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DEGREE OF APPROXIMATION OF FOURIER SERIES BY A NEW TRIGONOMETRIC MEAN $R_1(\beta)$, FOR $\beta = 2$

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Abstract. In this paper we have estimated the rate of convergence of Fourier series of functions belonging to the generalized Hölder metric space H_p^ω with the norm $||f||_p^{(\omega)} := \|f\|_p + A(f,\omega)$, by using a new trigonometric mean $R_1(\beta)$, for $\beta = 2$.

Keywords: Trigonometric mean; Fourier series; Generalized Hölder metric space

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1. Introduction

Many results are obtained in the theory of approximation, particularly in the study of approximation of continuous function by polynomials. Perhaps the best known result is due to Weierstrass [10].

Theorem 1.1. *If $f \in C[a,b]$, then for every $\varepsilon > 0$ there exists an algebraic polynomial P such that*

$$| f(x) - P | < \varepsilon.$$

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Later the study was extended to approximate the piecewise continuous periodic function by trigonometric polynomials.The study of the theory of trigonometric approximation is of great mathematical interest and of great practical importance. Basically signals are treated as function of single variable and images are represented by function of two variables. The study of these concepts is directly related to the trending area of information technology.Then the development happened for dealing with estimating the error which occurs in approximation. The investigation of this depend on the smoothness of the approximated function, as well as the study and comparison of the approximation properties of various approximation methods such as Euler mean, Cèsaro mean, Nörland mean, Matrix mean etc. which led to the creation of the approximation theory of functions, one of the most rapidly developing branches of mathematical analysis.The approximation of functions by Fourier series, conjugate series based on trigonometric polynomial is a very closely related topic in the recent development of engineering and mathematics. In the approximation process it was observed that we are getting minimum error by taking Fourier coefficients in the $n-$th degree trigonometric polynomial. The error approximation of periodic functions belonging to generalized Hölder metric space using different summability methods is also an active area of research in the last decades. The degree of approximation of functions belonging to H_p^{ω} using different summability methods have been studied by a number of researchers [9, 12, 8, 3, 2, 7] and many others. Recently Nigam and Rani [11] estimated the approximation of conjugate of a function in the generalized Hölder class H_p^{ω}.

In an attempt to make an advance study in this direction, we estimate the rate of convergence of Fourier series of functions belonging to the generalized Hölder metric space H_p^{ω} by using a trigonometric mean $R_1(\beta)$, for $\beta = 2$.

2. Definition and Notation

Trigonometric polynomial we mean a function of the form

$$T_N(f;x) = \sum_{n=0}^{N} (a_n cos(nx) + b_n sin(nx))$$

Where $x \in \mathbb{R}$, $a_n, b_n \in \mathbb{C}, \forall n$

And the degree of the trigonometric polynomial is N, provided $|a_N| + |b_N| \neq 0$ Let f be a 2π-periodic and integrable function in the sense of Lebesgue, then the Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k cos(kx) + b_k sin(kx)), \forall k \geq 1$$

where a_0, a_k and b_k are Fourier coefficients defined as:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx.$$
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx.$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx.$$

Let n-th partial sum of the above Fourier series is denoted by

$$s_n(x) = s_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k cos(kx) + b_k sin(kx)),$$

which is called as trigonometric polynomial of degree n, of the Fourier Series of f.

It is known that[14]

$$s_n(f;x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi_x(t) \frac{sin\left(n + \frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)} dt,$$

where $\phi_x(t) = \frac{1}{2}\left[f(x+t) + f(x-t) - 2f(x)\right]$.

Approximation of $f \in L_p[0, 2\pi]$ by trigonometric polynomials $T_n(f;x)$ of degree n, which is obtained from the Fourier series of f is called trigonometric Fourier approximation and the degree of approximation $E_n(f)$ is given by [14]

$$E_n(f) = \underset{n}{Min}\|f(x) - T_n(f;x)\|_p.$$

Let $C[0,2\pi]$ denote the Banach space of continuous periodic functions with period 2π under the supremum norm. The H^ω- class of functions has been defined by Leindler [9] as

$$H^\omega = \{f \in C[0,2\pi] : \omega(\delta) = O(\omega(\delta))\},$$

where ω is a modulus of continuity, which is a positive nondecreasing continuous function on $[0,2\pi]$. Modulus of continuity with $p-$norm is given by

$$\omega(\delta) = \sup_{0\leq h\leq\delta} \|f(x+h) - f(x)\|_c$$
$$\omega_p(\delta) = \sup_{0\leq h\leq\delta} \|f(x+h) - f(x)\|_p$$
$$\omega_p^{(2)}(\delta) = \sup_{0\leq h\leq\delta} \|f(x+h) + f(x-h) - 2f(x)\|_p$$

Some useful properties of the modulus of continuity are listed bellow:

(i) $\omega(0) = 0$.

(ii) If $0 < \delta_1 \leq \delta_2$, then $\omega(\delta_1) \leq \omega(\delta_2)$.

(iii) If $\lambda > 0$, then $\omega(\lambda\delta) \leq (1+\lambda)\omega(\delta)$.

Lemma 2.1. *let* $0, \delta \leq 1$ *and* $\frac{\omega(t)}{\nu(t)}$ *non decreasing. Then*

$$n^\delta \int_0^{\pi/n} \frac{\omega(t)}{\nu(t)} t^{\delta-1} dt \leq C \int_{\pi/n}^{\pi} \frac{\omega(t)}{t\nu(t)} dt,$$

wherev C *is a constant.*

Recently, the H^ω space was further generalized by Das, Nath and Ray [2] as follows:

$$H_p^\omega := \{f \in L_p[0,2\pi], p \geq 1 : A(f,\omega) < \infty\},$$

where

$$A(f,\omega) := \sup_{t\neq 0} \frac{||f(.+t) - f(.)||_p}{\omega(|t|)}.$$

The norm in the space H_p^{ω} is defined by

$$||f||_p^{(\omega)} := \|f\|_p + A(f,\omega).$$

The completeness of the space H_p^{ω} under the above defined norm can be proved by using the completeness of L_p-Space, $p \geq 1$.
Given the spaces H_p^{ω} and H_p^{ν}, if $\omega(t)/\nu(t)$ is non-decreasing then $\|f\|_p^{\nu} \leq \max\left\{1, \frac{\omega(2\pi)}{\nu(2\pi)}\right\} \|f\|_p^{\omega}$. Thus

$$H_p^{\omega} \subseteq H_p^{\nu} \subseteq L_p, p \geq 1.$$

Let $\sum a_n$ be a given infinite series with s_n for its n-th partial sum. Then the $R_1(\beta)$ mean,for $\beta = 2$ which was introduced by Das, Nath and Ray [1] is defined as follows:

$$\begin{aligned} B(x) &= \sum_{\nu=1}^{\infty} b_\nu(x) s_\nu \\ &:= \frac{1}{l(x)} \sum_{\nu=1}^{\infty} \left(\frac{\sin \nu x}{\nu x}\right) \frac{s_\nu}{\nu}. \end{aligned}$$

Where $l(x) = \log \frac{1}{2} \csc \frac{1}{2} x$ and $l_p(x)$ be the p-th integral of $l(x)$, then

$$l_p(x) = \frac{x^p}{p!} + O(x^p), p \geq 1.$$

Notice that $l(x) \sim \frac{1}{\log x}$ as $x \to 0^+$.
We know $R_1(\beta), \beta = 2$ is Fourier effective [4] as it satisfies the following conditions:

(i) $\sum_{\nu=1}^{\infty} b_\nu(x) \to 1$ as $x \to 0^+$.
(ii) $\sum_{\nu=1}^{\infty} |b_\nu(x)| = O(1)$ as $x \to 0^+$.
(iii) $b_\nu(x) \to 0$ as $x \to 0^+$ (for fixed ν).

Taking $x=\frac{1}{n}$ and $K=\frac{1}{\pi}$, we get

$$W_n = \frac{1}{\log n}\sum_{\nu=1}^{\infty}\left(\frac{\sin(\nu/n)}{(\nu/n)}\right)\frac{1}{\nu}s_\nu. \tag{2.1}$$

$$\theta_n - 1 = \frac{1}{\log n}\sum_{\nu=1}^{\infty}\left(\frac{\sin(\nu/n)}{(\nu/n)}\right)\frac{1}{\nu} - 1 = O\left(\frac{1}{\log n}\right). \tag{2.2}$$

$$K_n(t) = \frac{K}{\log n}\sum_{\nu=1}^{\infty}\left(\frac{\sin(\nu/n)}{(\nu/n)}\right)\frac{1}{\nu}\sin\left(\nu+\frac{1}{2}\right)t. \tag{2.3}$$

$$G_n(t) = \frac{K}{\log n}\sum_{\nu=1}^{n}\left(\frac{\sin(\nu/n)}{(\nu/n)}\right)\frac{1}{\nu}\sin\left(\nu+\frac{1}{2}\right)t. \tag{2.4}$$

$$H_n(t) = \frac{K}{\log n}\sum_{\nu=n+1}^{\infty}\left(\frac{\sin(\nu/n)}{(\nu/n)}\right)\frac{1}{\nu}\sin\left(\nu+\frac{1}{2}\right)t. \tag{2.5}$$

3. KNOWN RESULTS

To estimate the the degree of convergence of Fourier series Das et al. [2] have established the theorem.

Theorem 3.1. *Let $F = u : u : [0,\pi) \to [0,\infty), lim_{t\to 0^+} u(t) = u(0)$ and u is nondecreasing. Let $\omega, \nu \in F$ such that $(\omega/\nu) \in F$. If $f \in H_p^{\omega}$, $p \geq 1$, then*

$$\|W_n^*(f,.) - f(.)\| = O(1)\frac{1}{\log n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t\nu(t)}dt$$

where $W_n^*(x;f)$ is the (T^*) transformation of the Fourier series of f at x and $\omega(t)/\nu(t)$ is a modulus of continuity.

Next dealing with $T\triangle_H$ mean Nigam and Rani [11] proved the following theorem.

Theorem 3.2. *If $\tilde{g} \in H_r^{(\eta)}$ class; $r \geq 1$, then the error estimation of $\tilde{g}$ by $T\triangle_H$ mean of its conjugate Fourier series is*

$$\|\tilde{t}_l^{T\triangle_H} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{\log(l+1)+1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right),$$

where $T\triangle_H$ mean is the superimposing of matrix mean on Hölder mean, η and χ are moduli of continuity such that $\frac{\eta(t)}{\chi(t)}$ are nondecreasing provided

$$\sum_{j=1}^{l-1} |\triangle b_{l,j}| = O\left(\frac{1}{l+1}\right)$$

and

$$(l+1)b_{l,l} = O(1).$$

Further dealing with the degree of approximation by even-type delayed arithmetic mean of Fourier series in the generalized Hölder metric space, Jaeman Kim [6] have established the theorem.

Theorem 3.3. *Let ω and ν be moduli of continuity such that $\frac{\omega(t)}{\nu(t)}$ is nondecreasing. If $f \in H_p^\omega$, for $p \geq 1$, then the degree of approximation of f by even-type Delayed Arithmetic Mean $\sigma_{n,kn}, k \in \mathbb{Z}^+$ of its Fourier series is given by*

$$||\sigma_{n,kn}(f;.) - f(.)||_p^\nu = O\left(\frac{1}{kn}\right) + O(1)\frac{k}{n^2}\int_{\pi/n}^{\pi} \frac{\omega(t)}{t^3\nu(t)}dt.$$

4. Main Results

In this paper we have estimated the rate of convergence of Fourier series of functions belonging to the generalized Hölder metric space

$$H_p^\omega = \{f \in L_p(0,2\pi), p \geq 1 : A(f,\omega) < \infty\}$$

with the norm $||f||_p^{(\omega)} := \|f\|_p + A(f,\omega)$, by using a trigonometric mean $R_1(\beta)$, for $\beta = 2$.

First we will prove the following lemma which will be used in proving our theorem.

Lemma 4.1.

(i) $|G_n(t)| = O(1)\frac{1}{nt\log n}$.

(ii) $|G_n(t)| = O\left(\frac{1}{\log n}\right)$.

(iii) $|H_n(t)| = O\left(\frac{1}{\log n}\right)$.

(iv) $|H_n(t)| = O\left(\frac{t^\delta n^\delta}{\log n}\right)$, *for* $0 < \delta < 1$.

(v) $|H_n(t)| = O\left(\frac{1}{nt\log n}\right)$.

Proof.

(i)

$$\begin{aligned}
|G_n(t)| &= \left|\frac{K}{\log n}\sum_{\nu=1}^{n}\frac{1}{\nu}\left(\frac{\sin(\nu/n)}{(\nu/n)}\right)\sin\left(\nu+\frac{1}{2}\right)t\right| \\
&\leq \frac{1}{\log n}\left(\frac{\sin(1/n)}{(1/n)}\right)\max_{1<M,M^I<n}\left|\sum_{\nu=M}^{M^I}\frac{1}{\nu}sin\left(\nu+\frac{1}{2}\right)t\right| \\
&\leq \frac{n^{-1}}{\log n}\max_{M<M^{II},M^{III}<M^I}\left|\sum_{\nu=M^{II}}^{M^{III}}\sin\left(\nu+\frac{1}{2}\right)t\right| \\
&= O(1)\frac{1}{nt\log n}.
\end{aligned}$$

(ii)

$$\begin{aligned}
|G_n(t)| &= \left|\frac{K}{\log n}\sum_{\nu=1}^{n}\frac{1}{\nu}\left(\frac{\sin(\nu/n)}{(\nu/n)}\right)\sin\left(\nu+\frac{1}{2}\right)t\right| \\
&\leq \frac{1}{\log n}\left(\frac{\sin(1/n)}{(1/n)}\right)\max_{1<M^{IV},M^V<n}\left|\sum_{\nu=M^{IV}}^{M^V}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\nu}\right| \\
&= O\left(\frac{1}{\log n}\right).
\end{aligned}$$

(iii) Similarly as G_n, we can deduce that $|H_n(t)| = O(\frac{1}{\log n})$.

(iv)

$$
\begin{aligned}
|H_n(t)| &= \left| \frac{K}{\log n} \sum_{\nu=n+1}^{\infty} \frac{1}{\nu} \left(\frac{\sin(\nu/n)}{(\nu/n)} \right) \sin\left(\nu + \frac{1}{2}\right) t \right| \\
&= \frac{|K_n|}{\log n} O(t^{\delta}) \sum_{\nu=n+1}^{\infty} \frac{1}{\nu^2} \nu^{\delta} \\
&= \frac{O(n)}{\log n} t^{\delta} \sum_{\nu=n+1}^{\infty} \frac{1}{\nu^{2-\delta}} \\
&= O(n) \frac{t^{\delta} n^{-1+\delta}}{\log n} \\
&= O\left(\frac{t^{\delta} n^{\delta}}{\log n} \right).
\end{aligned}
$$

(v)

$$
\begin{aligned}
|H_n(t)| &\leq \frac{|K|}{\log n} \sum_{\nu=n+1}^{\infty} \left| \frac{1}{\nu} \left(\frac{\sin(\nu/n)}{(\nu/n)} \right) \sin\left(\nu + \frac{1}{2}\right) t \right| \\
&= O(1) \frac{1}{nt \log n}
\end{aligned}
$$

□

Theorem 4.2. *Let ω and ν be moduli of continuity such that $\nu(t)$ and $\frac{\omega(t)}{\nu(t)}$ are non-decreasing. If $f \in H_p^{\omega}$, for $p \geq 1$; then the degree of approximation of f by $R_1(\beta)$ trigonometric mean $W_n(f,x) = \frac{1}{l(x)} \sum_{\nu=1}^{\infty} \left(\frac{\sin \nu x}{\nu x} \right) \frac{s_{\nu}}{\nu}$., for $\beta = 2$ of its Fourier series is given by*

$$
\begin{aligned}
\|W_n(f,.) - f(.)\|_p^{(\nu)} &= \|l_n(.)\|_p^{(\nu)} \\
&= \|l_n(.)\|_p + \sup_{y \neq 0} \frac{||l_n(.+y) - l_n(.)||_p}{\nu(y)} \\
&= O(1) \max \left\{ \frac{1}{n \log n} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t \nu(t)} dt, \frac{1}{\log n} \right\}.
\end{aligned}
$$

Proof.

We know[14] that

$$s_n(f;.)-f(.)=\frac{1}{2\pi}\int_0^{\pi}\phi_.(t)\frac{sin\left(n+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}dt.$$

It can be easily verified that

$$\begin{aligned} l_n(.) &= W_n(f,.)-f(.)\\ &=\int_0^{\pi}\frac{\phi_.(t)}{2\sin(t/2)}\frac{1}{\pi\log n}\sum_{\nu=1}^{\infty}\left(\frac{\sin(\nu/n)}{(\nu/n)}\right)\sin\left(\nu+\frac{1}{2}\right)tdt+[\theta_n-1]f(.)\\ &=\int_0^{\pi}\frac{\phi_.(t)}{2\sin(t/2)}K_n(t)dt+f(.)[\theta_n-1], \end{aligned} \tag{4.1}$$

where θ_n-1 and K_n are defined in equation 2.2 and equation 2.3 respectively.

Using 4.1, we get

$$l_n(.+y)-l_n(.)=\int_0^{\pi}\frac{\phi_{.+y}(t)-\phi_.(t)}{2\sin(t/2)}K_n(t)dt+(f(.+y)-f(.))[\theta_n-1].$$

By generalized Minkowski's inequality [14]

$$\begin{aligned} \|l_n(.+y)-l_n(.)\|_p &\leq \int_0^{\pi}\frac{\|\phi_{.+y}(t)-\phi_.(t)\|_p}{2\sin(t/2)}|K_n(t)|dt+\|f(.+y)-f(.)\|_p|\theta_n-1|\\ &=\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right]\frac{\|\phi_{.+y}(t)-\phi_.(t)\|_p}{2\sin\left(\frac{t}{2}\right)}|K_n(t)|dt+\|f_{.+y}(t)-f_.(t)\|_p|\theta_{n-1}|\\ &=\int_0^{\pi/n}\frac{\|\phi_{.+y}(t)-\phi_.(t)\|_p}{2\sin\left(\frac{t}{2}\right)}|K_n(t)|dt\\ &\quad+\int_{\pi/n}^{\pi}\frac{\|\phi_{.+y}(t)-\phi_.(t)\|_p}{2\sin\left(\frac{t}{2}\right)}|K_n(t)|dt\\ &\quad+\|f(.+y)-f(.)\|_p|\theta_n-1|\\ &=I+J+K. \end{aligned} \tag{4.2}$$

Now, using Lemma 4.1

$$\begin{aligned} I &= \int_0^{\pi/n} \frac{\|\phi_{.+y}(t)-\phi_{.}(t)\|_p}{2\sin(t/2)}|K_n(t)|dt \\ &= O(1)\nu(y)\int_0^{\pi/n}\frac{\omega(t)}{t\nu(t)}\left(\frac{1}{\log n}+\frac{t^\delta n^\delta}{\log n}\right)dt. \end{aligned}$$

$$\begin{aligned} J &= \int_{\pi/n}^{\pi} \frac{\|\phi_{.+y}(t)-\phi_{.}(t)\|_p}{2\sin\left(\frac{t}{2}\right)}|K_n(t)|dt \\ &= O(1)\nu(y)\int_{\pi/n}^{\pi}\frac{\omega(t)}{t\nu(t)}\left(\frac{1}{nt\log n}\right)dt \qquad \text{(By Lemma 4.1)}. \end{aligned}$$

Lastly by using equation 2.2

$$\begin{aligned} K &= \|f(.+y)-f(.)\|_p|\theta_{n-1}| \\ &= \omega(y)O\left(\frac{1}{\log n}\right) \qquad (\because \|f(.+y)-f(.)\|_p = O(\omega(y))) \\ &= \frac{\omega(y)}{\nu(y)}O\left(\frac{\nu(y)}{\log n}\right) = O(1)\nu(y)\frac{1}{\log n}. \end{aligned}$$

By putting the values of I, J, K in equation 4.2, we obtain

$$\begin{aligned} \|l_n(.+y)-l_n(.)\|_p &= O(1)\nu(y)\int_0^{\pi/n}\frac{\omega(t)}{t\nu(t)}\left(\frac{1}{\log n}+\frac{t^\delta n^\delta}{\log n}\right)dt \\ &+O(1)\nu(y)\int_{\pi/n}^{\pi}\frac{\omega(t)}{t\nu(t)}\left(\frac{1}{nt\log n}\right)dt \\ &+O(1)\nu(y)\frac{1}{\log n}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{y\neq 0}\frac{\|l_n(.+y)-l_n(.)\|_p}{\nu(y)} &= O(\frac{1}{\log n})\int_0^{\pi/n}\frac{\omega(t)}{t\nu(t)}dt+\frac{n^\delta}{\log n}\int_0^{\pi/n}\frac{\omega(t)t^\delta}{t\nu(t)}dt \\ &+O\left(\frac{1}{n\log n}\right)\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2\nu(t)}dt+O\left(\frac{1}{\log n}\right) \end{aligned}$$

Thus,

$$\begin{aligned}
\|l_n\|_p^{(\nu)} &= \|l_n(.)\|_p + A(f;\nu) \\
&= \|l_n(.)\|_p + \sup_{y\neq 0} \frac{\|l_n(.+y) - l_n(.)\|_p}{\nu(y)} \\
&\leq \int_0^{\pi} \|\phi_.(t)\|_p \frac{|K_n(t)|}{2\sin\left(\frac{t}{2}\right)} dt + \|f(x)\|_p |[\theta_n - 1]| \\
&\quad + O(1)\frac{1}{\log n}\left(\int_0^{\pi/n} \frac{\omega(t)}{t\nu(t)}(1+t^{\delta}n^{\delta})dt + \int_{\pi/n}^{\pi} \frac{\omega(t)}{t\nu(t)}\left(\frac{1}{nt}\right)dt + 1\right)
\end{aligned}$$

$$\begin{aligned}
= O\left(\frac{1}{\log n}\right)\int_{\pi/n}^{\pi} \frac{\omega(t)}{t\nu(t)}dt + O\left(\frac{1}{n\log n}\right)\int_{\pi/n}^{\pi} \frac{\omega(t)}{t\nu(t)}dt \\
+ O\left(\frac{1}{n\log n}\right)\int_{\pi/n}^{\pi} \frac{\omega(t)}{t^2\nu(t)}dt \qquad \text{(By Lemma 2.1)}
\end{aligned}$$

$$= O(1)\max\left\{\frac{1}{n\log n}\int_{\pi/n}^{\pi} \frac{\omega(t)}{t\nu(t)}dt, \frac{1}{\log n}\right\}.$$

□

Corollary 1. *Let* $\omega(t) = t^{\alpha}$, $\nu(t) = t^{\beta}$, $0 \leq \beta \leq \alpha < 1$ *and* $f \in H_p^{\omega}$, $p \geq 1$. *Then*

$$\|W_n(f,.) - f(.)\|_p^{(\nu)} = O\left(\frac{1}{n\log n}\right).$$

Proof. For $\omega(t) = t^{\alpha}$ and $\nu(t) = t^{\beta}$, $0 \leq \beta \leq \alpha < 1$, We observe that

$$\begin{aligned}
\int_{\pi/n}^{\pi} \frac{\omega(t)}{t\nu(t)}dt &= \int_{\pi/n}^{\pi} \frac{t^{\alpha}}{tt^{\beta}}dt \\
&= \int_{\pi/n}^{\pi} t^{(\alpha-\beta-1)}dt \\
&= O(1).
\end{aligned}$$

Now,

$$\begin{aligned}
l_n(.) &= \|W_n(f,.)-f(.)\|_p^{(\nu)} \\
&= O(1)\max\left\{\frac{1}{n\log n}O(1), \frac{1}{\log n}\right\} \qquad \text{(By theorem 4.2)} \\
&= O(1)\max\left\{O\left(\frac{1}{n\log n}\right), \frac{1}{\log n}\right\} \\
&= O(1)O\left(\frac{1}{n\log n}\right) \\
&= O\left(\frac{1}{n\log n}\right).
\end{aligned}$$

□

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