infty} a_{nk}| < \infty, \tag{8.2}$$

$$A \in (b\nu_0,\ m) \Leftrightarrow (8.1). \tag{9}$$

$$A \in (\Gamma,\ m) \Leftrightarrow (10.1) \tag{10}$$

$$\sup_{n,k} |a_{nk}|^{\frac{1}{k}} < \infty \tag{10.1}$$

$$A \in (\Lambda,\ m) \Leftrightarrow (11.1) \tag{11}$$

$$\text{The sequence}\{f_n(z)\} \text{ where} f_n(z) = \sum_{p=1}^{\infty} a_{np}\ z^p (n = 1, 2,) \tag{11.1}$$

of integral functions is uniformly bounded on every compact set (of the complex plane).

$$A \in (E_r(m),\ m) = (E_r(c),\ m)(E_r(c_o),\ m) \Leftrightarrow (12.1), \tag{12}$$

$$\sup_n \sum_k |k^{-r}\ a_{nk}| < \infty, \tag{12.1}$$

$$A \in (E_r(l_p),\ m) \Leftrightarrow (13.1) \tag{13}$$

$$\sup_n \sum_k |k^{-r}\ a_{nk}|^q < \infty, \text{where } \frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty, \tag{13.1}$$

$$A \in (E_r(l),\ m) \Leftrightarrow (14.1) \tag{14}$$

$$\sup_{n,k} |k^{-r}\ a_{nk}| < \infty, \tag{14.1}$$

$$A \in (D_\infty^\Lambda,\ m) \Leftrightarrow (15.1) \tag{15}$$

$$\sup_n \sum_k |\frac{a_{nk}}{v_k}| < \infty, \tag{15.1}$$

$$A \in (D_1^\Lambda, m) \Leftrightarrow (16.1) \tag{16}$$

$$\sup_{n,\ k} |\frac{a_{nk}}{v_k}| < \infty \tag{16.1}$$

$$A \in (m(p),\ m) = (c(p),\ m) = (c_o(p),\ m) \Leftrightarrow 17.1 \tag{17}$$

$$\sup_n \sum_k |a_{nk}| N^{\frac{1}{p_k}} < \infty, \text{for every integer } N > 1. \tag{17.1}$$

$$\text{Let } 1 < p_k < H \text{ and } \frac{1}{p_k} + \frac{1}{q_k} = 1, \text{for every k,}$$

$$\text{Then A} \in (l(p),\ m) \Leftrightarrow (18.1) \tag{18}$$

$$\exists \ \text{ an integer } B > 1 \text{ such that} \sup_n \sum_k |a_{nk}|^{qk}\ B^{-q_k} < \infty, \text{Let } 0\ < p_k \leq 1 \text{ for every k} \tag{18.1}$$

$$\text{Then } A \in (l(p), m) \Leftrightarrow (18.2) \sup_{n,k} |a_{nk}|^{pk} < \infty \tag{18.2}$$

$$A \in (\omega, m) \Leftrightarrow (19.1) \tag{19}$$

$$\sup_n \sum_t 2^t \max_t |a_{nk}| < \infty \tag{19.1}$$

$$\text{Let } 0 < p_k \leq 1 \text{ for every kThen } A \in (\omega(p),\ m) \Leftrightarrow (20.1) \tag{20}$$

$$\exists \quad \text{an integer} \quad B > 1 \quad \text{such that,} \sup_n \sum_{t=0}^{\infty} \max_t \{(2^t B^{-1})^{\frac{1}{p_k}} |a_{nk}|\} < \infty \tag{20.1}$$

$$\text{Let } 0 < p < 1, \text{ Then }, A \in (w_p,\ m) \Leftrightarrow (21.1), (21.2) \tag{21}$$

$$\sup_n \sum_{t=0}^{\infty} 2^{\frac{t}{p}} \max_t |a_{nk}| < \infty, \tag{21.1}$$

$$\text{where maximum is taken over k such that } 2^t \leq k < 2^{t+1} (*)$$

$$\text{Let } 1 < p < \infty \text{ and } \frac{1}{p_k} + \frac{1}{q_k} = 1.$$

$$\text{Then, } A \in (w_p,\ m) \Leftrightarrow (21.2) \text{ holds} \sup_n \sum_{t=0}^{\infty} 2^{\frac{1}{p}} \{(|a_{nk}|^q)^{\frac{1}{q}}\} < \infty \tag{21.2}$$

where sum is taken over k such that * holds

$$A \in (m, c) \Leftrightarrow (22.1), (22.2) \Leftrightarrow (1.1), \tag{22}$$

$$(22.1), (22.3) \Leftrightarrow (22.1), (22.4)$$

$$\lim_n a_{nk} \text{ exists for all k,} \tag{22.1}$$

$$\lim_n \sum_k |a_{nk}| = \sum_k |\lim_n a_{nk}|, \tag{22.2}$$

$$\lim_n \sum_k |a_{nk} - \lim_n a_{nk}| = 0, \tag{22.3}$$

$$\sum_k |a_{nk}| \text{ converges uniformly in n} \tag{22.4}$$

$$A \in (c,\ c) \Leftrightarrow (1.1), (22.1), (23.1) \tag{23}$$

$$A \in (c,\ c,\ P) \Leftrightarrow (1.1), (23.2), (23.3)$$

$$\lim_n \sum_k a_{nk} \text{ exists,} \tag{23.1}$$

$$\lim_n a_{nk} = 0\ \forall\ k, \tag{23.2}$$

$$\lim_n \sum_k a_{nk} = 1 \tag{23.3}$$

$$A \in (c_0,\ c) \Leftrightarrow (1.1), (22.1) \tag{24}$$

$$A \in (m_s,\ c) \Leftrightarrow (2.1), (25.1), (25.2) \tag{25}$$

$$\Leftrightarrow (2.1), (2.2), (25.1), (25.3),$$

$$\Leftrightarrow (2.1), (22.1), (25.4),$$

$$\Leftrightarrow (2.1), (25.1), (25.4),$$

$$\lim_n (a_{nk} - a_{n,k+1}) \text{ exists for all k,} \tag{25.1}$$

$$\lim_n \sum_k |a_{nk} - a_{n,k+1}| = \sum_k |\lim_n (a_{nk} - a_{n,k+1})|, \tag{25.2}$$

$$\lim_n \sum_k |a_{nk} - a_{n,k+1} - \lim_n (a_{nk} - a_{n,k+1})| = 0, \tag{25.3}$$

$$\sum_k ||a_{nk} - a_{n,k+1}| \text{ converges uniformly in n} \tag{25.4}$$

$$A \in (c_s,\ c) \Leftrightarrow (2.2), (22.1) \tag{26}$$

$$A \in (c_s,\ c, P) \Leftrightarrow (2.2), (26.1)$$

$$\lim_n a_{nk} = 1 \text{ for all k} \tag{26.1}$$

$$A \in ((c_0)_s,\ c) \Leftrightarrow (2.2), (25.1) \tag{27}$$

$$A \in (l_p,\ c) \Leftrightarrow (5.1), (22.1), p > 1 \tag{28}$$

$$A \in (l,\ c) \Leftrightarrow (6.1), (22.1) \tag{29}$$

$$A \in (q^{\alpha},\ c) \Leftrightarrow (7.2), (22.1), (23.1) \tag{30}$$

$$A \in (b\nu,\ c) \Leftrightarrow (8.1), (22.1), (23.1) \tag{31}$$

$$A \in (b\nu_0,\ c) \Leftrightarrow (8.1), (22.1) \tag{32}$$

$$A \in (\Gamma,\ c) \Leftrightarrow (10.1), (22.1) \tag{33}$$

$$A \in (\Lambda,\ c) \Leftrightarrow (11.1), (22.1) \tag{34}$$

$$A \in (E_r(m),\ c) \Leftrightarrow (35.1), (35.2) \tag{35}$$

$$\lim_n \{k^{-r}\ a_{nk}\} \text{ exist for all k} \tag{35.1}$$

$$\sum_k |k^{-r}\ a_{nk}| \text{ converges uniformly in m} \tag{35.2}$$

$$A \in (E_r(c),\ c) \Leftrightarrow (12.1), (35.1), (36.1) \tag{36}$$

$$\lim_n \sum_k k^{-r} a_{nk} \text{ exists} \tag{36.1}$$

$$A \in (E_r(c_o),\ c) \Leftrightarrow (12.1), (35.1) \tag{37}$$

$$A \in (E_r(l_p),\ c) \Leftrightarrow (13.1), (35.1) \tag{38}$$

$$A \in (E_r(l),\ c) \Leftrightarrow (14.1), (35.1) \tag{39}$$

$$\text{If } A \in (c,\ c,\ P),$$

$$\text{Then}, A \in (\hat{c},\ c) \Leftrightarrow (40.1) \tag{40}$$

$$\lim_n \sum_k |a_{nk} - a_{n,k+1}| = 0 \tag{40.1}$$

$$A \in (m(p),\ c) \Leftrightarrow (41.1), (41.2) \tag{41}$$

$$\sum_k |a_{nk}| N^{\frac{1}{p_k}} \text{ converges uniformly in n, for every integer N} > 1 \tag{41.1}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha_k \tag{41.2}$$

$$A \in (c(p),\ c) \Leftrightarrow (41.2), (42.1), (42.2) \tag{42}$$

$$A \in (c(p),\ c,\ P) \Leftrightarrow (42.1), (42.3), (42.4)$$

There exist a constant $B > 1$ such that

$$\sup_n \sum_k a_{nk} B^{\frac{-1}{p_k}} < \infty \tag{42.1}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha \tag{42.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0 \tag{42.3}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 1 \tag{42.4}$$

$$A \in (c_o(p),\ c) \Leftrightarrow (41.2), (42.2) \tag{43}$$

$$A \in (l(p),\ c) \Leftrightarrow (18.1), (41.2),\ 1 \leq p_k \leq H \tag{44}$$

$$\Leftrightarrow (18.2), (41.2),\ 0 \leq p_k \leq 1$$

$$A \in (w,\ c) \Leftrightarrow (19.1), (41.2), (42.2) \tag{45}$$

$$A \in (w,\ c,\ P) \Leftrightarrow (19.1), (42.3), (42.4)$$

$$A \in (w(p),\ c) \Leftrightarrow (20.1, (41.2, (42.2, 0 < p_k \leq 1 \tag{46}$$

$$A \in (w(p),\ c, P) \Leftrightarrow (20.1), (41.2), (42.3), (42.2) 0 < p_k \leq 1$$

$$A \in (w_p,\ c) \Leftrightarrow (21.1), (41.2), 0 < p \leq 1 \tag{47}$$

$$A \in (w_p,\ c, P) \Leftrightarrow (21.1), (42.3), 0 < p \leq 1$$

$$A \in (w_p,\ c) \Leftrightarrow (21.1), (41.2), (42.2), 1 < p_k < \infty$$

$$A \in (w_p,\ c, P) \Leftrightarrow (21.2), (42.3), (42.4)$$

$$A \in (m,\ c_0) \Leftrightarrow (48.1) \tag{48}$$

$$\lim_n \sum_k |a_{nk}| = 0 \tag{48.1}$$

$$A \in (c,\ c_0) \Leftrightarrow (1.1), (23.2), (49.1) \tag{49}$$

$$\lim_n \sum a_{nk} = 0 \tag{49.1}$$

$$A \in (c_0,\ c_0) \Leftrightarrow (1.1), (23.2) \tag{50}$$

$$A \in (m_s,\ c_0) \Leftrightarrow (2.1), (51.1) \tag{51}$$

$$\lim_n \sum_k |a_{nk} - a_{n,k+1}| = 0 \tag{51.1}$$

$$A \in (c_s,\ c_0) \Leftrightarrow (2.2), (23.2) \tag{52}$$

$$A \in ((c_0)_s,\ c_0) \Leftrightarrow (2.2), (53.1) \tag{53}$$

$$\lim_n (a_{nk} - a_{n,k+1}) = 0 \text{ for all k} \tag{53.1}$$

$$A \in (l_p,\ c_0) \Leftrightarrow (5.1), (23.2), p > 1 \tag{54}$$

$$A \in (l,\ c_0) \Leftrightarrow (6.1), (23.2) \tag{55}$$

$$A \in (q^{\alpha},\ c_0) \Leftrightarrow (7.2), (23.2), (49.1) \tag{56}$$

$$A \in (b\nu,\ c_0) \Leftrightarrow (8.1), (23.2), (49.1), \tag{57}$$

$$\Leftrightarrow (8.2), (23.2), (49.1)$$

$$A \in (bv_0,\ c_0) \Leftrightarrow (8.1), (23.2) \tag{58}$$

$$A \in (\Gamma,\ c_o) \Leftrightarrow (10.1), (23.2) \tag{59}$$

$$A \in (\Lambda,\ c_o) \Leftrightarrow (11.1), (23.2) \tag{60}$$

$$A \in (E_r(m),\ c_0) \Leftrightarrow (61.1) \tag{61}$$

$$\lim_n \sum_k |k^{-r} a_{nk}| = 0 \tag{61.1}$$

$$A \in (E_r(c),\ c_0) \Leftrightarrow (12.1), (62.1), (62.2) \tag{62}$$

$$\lim_n |k^{-r}\ a_{nk}| = 0 \text{ for all k} \tag{62.1}$$

$$\lim_n \sum_k \{k^{-r} a_{nk}\} = 0 \tag{62.2}$$

$$A \in (E_r(c_o),\ c_o) \Leftrightarrow (12.1), (62.1) \tag{63}$$

$$A \in (E_r(l_p),\ c_o) \Leftrightarrow (13.1), (62.1) \tag{64}$$

$$A \in (E_r(l),\ c_o) \Leftrightarrow (14.1, (62.1) \tag{65}$$

$$\text{Let } A \in (c_o,\ c_o), \tag{66}$$

$$\text{Then } A \in (c_o,\ (\hat{c_o}) \Leftrightarrow (40.1)$$

$$A \in (m(p),\ c_o) \Leftrightarrow (41.1), (42.3) \tag{67}$$

$$A \in (c(p),\ c_o) \Leftrightarrow (42.1), (42.3), (68.1) \tag{68}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0 \tag{68.1}$$

$$A \in (c_o(p),\ c_o) \Leftrightarrow (42.1), (42.3) \tag{69}$$

$$A \in (l(p),\ c_o) \Leftrightarrow (18.1), (42.3),\ \ 1 < p_k \leq H < \infty \tag{70}$$

$$A \in (l(p),\ c_o) \Leftrightarrow (18.2), (42.3),\ \ 0 < p_k \leq 1$$

$$A \in (w,\ c_o) \Leftrightarrow (19.1), (42.3), (68.1) \tag{71}$$

$$A \in (w(p),\ c_o) \Leftrightarrow (20.1), (42.3), (68.1),\ \ 0 < p_k \leq 1 \tag{72}$$

$$A \in (w_p,\ c_o) \Leftrightarrow (21.1), (42.3), 0 < p \leq 1 \tag{73}$$

$$A \in (w_p,\ c_o) \Leftrightarrow (21.1), (42.3), (68.1), 1 < p < \infty$$

$$A \in (m,\ m_s) = (c,\ m_s) = (c_0,\ m_s) \Leftrightarrow (74.1) \tag{74}$$

$$\sup_m \sum_k |\sum_{n=0}^{m} a_{nk}| < \infty \tag{74.1}$$

$$A \in (m_s,\ m_s) \Leftrightarrow (2.1), (75.1) \tag{75}$$

$$\sup_m \sum_k |\sum_{n=0}^{m} a_{nk} - a_{n,k+1}| < \infty \tag{75.1}$$

$$A \in (c_s,\ m_s) \Leftrightarrow (75.1), (76.1) \Leftrightarrow (76.2) \tag{76}$$

$$\sup_m |\lim_k \sum_{n=0}^{m} a_{nk}| < \infty \tag{76.1}$$

$$\lim_m \sum_k |\sum_{n=0}^{m} a_{nk} - a_{n,k+1}| < \infty \tag{76.2}$$

$$A \in ((c_0)_s,\ m_s) \Leftrightarrow (75.1) \tag{77}$$

$$A \in (l_p,\ m_s) \Leftrightarrow (78.1), p > 1 \tag{78}$$

$$\sup_n \sum_k |\sum_{n=0}^{m} a_{nk}| q \ < \ \infty,\ \frac{1}{p} + \frac{1}{q} = 1 \tag{78.1}$$

$$A \in (l,\ m_s) \Leftrightarrow (79.1) \tag{79}$$

$$\sup_{m,k} |\sum_{n=0}^{m} a_{nk}| < \infty \tag{79.1}$$

$$A \in (q^{\alpha},\ m_s) \Leftrightarrow (80.1), (80.2) \tag{80}$$

$$\sup_m |\sum_{n=0}^{m} \sum_k a_{nk}| < \ \infty \tag{80.1}$$

$$\sup_{m,\ l} |\binom{l+\alpha-1}{l})^{-1} \sum_{n=0}^{m} \sum_{k=0}^{l} \binom{l-k+\alpha-1}{l-k} a_{nk}| < \infty \tag{80.2}$$

$$A \in (bv,\ m_s) \Leftrightarrow (80.1), (81.1) \Leftrightarrow (81.2) \tag{81}$$

$$\sup_{m,\ l} |\sum_{n=0}^{\infty} \sum_{k=0}^{l} a_{nk}| < \ \infty \tag{81.1}$$

$$\sup_{m,\ l} |\sum_{n=0}^{m} \sum_{k=l}^{\infty} a_{nk}| < \ \infty \tag{81.2}$$

$$A \in (b\nu_0,\ m_s) \Leftrightarrow (81.1) \tag{82}$$

$$A \in (m(p),\ m_s) \Leftrightarrow (83.1) \tag{83}$$

$$\sup_m \sum_k |\sum_{n=0}^{m} a_{nk}| N^{\frac{1}{p_k}} \text{ for every integer } N > 1 \tag{83.1}$$

$$A \in (m,\ c_s) \Leftrightarrow (84.1) \ \Leftrightarrow \ (84.2) \tag{84}$$

$$\Leftrightarrow (84.3), (84.4)$$

$$\lim_m \sum_k |\sum_{n=0}^{m} a_{nk}| = \sum_k |\sum_n a_{nk}| \tag{84.1}$$

$$\lim_m \sum_k |\sum_{n=0}^{\infty} a_{nk}| = 0 \tag{84.2}$$

$$\sum_k |\sum_{n=0}^{m} a_{nk}| \text{ converges uniformly in n} \tag{84.3}$$

$$\sum_{n=0}^{\infty} a_{nk} \text{ converges for all k} \tag{84.4}$$

$$A \in (c,\ c_s) \Leftrightarrow (74.1), (84.4), (85.1) \tag{85}$$

$$A \in (c,\ c_s,\ p) \Leftrightarrow (74.1), (85.2), (85.3)$$

$$\sum_n \sum_k a_{nk} \text{ converges} \tag{85.1}$$

$$\sum_n a_{nk} = 0 \text{ for all k} \tag{85.2}$$

$$\sum_n \sum_k a_{nk} = 1 \tag{85.3}$$

$$A \in (c_0,\ c_s) \Leftrightarrow (74.1), (84.1) \tag{86}$$

$$A \in (m_s,\ c_s) \Leftrightarrow (2.1), (87.1) \Leftrightarrow (2.1), (87.2) \tag{87}$$

$$\Leftrightarrow (2.1), (84.1), (87.3)$$

$$\Leftrightarrow (2.1), (87.3), (87.4)$$

$$\lim_m \sum_k |\sum_{n=0}^{m} (a_{nk} - a_{n,k+1}| \tag{87.1}$$

$$= \sum_k |\sum_n (a_{nk} - a_{n,k+1}|$$

$$\lim_m \sum_k |\sum_{n=m}^{\infty} (a_{nk} - a_{n,k+1}| = 0 \tag{87.2}$$

$$\sum_k |\sum_{n=0}^{m} (a_{nk} - a_{n,k+1}| \text{ converges uniformly in m} \tag{87.3}$$

$$\sum_n (a_{nk} - a_{n,k+1}) \text{ converges for all k} \tag{87.4}$$

$$A \in (c_s,\ c_s) \Leftrightarrow (76.2), (84.4) \tag{88}$$

$$A \in (c_s,\ c_s,\ p) \Leftrightarrow (76.2), (88.1)$$

$$\sum_n (a_{nk}) = 1 \ \text{ for all k} \tag{88.1}$$

$$A \in ((c_0)_s,\ c_s) \Leftrightarrow (75.1), (87.4) \tag{89}$$

$$A \in (l_p,\ c_s) \Leftrightarrow (78.1), (84.4), p > 1 \tag{90}$$

$$A \in (l,\ c_s) \Leftrightarrow (79.1), (84.4) \tag{91}$$

$$A \in (q^{\alpha},\ c_s) \Leftrightarrow (80.2), (84.4), (85.1) \tag{92}$$

$$A \in (b\nu,\ c_s) \Leftrightarrow (81.1), (84.4), (85.1) \tag{93}$$

$$A \in (b\nu_0,\ c_s) \Leftrightarrow (81.1), (84.4) \tag{94}$$

$$A \in (m(p),\ c_s) \Leftrightarrow (95.1), (95.2) \tag{95}$$

$$\sum_k |\sum_{n=0}^{m} a_{nk}| N^{\frac{1}{p_k}} \text{ converges uniformly in m, for every integer } N > 1, \tag{95.1}$$

$$\sum_n a_{nk} \text{ converges for all k} \tag{95.2}$$

$$A \in (c(p),\ c_s) \Leftrightarrow (96.1), (96.2), (96.3) \tag{96}$$

$$\sup_n \sum_k |\sum_{i=1}^{n} a_{ik}| B^{\frac{-1}{p_k}} < \infty \text{ for some integer} B > 1, \tag{96.1}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_{nk} \text{ exists for all k} \tag{96.2}$$

$$\lim_{n\to\infty} \sum_k \sum_{i=1}^{n} a_{nk} \text{ exists} \tag{96.3}$$

$$A \in (l(p),\ c_s) \Leftrightarrow (97.1), (97.2) \tag{97}$$

There exists an integer $B > 1$ such that

$$\sup_n \sum_{k=1}^{\infty} |\sum_{i=1}^{n} a_{ik}|^{q_k} \ B^{-qk} < \infty, (1 < p_k < \infty) \tag{97.1}$$

$$\sup_{n,k} |\sum_{i=1}^{n} a_{ik}|^{p_k} < \infty, (0 \ < p_k \ < 1)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_{ik} \text{ exists for all k} \tag{97.2}$$

$$A \in (m, \ (c_0)_s) \Leftrightarrow (98.1) \tag{98}$$

$$\lim_m \sum_k |\sum_{n=0}^{m} a_{nk}| = 0 \tag{98.1}$$

$$A \in (c, \ (c_0)_s) \Leftrightarrow (74.1), (85.2), (99.1) \tag{99}$$

$$\sum_n \ \sum_k a_{nk} = 0 \tag{99.1}$$

$$A \in (c_0, \ (c_0)_s) \Leftrightarrow (74.1), (85.1) \tag{100}$$

$$A \in (m_s, \ (c_0)_s) \Leftrightarrow (2.1), (101.1) \tag{101}$$

$$\lim_m \sum_k |\sum_{n=0}^{m} a_{nk} - a_{n,n+1}| = 0 \tag{101.1}$$

$$A \in (c_s, \ (c_0)_s) \Leftrightarrow (75.1), (85.2) \tag{102}$$

$$A \in ((c_0)_s, \ (c_0)_s) \Leftrightarrow (75.1), (103.1) \tag{103}$$

$$\sum_n (a_{nk} - a_{n,k+1}) = 0 \text{ for all k} \tag{103.1}$$

$$A \in (l_p, \ (c_0)_s) \Leftrightarrow (78.1), (85.2), p > 1 \tag{104}$$

$$A \in (l, \ (c_0)_s) \Leftrightarrow (79.1), (85.2) \tag{105}$$

$$A \in (q^{\alpha}, \ (c_0)_s) \Leftrightarrow (80.1), (85.2), (99.1) \tag{106}$$

$$A \in (b\nu, \ (c_0)_s) \Leftrightarrow (81.1), (85.2), (99.1) \tag{107}$$

$$\Leftrightarrow (81.2), (85.2), (99.1)$$

$$A \in (b\nu_0,\ (c_0)_s) \Leftrightarrow (81.1), (85.2) \tag{108}$$

$$A \in (m(p), (c_o)_s) \Leftrightarrow (95.1), (109.1) \tag{109}$$

$$\sum_n a_{nk} \text{ converges to zero for all k} \tag{109.1}$$

$$A \in (m,\ l_r) = (c,\ l_r) = (c_0,\ l_r) \Leftrightarrow (110.1) \tag{110}$$

$$\Leftrightarrow (110.2), r > 1$$

$$\sup_k \sum_n |\sum_{k \in K} a_{nk}|^r < \infty \tag{110.1}$$

$$\sum_n |\sum_{k \in K} a_{nk}|^r \text{converges for all k} \tag{110.2}$$

$$A \in (m_s,\ l_r) \Leftrightarrow (2.1), (111.1); r > 1 \tag{111}$$

$$\sup_k \sum_n |\sum_{k \in K} (a_{nk} - a_{n,k+1}|^r < \infty \tag{111.1}$$

$$A \in (c_s,\ l_r) \Leftrightarrow (112.1), r > 1 \tag{112}$$

$$\sup_k \sum_n |\sum_{k \in K} (a_{nk} - a_{n,k+1}|^r < \infty \tag{112.1}$$

$$A \in ((c_0)_s,\ l_r) \Leftrightarrow (111.1), r > 1 \tag{113}$$

$$A \in (l_p,\ l_r), p > 1, r > 1 \text{ is unknown} \tag{114}$$

$$A \in (l,\ l_r) \Leftrightarrow (115.1), r > 1 \tag{115}$$

$$\sup_k \sum_n |(a_{nk}|^r < \infty \tag{115.1}$$

$$A \in (q^\alpha,\ l_r) \Leftrightarrow (116.1), (116.2), r > 1 \tag{116}$$

$$\sum_n |\sum_k (a_{nk}|^r \text{ converges} \tag{116.1}$$

$$\sup_l \sum_n |\binom{l+\alpha-1}{l})^{-1} \sum_{k=0}^{l} \binom{l-k+\alpha-1}{l-k})a_{nk}|^r < \infty \tag{116.2}$$

$$A \in (b\nu,\ l_r) \Leftrightarrow (116.1), (117.1) \Leftrightarrow (117.2), r > 1 \tag{117}$$

$$\sup_l \sum_n |\sum_{k=0}^{l} (a_{nk}|^r < \infty \tag{117.1}$$

$$\sup_l \sum_n |\sum_{k=l}^{\infty} a_{nk}|^r < \infty \tag{117.2}$$

$$A \in (b\nu_0,\ l_r) \Leftrightarrow (117.1), r > 1 \tag{118}$$

$$A \in (D_1^{\Lambda},\ l_r) \Leftrightarrow (119.1) \tag{119}$$

$$\sup_k \sum_n |\frac{a_{nk}}{\nu_k}|^r < \infty \tag{119.1}$$

$$A \in (m(p),\ l_r) = (c(p),\ l_r) = (c_o(p),\ l_r) \Leftrightarrow (120.1), 1 < r < \infty \tag{120}$$

$$\sum_n (\sum_k |a_{nk}| N^{\frac{1}{p_k}})^r \text{ for every integer } N > 1, \tag{120.1}$$

$$A \in (c_o,\ m(r)) \Leftrightarrow (121.1) \tag{121}$$

$$\sup_n (\sum_k |a_{nk}|)^{r_n} < \infty \tag{121.1}$$

$$A \in (c_o(p),\ m(r)) \Leftrightarrow (122.1) \tag{122}$$

There exists an integer $B > 1$ such that

$$\sup_n (\sum_k |a_{nk}|\ B^{\frac{-1}{p_k}})^{r_n} < \infty,\ p, r \in m \tag{122.1}$$

$$A \in (c_o,\ c_o(r)) \Leftrightarrow (121.1), (123.1) \tag{123}$$

$$\lim_{n \to \infty} |a_{nk}|^{r_n} = 0 \tag{123.1}$$

$$A \in (c_o(p),\ c_o(r)) \Leftrightarrow (122.1), (123.1) \tag{124}$$

$$A \in (m,\ l) = (c,\ l) = (c_o,\ l) \Leftrightarrow (125.1) \tag{125}$$

$$\Leftrightarrow (125.2)\ \ \Leftrightarrow (125.3)\ \ \Leftrightarrow (125.4)$$

$$\sup_{n,\ k} |\sum_{n \in N} \sum_{k \in K} a_{nk}| < \infty \tag{125.1}$$

$$\sup_N \sum_k |\sum_{n \in N} a_{nk}| < \infty \tag{125.2}$$

$$\sup_{K}\sum_{n}|\sum_{k\in K} a_{nk}|<\infty \tag{125.3}$$

$$\sum_{n}|\sum_{k\in K} a_{nk}|<\infty \tag{125.4}$$

$$A\in(m_s,\ l)\Leftrightarrow(2.1),(126.1) \tag{126}$$

$$\sup_{N,K}|\sum_{n\in N}\sum_{k\in K}(a_{nk}-a_{n,k+1})|<\infty \tag{126.1}$$

$$A\in(c_s,\ l)\Leftrightarrow(127.1) \tag{127}$$

$$\sup_{N,\ K}|\sum_{n\in N}\sum_{k\in K}(a_{nk}-a_{n,k-1})|<\infty \tag{127.1}$$

$$A\in((c_0)_s,\ l)\Leftrightarrow(126.1) \tag{128}$$

$$A\in(l_p,\ l)\Leftrightarrow(129.1),p<1 \tag{129}$$

$$\sup_{N}\sum_{k}|\sum_{n\in N} a_{nk}|^q<\infty \tag{129.1}$$

$$A\in(l,\ l)\Leftrightarrow(130.1) \tag{130}$$

$$\sup_{k}\sum_{n}|a_{nk}|<\infty \tag{130.1}$$

$$A\in(q^{\alpha},\ l)\Leftrightarrow(131.1),(131.2) \tag{131}$$

$$\sum_{n}|\sum_{k} a_{nk}|\text{ convergent} \tag{131.1}$$

$$\sup_{l}\sum_{n}|\binom{l+\alpha-1}{l})^{-1}\sum_{k=0}^{l}\binom{l-k+\alpha-1}{l-k}a_{nk}<\infty \tag{131.2}$$

$$A\in(bv,\ l)\Leftrightarrow(131.1),(132.1)\Leftrightarrow(132.2) \tag{132}$$

$$\sup_{l}\sum_{n}|\sum_{k=0}^{\infty}a_{nk}|<\infty \tag{132.1}$$

$$\sup_{l}\sum_{n}|\sum_{k=l}^{\infty}a_{nk}|<\infty \tag{132.2}$$

$$A\in(b\nu_0,\ l)\Leftrightarrow(132.1) \tag{133}$$

$$A\in(E_r(m),\ l)=(E_r(c),\ l)=(E_r(c_o),\ l)\Leftrightarrow(134.1) \tag{134}$$

$$\sup_{N,K} |\sum_{n\in N}\sum_{k\in K} k^{-r}\ a_{nk}| < \infty \tag{134.1}$$

$$A \in (E_r(l_p), l) \Leftrightarrow (135.1) \tag{135}$$

$$\sup_{N}\sum_{k} |\sum_{n\in N}\ k^{-r}\ a_{nk}|^q < \infty \tag{135.1}$$

$$A \in (E_r(l),\ l) \Leftrightarrow (136.1) \tag{136}$$

$$\sup_{k}\ \sum_{n} |k^{-r}\ a_{nk}| < \infty \tag{136.1}$$

$$A \in (D^{\Lambda}_{\infty},\ l) \Leftrightarrow (137.1) \tag{137}$$

$$\sum_{n}\sum_{k} |a_{nk},\ \vartheta_n| < \infty \tag{137.1}$$

$$A \in (m(p),\ l) = (c(p),\ l) = (c_o(p),\ l) \Leftrightarrow (138.1) \tag{138}$$

$$\sum_{n}\sum_{k} |a_{nk}| N^{\frac{1}{p_k}} < \infty \tag{138.1}$$

for every integer $N > 1$

$$A \in (m,\ q^{\beta}) \Leftrightarrow (22.1), (22.4), (139.1) \tag{139}$$

$$\sup_{N,\ K} |\sum_{n\in N}\sum_{k\in K} \binom{n+\beta-1}{n}) \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r}) a_{n+r,k}| < \infty \tag{139.1}$$

$$A \in (c,\ q^{\beta}) \Leftrightarrow (1.1), (22.1), (23.1), (139.1) \tag{140}$$

$$A \in (c_0,\ q^{\beta}) \Leftrightarrow (1.1), (22.1), (139.1) \tag{141}$$

$$A \in (m_s,\ q^{\beta}) \Leftrightarrow (2.1), (25.1), (25.4), (142.1) \tag{142}$$

$$\sup_{N,K} |\sum_{n\in N}\sum_{k\in K} \binom{n+\beta-1}{n} \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r}) a_{n+r,k} - a_{n+r,k+1}| < \infty \tag{142.1}$$

$$A \in (c_s,\ q^{\beta}) \Leftrightarrow (2.2), (22.1), (142.1) \tag{143}$$

$$A \in ((c_0)_s,\ q^{\beta}) \Leftrightarrow (2.2), (25.1), (142.1) \tag{144}$$

$$A \in (l_p,\ q^{\beta}) \Leftrightarrow (5.1), (22.1), (145.1) \tag{145}$$

$$\binom{sup_N \sum_k |\sum_{n\in N}(n+\beta-1}{n) \sum_{r=0}^{\beta}(-1)^r \binom{\beta}{r} a_{n+r,k}|^q} \tag{145.1}$$

$$A \in (l,\ q^{\beta}) \Leftrightarrow (6.1), (22.1), (146.1) \tag{146}$$

$$\sup_{k} \sum_{n} (\frac{n+\beta-1}{n})\ |\sum_{r=0}^{\beta}(-1)^{r}(\frac{\beta}{r})a_{n+r,k}| < \infty \tag{146.1}$$

$$A \in (q^{\alpha},\ q^{\beta}) \Leftrightarrow (7.2), (22.1), (23.1), (147.1), (147.2) \tag{147}$$

$$\sum_{n} (\frac{n+\beta-1}{n})|\sum_{r=0}^{\beta}(-1)^{r}(\frac{\beta}{r})a_{n+r,k}| \text{ convergent} \tag{147.1}$$

$$\sup\ \sum_{n} (\frac{n+\beta-1}{n})|\sum_{r=0}^{\beta}(-1)^{r}(\frac{\beta}{r})(\frac{l+\alpha-1}{l})^{-1} \tag{147.2}$$

$$\sum_{k=0}^{l} a_{n+r,k}(\frac{l-k+\alpha-1}{l-k})a_{nk}| < \infty$$

$$A \in (b\nu,\ q^{\beta}) \Leftrightarrow (8.1), (22.1), (23.1), (147.1), (148.1) \tag{148}$$

$$\Leftrightarrow (8.1), (22.1), (23.1), (148.2)$$

$$\sup_{l} \sum_{n} (\frac{n+\beta-1}{n})|\sum_{r=0}^{\beta}(-1)^{r}(\frac{\beta}{r})\sum_{k=0}^{l} a_{n+r,k}| < \infty \tag{148.1}$$

$$\sup_{l} \sum_{n} (\frac{n+\beta-1}{n})|\sum_{r=0}^{\beta}(-1)^{r}(\frac{\beta}{r})\sum_{k=l}^{\infty} a_{n+r,k}| < \infty \tag{148.2}$$

$$A \in (b\nu_0,\ q^{\beta}) \Leftrightarrow (8.1), (22.1), (148.1) \tag{149}$$

$$A \in (m,\ b\nu) = (c,\ b\nu) = (c_o, b\nu) \Leftrightarrow (150.1) \tag{150}$$

$$\sup_{N,\ K} |\sum_{n \in N}\sum_{k \in K}[(a_{nk} - a_{n-1,k})| < \infty \tag{150.1}$$

$$A \in (m_s,\ b\nu) \Leftrightarrow (2.1), (151.1) \tag{151}$$

$$\sup N,\ K|\sum_{n \in N}\sum_{k \in K}[(a_{nk} - a_{n,k+1}) - (a_{n,k-1} - a_{n-1,k+1})]| < \infty \tag{151.1}$$

$$A \in (c_s,\ b\nu) \Leftrightarrow (152.1) \tag{152}$$

$$\sup_{N,\ K} |\sum_{n \in N}\sum_{k \in K}[(a_{nk} - a_{n,k-1}) - (a_{n,k-1} - a_{n-1,k-1})]| < \infty \tag{152.1}$$

$$A \in ((c_0)_s,\ bv) \Leftrightarrow (151.1) \tag{153}$$

$$A \in (l_p,\ bv) \Leftrightarrow (154.1), p > 1 \tag{154}$$

$$\sup_N \sum_k |\sum_{n\in N}(a_{nk}-a_{n-1,k})|^q < \infty \tag{154.1}$$

$$A \in (l,\ b\nu) \Leftrightarrow (155.1) \tag{155}$$

$$\sup_k \sum_n |(a_{nk}-a_{n-1,k})| < \infty \tag{155.1}$$

$$A \in (q^\alpha,\ bv) \Leftrightarrow (156.1),(156.2) \tag{156}$$

$$\sum_n |\sum_k (a_{nk}-a_{n-1,k})| \text{ convergent} \tag{156.1}$$

$$\sup \quad \sum_n |(\frac{l+\alpha-1}{l})\sum_{k=0}^{l}(a_{nk}-a_{n-1,k})(\frac{l-k+\alpha-1}{l-k})| < \infty \tag{156.2}$$

$$A \in (b\nu,\ b\nu) \Leftrightarrow (156.1),(157.1) \Leftrightarrow (157.2) \tag{157}$$

$$\Leftrightarrow (157.1),(157.3)$$

$$\sup_l \quad \sum_k |\sum_{k=0}^{l}(a_{nk}-a_{n-1,k})| < \infty \tag{157.1}$$

$$\sup_l \quad \sum_k |\sum_{k=l}^{\infty}(a_{nk}-a_{n-1,k})| < \infty \tag{157.2}$$

$$\sum_l a_{nk} \text{ converges for all n} \tag{157.3}$$

$$A \in (b\nu_0,\ b\nu) \Leftrightarrow (157.1) \tag{158}$$

$$A \in (m,\ b\nu_0) \Leftrightarrow (48.1),(150.1) \tag{159}$$

$$A \in (c,\ b\nu_0) \Leftrightarrow (23.2),(49.1),(150.1) \tag{160}$$

$$A \in (c_0,\ b\nu_0) \Leftrightarrow (23.2),(150.1) \tag{161}$$

$$A \in (m_s,\ b\nu_0) \Leftrightarrow (2.1),(51.1),(151.1) \tag{162}$$

$$A \in (c_s,\ b\nu_0) \Leftrightarrow (23.2),(152) \tag{163}$$

$$A \in ((c_0)_s,\ b\nu_0) \Leftrightarrow (53.1),(151.1) \tag{164}$$

$$A \in (l_p,\ b\nu_0) \Leftrightarrow (23.2),(154.1) \tag{165}$$

$$A \in (l,\ b\nu_0) \Leftrightarrow (23.2), (155.1) \tag{166}$$

$$A \in (q^{\alpha},\ b\nu_0) \Leftrightarrow (23.2), (49.1), (156.1), (156.2) \tag{167}$$

$$A \in (b\nu,\ b\nu_0) \Leftrightarrow (23.2), (49.1), (157.1) \tag{168}$$

$$\Leftrightarrow (23.2), (49.1), (157.2)$$

$$A \in (bv_0,\ bv_0) \Leftrightarrow (23.2), (157.1) \tag{169}$$

$$A \in (m,\ \Gamma) = (c(p),\ \Gamma) = (c_o,\ \Gamma) \Leftrightarrow (170.1) \tag{170}$$

$$\lim_n (\sum_k |a_{nk}|)^{\frac{1}{n}} = 0 \tag{170.1}$$

$$A \in (l,\ \Gamma) \Leftrightarrow (171.1) \tag{171}$$

$$\lim_n |a_{nk}|^{\frac{1}{n}} = 0 \text{ uniformly in k} \tag{171.1}$$

$$A \in (\Gamma,\ \Gamma) \Leftrightarrow (172.1) \tag{172}$$

For each positive integer q, $\exists\ p(q) \ \geq\ q$ and a constant $M(p,q)$ such that for $k = 1, 2, ...$

$$\sum_n |a_{nk}| q^n p^{-k} \leq M \ \text{ independently of k} \tag{172.1}$$

$$A \in (\Lambda, \Gamma) \Leftrightarrow (173.1) \tag{173}$$

$$|f_n(z)|^{\frac{1}{n}} \to 0 \text{ as n} \to \infty \tag{173.1}$$

uniformly on every compact set (of the complex plane) where $\{f_n(z)\}$ is the sequence of all integral functions $f_n(z) = \sum_k a_{nk} z^k, (n = 1, 2, ...)$

$$A \in (E_r(m),\ \Gamma) = (E_r(c),\ \Gamma) = (E_r(c_o),\ \Gamma) \Leftrightarrow (174.1) \tag{174}$$

$$\lim_n (\sum_k k^{-r} |a_{nk}|)^{\frac{1}{n}} = 0 \tag{174.1}$$

$$A \in (E_r(l),\ \Gamma) \Leftrightarrow (175.1) \tag{175}$$

$$\lim_n |k^r a_{nk}|^{\frac{1}{n}} = 0 \tag{175.1}$$

where $n \to \infty$ uniformly in k

$$A \in (m,\ \Lambda) = (c,\ \Lambda) = (c_o,\ \Lambda) \Leftrightarrow (176.1) \tag{176}$$

$$\sup_n (\sum_k |a_{nk}|)^{\frac{1}{n}} = 0 \tag{176.1}$$

$$A \in (l,\ \Lambda) \Leftrightarrow (177.1) \tag{177}$$

$$\sup_{n,\ k} |a_{nk}|^{\frac{1}{n}} = \infty \tag{177.1}$$

$$A \in (\Gamma,\ \Lambda) \Leftrightarrow (178.1) \tag{178}$$

$$\sup_{n,\ k} |a_{nk}|^{\frac{1}{n+k}} = \infty \tag{178.1}$$

$$A \in (\Lambda,\ \Lambda) \Leftrightarrow (179.1) \tag{179}$$

For all $\epsilon > 0$, there exist $M = M(\epsilon)$ such that:

$$|a_{nk}| \leq\ \epsilon^k M^n (n, k = 1, 2, ...) \tag{179.1}$$

$$A \in (E_r(m),\ \Lambda) = (E_r(c),\ \Lambda) = (E_r(c_o),\ \Lambda) \Leftrightarrow (180.1) \tag{180}$$

$$\sup_n (\sum_k |k^{-r} a_{nk}|)^{\frac{1}{n}} = \infty \tag{180.1}$$

$$A \in (E_r(l),\ \Gamma) \Leftrightarrow (181.1) \tag{181}$$

$$\sup_{n,\ k} |k^{-r}\ a_{nk}|)^{\frac{1}{n}} = \infty \tag{181.1}$$

$$A \in (m,\ E_s(m)) = (c,\ E_s(m)) = (c_o,\ E_s(m)) \Leftrightarrow (182.1) \tag{182}$$

$$\sup_n (\sum_k |n^s\ a_{nk}|) < \infty \tag{182.1}$$

$$A \in (l,\ E_s(m)) \Leftrightarrow (183.1) \tag{183}$$

$$\sup_{n,\ k} |n^s a_{nk}| < \infty \tag{183.1}$$

$$A \in (\Gamma,\ E_s(m)) \Leftrightarrow (184.1) \tag{184}$$

$$\sup_{n,\ k} |n^s\ a_{nk}|^{\frac{1}{k}} = \infty \tag{184.1}$$

$$A \in (\Lambda,\ E_s(m)) \Leftrightarrow (185.1), (185.2) \tag{185}$$

The sequence $\{f_n(z)\}$ where

$$f_n(z) = \sum_{k=1}^{\infty} n^s a_{nk}\ z^k\ (n = 1, 2, ...) \tag{185.1}$$

of integral functions is uniformly bounded on every compact set of the complex plane

$$\lim_n \{n^s\ a_{nk}\} \text{ exists for each k} \tag{185.2}$$

$$A \in (E_r(m),\ E_s(m)) = (E_r(c), E_s(m)) = (E_r(c_o), E_s(m)) \Leftrightarrow (186.1) \tag{186}$$

$$\sup_n \sum_k |n^s\ a_{nk}| < \infty \tag{186.1}$$

$$A \in (E_r(l_p),\ E_s(m)) \Leftrightarrow (187.1) \tag{187}$$

$$\sup_n \sum_k |n^s\ k^{-r}\ a_{nk}|^q < \infty, \tag{187.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$

$$A \in (E_r(l),\ E_s(m)) \Leftrightarrow (187.1) \tag{188}$$

$$\sup_{n,\ k} |n^s\ k^{-r}\ a_{nk}| < \infty \tag{188.1}$$

$$A \in (m,\ E_s(c)) \Leftrightarrow (189.1)(189.2) \tag{189}$$

$$\lim_n |n^s\ a_{nk}| \text{ exists for all k} \tag{189.1}$$

$$\sum_k |n^s\ a_{nk}| \text{ converges uniformly in n} \tag{189.2}$$

$$A \in (c,\ E_s(c)) \Leftrightarrow (182.1), (189.1), (190.1) \tag{190}$$

$$\lim_n \sum_n |n^s\ a_{nk}| \text{ exists} \tag{190.1}$$

$$A \in (c_o,\ E_s(c)) \Leftrightarrow (182.1), (189.1) \tag{191}$$

$$A \in (l,\ E_s(c)) \Leftrightarrow (183.1), (189.1) \tag{192}$$

$$A \in (\Gamma,\ E_s(c)) \Leftrightarrow (184.1), (189.1) \tag{193}$$

$$A \in (\Lambda,\ E_s(c)) = (\Lambda, E_s(c_o)) \Leftrightarrow (185.1), (189.1) \tag{194}$$

$$A \in (E_r(m),\ E_s(c)) \Leftrightarrow (195.1), (195.2) \tag{195}$$

$$\lim_n |n^s\ k^{-r}\ a_{nk}| \ \text{ exists for all k} \tag{195.1}$$

$$\sum_k |n^s\ k^{-r}\ a_{nk}| \text{ converges uniformly in n} \tag{195.2}$$

$$A \in (E_r(c),\ E_s(c)) \Leftrightarrow (186.1), (195.1), (196.1) \tag{196}$$

$$\lim_n \sum_k |n^s\ k^{-r} a_{nk}| \text{ exists} \tag{196.1}$$

$$A \in (E_r(c_o),\ E_s(c)) \Leftrightarrow (186.1), (195.1) \tag{197}$$

$$A \in (E_r(l_p),\ E_s(c)) \Leftrightarrow (187.1), (195.1) \tag{198}$$

$$A \in (E_r(l),\ E_s(c)) \Leftrightarrow (186.1), (195.1) \tag{199}$$

$$A \in (m,\ E_s(c_o)) \Leftrightarrow (200.1) \tag{200}$$

$$\lim_n \sum_k |n^s\ a_{nk}| = 0 \tag{200.1}$$

$$A \in (c,\ E_s(c_o)) \Leftrightarrow (182.1), (201.1), (201.2) \tag{201}$$

$$\lim_n \{n^s\ a_{nk}\} = 0 \ \text{ for all k} \tag{201.1}$$

$$\lim_n \sum_k |n^s\ a_{nk}| = 0 \tag{201.2}$$

$$A \in (c_o,\ E_s(c_o)) \Leftrightarrow (182.1), (201.1) \tag{202}$$

$$A \in (l,\ E_s(c_o)) \Leftrightarrow (183.1), (201.1) \tag{203}$$

$$A \in (\Gamma,\ E_s(c_o)) \Leftrightarrow (184.1), (201.1) \tag{204}$$

$$A \in (E_r(m),\ E_s(c_o)) \Leftrightarrow (205.1) \tag{205}$$

$$\lim_n \sum_k |n^s\ k^{-r} a_{nk}| = 0 \tag{205.1}$$

$$A \in (E_r(c),\ E_s(c_o)) \Leftrightarrow (186.1), (206.1), (206.2) \tag{206}$$

$$\lim_n \{n^s k^{-r}\ a_{nk}\} = 0 \ \text{ for all k} \tag{206.1}$$

$$\lim_n \sum_k |n^s\ k^{-r} a_{nk}| = 0 \tag{206.2}$$

$$A \in (E_r(c_o),\ E_s(c_o)) \Leftrightarrow (186.1), (206.1) \tag{207}$$

$$A \in (E_r(l_p),\ E_s(c_o)) \Leftrightarrow (187.1), (206.1) \tag{208}$$

$$A \in (E_r(l),\ E_s(c_o)) \Leftrightarrow (188.1), (206.1) \tag{209}$$

$$A \in (m,\ E_s(l_t)) = (c,\ E_s(l_t)) = (c_o,\ E_s(l_t)) \Leftrightarrow (210.1) \tag{210}$$

$$\sup_k \sum_n |\sum_{k \in K} n^s\ a_{nk}|^t < \infty \tag{210.1}$$

$$A \in (l,\ E_s(l_t)) \Leftrightarrow (211.1) \tag{211}$$

$$\sup_k \sum_n |n^s\ a_{nk}|^t < \infty \tag{211.1}$$

$$A \in (E_r(m),\ E_s(l_t)) = (E_r(c),\ E_s(l_t)) = (E_r(c_o),\ E_s(l_t)) \Leftrightarrow (212.1) \tag{212}$$

$$\sup_k \sum_n |\sum_{k \in K} n^s k^{-r}\ a_{nk}|^t < \infty \tag{212.1}$$

$$A \in (E_r(l),\ E_s(l_t)) \Leftrightarrow (213.1) \tag{213}$$

$$\sup_k \sum_n |n^s k^{-r}\ a_{nk}|^t < \infty \tag{213.1}$$

$$A \in (m,\ E_s(l)) = (c,\ E_s(l)) = (c_o,\ E_s(l)) \Leftrightarrow (214.1) \tag{214}$$

$$\sup_{n,\ k} |\sum_{n \in N} \sum_{k \in K} |n^s\ a_{nk}| < \infty \tag{214.1}$$

$$A \in (l,\ E_s(l)) \Leftrightarrow (215.1) \tag{215}$$

$$\sup_k \sum_n |n^s\ a_{nk}| < \infty \tag{215.1}$$

$$A \in (E_r(m),\ E_s(l)) = (E_r(c),\ E_s(l)) = (E_r(c_o),\ E_s(l)) \Leftrightarrow (216.1) \tag{216}$$

$$\sup_{N,K} |\sum_{n\in N}\sum_{k\in K} n^s\ k^{-r}\ a_{nk}| < \infty \tag{216.1}$$

$$A \in (E_r(l_p),\ E_s(l)) \Leftrightarrow (217.1) \tag{217}$$

$$\sup_{N} \sum_{k} |\sum_{n\in N} n^s\ k^{-r}\ a_{nk}|^q < \infty \tag{217.1}$$

where, $\frac{1}{p}+\frac{1}{q}=1$

$$A \in (E_r(l),\ E_s(l)) \Leftrightarrow (218.1) \tag{218}$$

$$\sup_{k} \sum_{n} |n^s\ k^{-r}\ a_{nk}| < \infty \tag{218.1}$$

$$A \in (m,\ D_\infty^\Lambda \Leftrightarrow (219.1) \tag{219}$$

$$\sup_{n} \sum_{k} |a_{n,k},\ y_n| < \infty \tag{219.1}$$

$$A \in (l,\ D_\infty^\Lambda) \Leftrightarrow (220.1) \tag{220}$$

$$\sup_{n,\ k} |a_{n,k},\ y_n| < \infty \tag{220.1}$$

$$A \in (D_\infty^\Lambda,\ D_\infty^\Lambda) \Leftrightarrow (222) \tag{221}$$

$$A \in (D_\infty^\Lambda(p),\ D_\infty^\Lambda) \Leftrightarrow (222.1) \tag{222}$$

$$\sup_{n,\ k} \sum_{k} |a_{n,\ k}, \frac{\vartheta_n}{\vartheta_k}| N^{\frac{1}{p_k}} < \infty \text{ for every integer} N > 1 \tag{222.1}$$

$$A \in (D_\infty^\Lambda,\ D_\infty^\Lambda) \Leftrightarrow (223.1) \tag{223}$$

$$\sup_{n,\ k} |a_{n,\ k}|, |\frac{\vartheta_n}{\vartheta_k}| < \infty \tag{223.1}$$

$$A \in (l(p),\ D_\infty^\Lambda) \Leftrightarrow (224.1) \tag{224}$$

There exists an integer $B > 1$ such that

$$C(B) = \sup_{n} \sum_{k} |a_{n,\ k}, \vartheta_n|^{q_k} < \infty \tag{224.1}$$

where, $\frac{1}{p_k}+\frac{1}{q_k}=1, 1 < p_k \le H < \infty$

$$A \in (c_o(p),\ D_\infty^\Lambda(q)) \Leftrightarrow (225.1) \tag{225}$$

Let p, q $\in l_\infty$ there exists an absolute constant $B > 1$ such that

$$B = \sup_n \{\sum_k |a_{n,\ k}, \vartheta_n|^{\frac{-1}{p_k}}\}_n^q < \infty \tag{225.1}$$

$$A \in (l,\ D_\infty^\Lambda) \Leftrightarrow (226.1) \tag{226}$$

$$\sup_k \sum_n |a_{n,\ k}, \vartheta_n|^p < \infty \tag{226.1}$$

$$A \in (D_1^\Lambda,\ D_p^\Lambda) \Leftrightarrow (227.1) \tag{227}$$

$$\sup_k \sum_n |a_{n,\ k}|^p\ |\frac{\vartheta_n}{\vartheta_k}|^p < \infty, 1 \le p < \infty \tag{227.1}$$

$$A \in (l_\infty,\ D_1^\Lambda) \Leftrightarrow (228.1) \tag{228}$$

$$\sum_n \sum_k |a_{n,\ k}, \vartheta_k| < \infty \tag{228.1}$$

$$A \in (D_\infty^\Lambda,\ D_1^\Lambda) \Leftrightarrow (229.1) \tag{229}$$

$$\sum_n \sum_k |a_{n,\ k}, \frac{\vartheta_n}{\vartheta_k}| < \infty \tag{229.1}$$

$$A \in (D_1^\Lambda,\ D_1^\Lambda, p) \Leftrightarrow (230.1), (230.1) \tag{230}$$

$$\sup_k |a_{n,\ k}, \frac{\vartheta_n}{\vartheta_k}| < \infty \tag{230.1}$$

$$\sum_n |a_{n,\ k}, \frac{\vartheta_n}{\vartheta_k}| = 1 \text{ for all k} \tag{230.2}$$

$$A \in (m,\ \hat{c}) \Leftrightarrow (231.1), (231.2), (231.3) \tag{231}$$

$$\sup_m \sum_k |a_{n,k,m}| < \infty \tag{231.1}$$

$$\lim_{m\to\infty} a(n,k,m)| = \alpha_k \text{ uniformly in n} \tag{231.2}$$

$$\lim_{m\to\infty} \sum_k |a(n,k,m)| - \alpha_k| = 0 \text{ uniformly in n} \tag{231.3}$$

$$A \in (c,\ \hat{c}) \Leftrightarrow (231.1), (231.2), (232.1) \tag{232}$$

$$A \in (c,\ \hat{c}, p) \Leftrightarrow (231.1), (231.2), (232.3)$$

$$\lim_{m\to\infty}\sum_k a(n,k,m)| = \alpha \text{ uniformly in n} \tag{232.1}$$

$$\lim_{m\to\infty} a(n,k,m)| = 0 \text{ uniformly in n} \tag{232.2}$$

$$\lim_{m\to\infty}\sum_k a(n,k,m)| = 1 \text{ uniformly in n} \tag{232.3}$$

$$A \in (c_o,\ \hat{c}) \Leftrightarrow (231.1), (231.2) \tag{233}$$

$$A \in (l_p,\ \hat{c}) \Leftrightarrow (231.2), (234.2),\ 1 < p < \infty\ \frac{1}{p} + \frac{1}{q} = 1 \tag{234}$$

$$A \in (l_p,\ \hat{c}) \Leftrightarrow (231.2), (234.2), 0 < p < 1$$

$$\sup_m \sum_k |a_{n,k,m}|^p < \infty, (n = 1, 2, ...) \tag{234.1}$$

$$\sup_{m,k} |a_{n,k,m}|^p < \infty, (n = 1, 2, ...) \tag{234.2}$$

$$A \in (l,\ \hat{c}) \Leftrightarrow (231.2), (234.2) \tag{235}$$

$$A \in (b\nu,\ \hat{c}) \Leftrightarrow (231.2), (232.2), (236.1) \tag{236}$$

$$\sup_m |\sum_{k=r}^{\infty} a_{n,k,m}| < \infty, (n, r = 1, 2, ...) \tag{236.1}$$

$$A \in (b\nu_o,\ \hat{c}) \Leftrightarrow (231.2), (236.1) \tag{237}$$

$$\text{Let}, A \in (c,\ \hat{c}, P) \text{ then}, A \in (\hat{c},\ \hat{c}) \Leftrightarrow (238.1) \tag{238}$$

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_k |\sum_{i=0}^{m} (a_{n+i,k} - a_{n+i,k+i})| = 0 \tag{238.1}$$

$$A \in (m(p),\ \hat{c}) \Leftrightarrow (231.1), (231.2), (239.1) \tag{239}$$

$$\lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k|\ N^{\frac{1}{p_k}} = 0 \text{ uniformly in n for every integer } N > 1 \tag{239.1}$$

$$A \in (c(p),\ \hat{c}) \Leftrightarrow (231.2), (231.2), (239.1) \tag{240}$$

There exist some integer $B > 1$such that

$$\sup_m \sum_k |a(n,k,m)| B^{\frac{-1}{p_k}} < \infty, (n = 1, 2, ...) \tag{240.1}$$

$$A \in (c_0(p),\ \hat{c}) \Leftrightarrow (231.2), (240.1) \tag{241}$$

$$A \in (l(p),\ \hat{c}) \Leftrightarrow (231.2), (242.1),\ 1 < p_k \leq H < \infty\ \frac{1}{p_k} + \frac{1}{q_k} = 1 \text{ for every k} \tag{242}$$

$$A \in (l(p),\ \hat{c}) \Leftrightarrow (231.2), (242.2),\ 0 < p_k < 1$$

$$\sup_m \sum_k |a(n,k,m)|^{q_k} B^{-q_k} < \infty, (n = 1, 2, ...) \tag{242.1}$$

$$\sup_{m,\ k} |a(n,k,m)|^{p_k} < \infty, (n = 1, 2, ...) \tag{242.2}$$

$$A \in (w,\ \hat{c}) \Leftrightarrow (231.2), (232.1), (243.1) \tag{243}$$

$$\sup_m \sum_{t=0}^{\infty} \max_t\ 2^t |a(n,k,m)| < \infty \tag{243.1}$$

$$A \in (w(p),\ \hat{c}) \Leftrightarrow (231.2), (232.1), (244.1) \tag{244}$$

$$\sup_m \sum_{t=0}^{\infty} \max_t\ (2^t B^{-1})^{\frac{1}{p_k}}\ |a(n,k,m)| < \infty, 0 < p_k \leq 1 \tag{244.1}$$

$$A \in (w_p,\ \hat{c}) \Leftrightarrow (231.2), (232.1), (245.1) \tag{245}$$

$$\sup_m \sum_{t=0}^{\infty} 2^{\frac{t}{p}} (\sum_t |a(n,k,m)|^q)^{\frac{1}{q}} < \infty, 1 < p < \infty, \text{ and } \frac{1}{p_k} + \frac{1}{q_k} = 1 \tag{245.1}$$

[The summation is taken over k satisfying $2^t \leq 2^{t+1}$]

$$\sup_m \sum_{t=0}^{\infty}\ \max_t\ 2^{\frac{t}{p}} |a(n,k,m)| < \infty,\ 0 < p < 1 \tag{245.2}$$

[The maximum is taken over k satisfying $2^t \leq 2^{t+1}$]

$$A \in (m,\ c_o^{\Lambda}) \Leftrightarrow (231.1), (246.1) \tag{246}$$

$$\lim_{m \to \infty}\ \sum_k |a(n,k,m)| = 0 \text{ uniformly in n} \tag{246.1}$$

$$A \in (c,\ c_o^{\Lambda}) \Leftrightarrow (231.1), (232.2), (247.1) \tag{247}$$

$$\lim_{m \to \infty} \sum_k a(n,k,m) = 0 \text{ uniformly in n} \tag{247.1}$$

$$A \in (c_o,\ c_o^{\Lambda}) \Leftrightarrow (231.1), (231.2) \tag{248}$$

$$A \in (l_p,\ c_o^{\Lambda}) \Leftrightarrow (232.2), (234.1), 1 < p < \infty \tag{249}$$

$$A \in (l_p,\ c_o^\Lambda) \Leftrightarrow (232.2), (234.2), 1 < p < 1$$

$$A \in (l,\ c_o^\Lambda) \Leftrightarrow (234.2), (232.2) \tag{250}$$

$$A \in (b\nu,\ c_o^\Lambda) \Leftrightarrow (232.2), (236.1), (247.1) \tag{251}$$

$$A \in (b\nu_o,\ c_o^\Lambda) \Leftrightarrow (232.2), (236.1) \tag{252}$$

$$\text{Let } A \in (c_o,\ c_o^\Lambda),\ \text{then } A \in (c_o^\Lambda,\ c_o^\Lambda) \Leftrightarrow (238.1) \tag{253}$$

$$A \in (m(p),\ c_o^\Lambda) \Leftrightarrow (231.1), (254.1) \tag{254}$$

$$\lim_{m\to\infty} (\sum_k |a(n,k,m)|N)^{\frac{1}{p_k}} = 0 \tag{254.1}$$

uniformly in n for every integer $N > 1$

$$A \in (c(p),\ c_o^\Lambda) \Leftrightarrow (232.2, (240.1), (247.1) \tag{255}$$

$$A \in (c_o(p),\ c_o^\Lambda) \Leftrightarrow (232.2), (240.1) \tag{256}$$

$$A \in (l(p),\ c_o^\Lambda) \Leftrightarrow (232.2), (242.2), 0 < p_k \le 1 \tag{257}$$

$$A \in (w,\ c_o^\Lambda) \Leftrightarrow (232.2), (243.1), (247.1) \tag{258}$$

$$A \in (w(p),\ c_o^\Lambda) \Leftrightarrow (232.2), (244.1), (247.1) \tag{259}$$

$$A \in (w_p,\ c_o^\Lambda) \Leftrightarrow (232.2), (245.1), (247.1), 1 < p < \infty \tag{260}$$

$$A \in (w_p,\ c_o^\Lambda) \Leftrightarrow (232.2), (245.2), 0 < p < 1$$

$$A \in (m,\ l_r^\Lambda) = (c_o,\ l_r^\Lambda) = (c^\Lambda,\ l_r^\Lambda) = (c_o^\Lambda,\ l_r^\Lambda) \Leftrightarrow (261.1) \tag{261}$$

$$\sup_n \sum_m (\sum_k |b(n,k,m)|)^r < \infty \tag{261.1}$$

$$A \in (l,\ l_r^\Lambda) \Leftrightarrow (262.1) \tag{262}$$

$$\sup_{n,k} \sum |b(n,k,m)|)^r < \infty, 1 < r < \infty \tag{262.1}$$

$$A \in (m(p),\ l_r^\Lambda) = (c(p), l_r^\Lambda), = (c_o(p),\ l_r^\Lambda) \Leftrightarrow (263.1) \tag{263}$$

$$\sup_n \sum_m (\sum_k |b(n,k,m)| N^{\frac{1}{p_k}}) < \infty \tag{263.1}$$

for every integer $N > 1$

$$A \in (m,\ l^{\Lambda}) = (c,\ l^{\Lambda}), = (c_o^{\Lambda},\ l^{\Lambda}) = (c_o^{\Lambda},\ l^{\Lambda}) \Leftrightarrow (265) \tag{264}$$

$$A \in (l_p,\ l^{\Lambda}) \Leftrightarrow (265.1) \tag{265}$$

$$\sum_{n,\ m} \sum_k |\sum_m b(n,k,m)|^q < \infty, 1 < p < \infty, \frac{1}{p_k} + \frac{1}{q_k} = 1 \tag{265.1}$$

$$A \in (l,\ l^{\Lambda}) \Leftrightarrow (266.1) \tag{266}$$

$$\sup_{n,\ k} \sum_m |b(n,k,m)| < \infty \tag{266.1}$$

$$A \in (m(p),\ l^{\Lambda}) = (c(p),\ l^{\Lambda}) = (c_o(p),\ l^{\Lambda}) \Leftrightarrow (267.1) \tag{267}$$

$$\sup_{n,\ m} \sum_k |b(n,k,m)| N^{\frac{1}{p_k}}) < \infty \tag{267.1}$$

for every integer $N > 1$

$$A \in (m,\ b\nu^{\Lambda}) = (c,\ b\nu^{\Lambda}) = (c_o,\ b\nu^{\Lambda}) = (c^{\Lambda},\ b\nu^{\Lambda}) \tag{268}$$

$$= (c_o^{\Lambda},\ b\nu^{\Lambda}) \Leftrightarrow (268.1), (268.2), (268.3)$$

There is a constant $K > 0$ such that

$$\sum_m \sum_k |c(n,k,m)| \leq K \text{ for all n} \tag{268.1}$$

$$\lim_{m \to \infty} \sum_k |c(n,\ k,\ m)| = \alpha_k \text{ uniformly in n} \tag{268.2}$$

$$\lim_{m \to \infty} \sum_k |c(n,\ k,\ m) - \alpha_k| = 0 \text{ uniformly in n} \tag{268.3}$$

$$A \in (b\nu,\ b\nu^{\Lambda}) \Leftrightarrow (269.1), (269.2), (268.2) \tag{269}$$

There is a constant K such that

$$\sum_m \sum_{k=0}^{r} |c(n,\ k,\ m)| \leq K(r, n = 1, 2, ...) \tag{269.1}$$

$$\lim_{m\to\infty}\sum_k \ |c(n,\ k,\ m)| = \alpha_k \text{ uniformly in n} \tag{269.2}$$

$$A \in (c_o,\ c_o^{\Lambda}(r)) \Leftrightarrow (270.1), (270.2) \tag{270}$$

$$\sup_m\{\sum_m |a(n,\ k,\ m)|\}^{r_m} < \infty (n = 1, 2, ...) \tag{270.1}$$

$$\lim_{m\to\infty} |a(n,\ k,\ m)|^{r_m} = 0 \ \text{ uniformly in n} \tag{270.2}$$

$$A \in (c_o(p),\ c_o^{\Lambda}(r)) \Leftrightarrow (270.2), (271.1) \tag{271}$$

There exists an integer $n > 1$ such that

$$\sup_m\{\sum_k |a(n,\ k,\ m)|B^{\frac{-1}{p_k}}\}^{r_m} < \infty \tag{271.1}$$

$$A \in (m,\ c_1) \Leftrightarrow (272.1), (272.2) \tag{272}$$

$$\sum_k |b_{nk}| \text{ converge uniformly in n} \tag{272.1}$$

There exists $\alpha_k \in C$ such that

$$\lim_{n\to\infty} b_{nk} = \alpha_k \text{ for all k} \tag{272.2}$$

$$A \in (m(p),\ c_1) \Leftrightarrow (272.2), (273.1) \tag{273}$$

For all integer $N > 1$

$$\sum_k |b_{nk}|N^{\frac{1}{p_k}} \text{ converges uniformly in n} \tag{273.1}$$

$$A \in (c(p),\ c_1) \Leftrightarrow (272.2), (274.1), (274.2) \tag{274}$$

$$D = \sup_n \sum_k |b_{nk}|B^{\frac{-1}{p_k}} < \infty \ \text{ for some integer} B > 1 \tag{274.1}$$

There exists $\alpha \in C$ such that

$$\lim_{n\to\infty}\sum_k b_{nk} = \alpha \tag{274.2}$$

$$A \in (l(p),\ c_1) \Leftrightarrow (272.2), (275.1) \tag{275}$$

There exist an integer $B > 1$ such that

$$C(B) < \infty \ (1 < p_k < \infty) \ \sup_{n,k} |b_{nk}|^{p_k} < \infty \ (0 < p_k < 1) \tag{275.1}$$

$$A \in (m,\ |c_1|_p) \Leftrightarrow (276.1) \tag{276}$$

$$\sum_n (\sum_k |a_{nk}|)^p < \infty \tag{276.1}$$

$$\text{Let}, p \ \geq \ 1, A \in (l,\ |c_1|_p) \Leftrightarrow (278) \tag{277}$$

$$A \in (m(p),\ |c_1|_p) \Leftrightarrow (278.1) \tag{278}$$

For every integer $N > 1$

$$\sum_n (\sum_k |c_{nk}| N^{\frac{1}{p_k}})^p < \infty \tag{278.1}$$

$$A \in (l_1,\ |c_1|) \Leftrightarrow (280) \tag{279}$$

$$A \in (l_1,\ |c_1|,\ p) \Leftrightarrow (280.1), (280.2) \tag{280}$$

$$\sup_k \sum_n |c_{nk}| < \infty \tag{280.1}$$

$$\sum_n c_{nk} = 1 \text{ for all k} \tag{280.2}$$

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$X \backslash Y$	m	c	c_o	m_s	c_s	$(c_o)_s$	l_r	m_r	$c_o(r)$	l	q^β	$b\upsilon$	$b\upsilon_o$	Γ	Λ	$E_s(m)$	$E_s(c)$	$E_s(c_o)$	$E_s(l_t)$	$E_s(l)$	D_∞^Λ	$D_\infty^\Lambda(q)$	D_p^Λ	D_1^Λ	c^Λ	c_o^Λ	l_r^Λ	l^Λ	$\overset{\Lambda}{b\upsilon}$	$c_o^\Lambda(r)$	c_1	$\lvert c_1\rvert_\infty$	$\lvert c_1\rvert$
m	1	22	48	74	84	98	110	-	-	125	139	150	159	170	176	182	189	200	210	214	219	-	-	228	230	246	261	264	268	-	272	276	-
c	1	23	49	74	85	99	110	-	-	125	140	150	160	170	176	182	190	201	210	214	-	-	-	-	231	247	261	264	268	-	-	276	-
c_o	1	24	50	74	86	100	110	121	123	125	141	150	161	170	176	182	191	202	210	214	-	-	-	-	232	248	261	264	268	270	-	-	-
m_s	2	25	51	75	87	101	111	-	-	126	142	151	162	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
c_s	3	26	52	76	88	102	112	-	-	127	143	152	163	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$(c_o)_s$	4	27	53	77	89	103	113	-	-	128	144	153	164	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
l_o	5	28	54	78	90	104	114	-	-	129	145	154	165	-	-	-	-	-	-	-	-	-	-	-	234	249	-	265	-	-	-	-	-
l	6	29	55	79	91	105	115	-	-	130	146	155	166	171	177	183	192	203	211	215	220	-	226	-	235	250	262	266	-	-	-	277	279
q^α	7	30	56	80	92	106	116	-	-	131	147	156	167	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$b\upsilon$	8	31	57	81	93	107	117	-	-	132	148	157	168	-	-	-	-	-	-	-	-	-	-	-	236	252	-	-	269	-	-	-	-
$b\upsilon_o$	9	32	58	82	94	108	118	-	-	133	149	158	169	-	-	-	-	-	-	-	-	-	-	-	237	252	-	-	-	-	-	-	-
Γ	10	33	59	-	-	-	-	-	-	-	-	-	-	172	178	184	193	204	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
Λ	11	34	60	-	-	-	-	-	-	-	-	-	-	173	179	185	194	194	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$E_r(m)$	12	35	61	-	-	-	-	-	-	134	-	-	-	174	180	186	195	205	212	216	-	-	-	-	-	-	-	-	-	-	-	-	-
$E_r(c)$	12	36	62	-	-	-	-	-	-	134	-	-	-	174	180	186	196	206	212	216	-	-	-	-	-	-	-	-	-	-	-	-	-
$E_r(c_o)$	12	37	63	-	-	-	-	-	-	134	-	-	-	174	180	186	197	207	212	216	-	-	-	-	-	-	-	-	-	-	-	-	-
$E_r(lp)$ $1 < p < \infty$	13	38	64	-	-	-	-	-	-	135	-	-	-	-	-	187	198	208	-	217	-	-	-	-	-	-	-	-	-	-	-	-	-
$E_r(l)$	14	39	65	-	-	-	-	-	-	136	-	-	-	175	181	188	199	209	213	218	-	-	-	-	-	-	-	-	-	-	-	-	-
D_∞^Λ	15	-	-	-	-	-	-	-	-	137	-	-	-	-	-	-	-	-	-	-	221	-	-	229	-	-	-	-	-	-	-	-	-
$D_\infty^\Lambda(p)$	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
D_p^Λ	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
D_1^Λ	16	-	-	-	-	-	119	-	-	-	-	-	-	-	-	-	-	-	-	-	223	-	227	230	-	-	-	-	-	-	-	-	-
c^Λ	-	40	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	238	-	261	264	263	-	-	-	-
c_o^Λ	-	-	66	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	253	261	264	268	-	-	-	-
$m(p)$	17	41	67	83	95	109	120	-	-	138	-	-	-	-	-	-	-	-	-	-	-	-	-	-	239	254	263	267	-	-	273	278	-
$c(p)$	17	42	68	83	96	-	120	-	-	138	-	-	-	-	-	-	-	-	-	-	-	-	-	-	240	255	263	267	-	-	274	-	-
$c_o(p)$	17	43	69	83	-	-	120	122	124	138	-	-	-	-	-	-	-	-	-	-	-	225	-	-	241	256	263	267	-	271	-	-	-
$l(p)$	18	44	70	-	97	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	224	-	-	-	242	257	-	-	-	-	275	-	-
ω	19	45	71	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	243	258	-	-	-	-	-	-	-
$\omega(p)$	20	46	72	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	244	259	-	-	-	-	-	-	-
ω_p	21	47	73	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	245	260	-	-	-	-	-	-	-

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Contents