Journal of the Orissa Mathematical Society ISSN:0975-2323
Volume 40, No.01-02, 2021, 100-113

MULTIOBJECTIVE MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS: PSEUDOLINEAR CASE

J. K. MAURYA[1,*] AND S. K. MISHRA[2]

[1] *Kashi Naresh Government Post Graduate College Gyanpur, Bhadohi, India*
[2] *Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India*

Abstract. In this paper, we consider pseudolinear multiobjective mathematical programs with equilibrium constraints. We establish necessary and sufficient strong efficient S-stationary conditions for a feasible point without using any constraint qualification. Although, necessary optimality conditions required constraint qualification, but in pseudolinear case there is no requiement of constraint qualification due to its own characterization. Since duality provide lower bound to the objective function therefore it have good advantage, so we propose Mond-Weir type dual models for a pseudolinear multiobjective mathematical program with equilibrium constraints and deduce usual duality results. Furthermore, some examples are presented to illustrate our results.

Keywords: Pseudolinear multiobjective programming; Mathematical programs with equilibrium constraints; Efficient solution and Duality

2020 AMS Subject Classification: 90C30, 90C33 ,90C46

1. Introduction

A mathematical programs with equilibrium constraints (MPEC) is a constrained optimization problem where constraints include equilibrium constraints, such as variational inequalities or complementarity conditions. MPECs has various applications

*Corresponding author.
E-mail addresses: jitendramourya150@gmail.com, bhu.skmishra@gmail.com

in process engineering [1], hydro-economic river basin model [2], traffic and telecommunications networks [3, 4]. MPECs are difficult to solve because the feasible region is not necessarily convex or connected, which cause MFCQ to not hold at every feasible point. This leads to linearly dependent active constraints, the set of multipliers being unbounded, and inconsistent linearizations arbitrarily close to a stationary point [5] also Abadie constraint qualification (ACQ) do not satisfy (see, [6]). Scheel and Scholtes [7] discussed several stationarity conditions like Bouligand stationary (B-stationary), weakly stationary (W-stationary), Clarke stationary (C-stationary), strong stationary (S-stationary) in equilibrium constraints sense. Flegel and Kanzow [16, 17] proposed new constraint qualifications for the MPEC and introduced Karush-Kuhn-Tucker type stationary conditions for the MPEC. Ye [6] established the relationship among Mordukhovich stationary (M-stationary) and several other stationary conditions along with necessary and sufficient optimality conditions for the MPEC.

In real-life, the formation of problems are usually multiobjective. Mordukhovich [18] and Bao *et al.* [19] studied multiobjective optimization problems with equilibrium constraints and established necessary optimality conditions. Recently, Zhang *et al.* [20] extended the existing constraint qualifications from a single objective to the multiobjective case and established various stationarity conditions under the proper Pareto sense. Further, Zhang *et al.* [21] established strong Pareto S-stationarity optimality conditions for multiobjective mathematical programs with equilibrium constraints. Pandey and Mishra [8] proposed strong KKT type sufficient optimality conditions for nonsmooth multiobjective semiinfinte MPEC. Further, Mishra *et al.* [9] established the duality results for MPVC. For a more treatment of the probable applications of MPEC and MPVC, we refer [10, 11, 12, 13, 14, 15] and references therein.

In 1967, Kortanek and Evans [22] studied some properties of a class of functions, which are both pseudoconvex as well as pseudoconcave. This class of functions was later named as pseudolinear functions by Chew and Choo [23] and established first and second order characterizations for this class of functions. Mishra *et al*. [24] characterised the Lagrange multiplier of solution sets of a nonsmooth pseudolinear optimization problem. For a comprehensive survey on pseudolinear functions and their applications, we refer to the monograph of Mishra and Upadhyay [25].

Therefore, in this paper we consider the pseudolinear multiobjective mathematical programs with equilibrium constraints (PMMPEC) as follows:

$$\begin{aligned} \min\ & f(z) = (f_1(z), ..., f_p(z)), \\ \text{subject to}\ & g_i(z) \leqq 0\ (i = 1, ..., q),\ \ h_i(z) = 0\ (i = 1, ..., r), \\ & G_i(z) \geqq 0,\ H_i(z) \geqq 0,\ G_i(z)H_i(z) = 0\ (i = 1, ..., m), \end{aligned} \tag{1.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $h_i : \mathbb{R}^n \to \mathbb{R}$, $G_i : \mathbb{R}^n \to \mathbb{R}$ and $H_i : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable pseudolinear functions on $\mathbb{R}^n$. There are some applications of multiobjective mathematical programs with equilibrium constraints in healthcare management [26] and determining energy and climate market policy [27]. Duality plays an important role in finding the lower bound of objective functions, thus application point of view duality have great importance. Duality of multiobjective programming problems has been given by several authors (see, [28, 29, 30, 14]).

In Sect. 2, we recall some preliminary results and definitions which will be used throughout the paper. In Sect. 3, we establish necessary and sufficient optimality conditions, formulate Mond-Weir type dual models and establish duality results. We also illustrate duality results under suitable examples.

2. **Preliminaries**

In this section, we recall some basic definitions and results which will be used throughout the paper. Inequalities between two vectors $y,\ z \in \mathbb{R}^n$, follow the following conventions:

$$\begin{aligned} y &\leqq z \Longleftrightarrow y_i \leqq z_i,\ i=1,...,n, \\ y &\leq z \Longleftrightarrow y \leqq z \ \text{ and } \ y \neq z, \\ y &< z \Longleftrightarrow y_i < z_i,\ i=1,...,n. \end{aligned}$$

Let S denotes the feasible region of the PMMPEC (1.1) is given as:

$$S = \{z \in \mathbb{R}^n : g(z) \leqq 0,\ h(z) = 0,\ G(z) \geqq 0, H(z) \geqq 0,\ G(z)^T H(z) = 0\}.$$

Let z^* denotes the feasible point of the PMMPEC (1.1). For further discussion we need to define index sets as follows:

$$\alpha(z^*) = \{i : G_i(z^*) = 0,\ H_i(z^*) > 0\},\ \gamma(z^*) = \{i : G_i(z^*) > 0, H_i(z^*) = 0\},$$
$$\beta(z^*) = \{i : G_i(z^*) = 0, H_i(z^*) = 0\},\ I_f = \{1,2,...,p\},$$
$$I_g(z^*) = \{i : g_i(z^*) = 0\},\ I_g = \{1,2,...,q\},\ I_h = \{1,2,...,r\},$$
$$I_h(z^*) = \{i : h_i(z^*) = 0\},\ I_f^i = \{1,2,...,p\} \setminus \{i\}.$$

Definition 2.1. [32] A feasible point z^* is said to be an efficient solution of the PMMPEC(1.1), if there is no other feasible point z, such that

$$f(z) \leq f(z^*).$$

Definition 2.2. [25] Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on an open convex set S. The function f is said to be

(1) Pseudoconvex at $z^* \in S$, if $\forall z \in S$,

$$\langle \nabla f(z^*), z - z^* \rangle \geqq 0 \Rightarrow f(z) \geqq f(z^*),$$

(2) Pseudoconcave at $z^* \in S$, if $\forall z \in S$,

$$\langle \nabla f(z^*), z - z^* \rangle \leqq 0 \Rightarrow f(z) \leqq f(z^*),$$

The function is said to be pseudoconvex (pseudoconcave) on S if it is pseudoconvex (pseudoconcave) at every $z \in S$. Moreover, the function is said to be pseudolinear on S if it is both pseudoconvex and pseudoconccve on S.

More precisely, a differentiable function $f : S \to \mathbb{R}$ on an open convex subset $S \subseteq \mathbb{R}^n$, is said to be pseudolinear if $\forall z_1, z_2 \in S$, one has

$$\langle \nabla f(z_1), z_2 - z_1 \rangle \geqq 0 \Rightarrow f(z_2) \geqq f(z_1),$$

and

$$\langle \nabla f(z_1), z_2 - z_1 \rangle \leqq 0 \Rightarrow f(z_2) \leqq f(z_1).$$

The following characterization of psudolinear function given by Chew and Choo [23], which is very useful in the derivation of further results.

Theorem 2.1. *Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be an open convex set. Then, f is a differentiable pseudolinear function on S if and only if $\forall\ z_1,\ z_2 \in S$ there exists a function $p : S \times S \to \mathbb{R}_+$, where $\mathbb{R}_+$ denotes positive real number, such that*

$$f(z_2) = f(z_1) + p(z_1, z_2)\langle \nabla f(z_1), z_2 - z_1 \rangle.$$

The function p is called proportional function.

3. Necessary and Sufficient Optimality Conditions

Motivated by strong Pareto S-stationary point of Zhang *et al.* [21], we define strong efficient S-stationary point for pseudolinear multiobjective mathematical programs with equilibrium constraints (1.1) as follows.

Definition 3.1. A feasible point z^* is called strong efficient S-stationary point of the PMMPEC (1.1) if there exist multipliers $(\lambda^f, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ satisfying the following conditions

$$\sum_{i=1}^{p} \lambda_i^f \nabla f_i(z^*) + \sum_{i=1}^{q} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{r} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} \lambda_i^G \nabla G_i(z^*) - \sum_{i=1}^{m} \lambda_i^H \nabla H_i(z^*) = 0,$$

$$\lambda_i^f > 0,\ \lambda_i^g \geqq 0,\ g(z^*)^T \lambda^g = 0, \lambda_i^G = 0\ (i \in \gamma(z^*)),\ \lambda_i^G \geqq 0\ (i \in \alpha(z^*)),$$

$$\lambda_i^H = 0\ (i \in \alpha(z^*)),\ \lambda_i^H \geqq 0\ (i \in \gamma(z^*)),\ \lambda_i^G \geqq 0,\ \lambda_i^H \geqq 0\ (i \in \beta(z^*)).$$

In the following result, we present relationship between strong efficient S-stationary point and efficient solution of pseudolinear multiobjective mathematical programs with equilibrium constraints

Theorem 3.1. *A feasible point z^* is an efficient solution of PMMPEC(1.1) if and only if z^* is an strong efficient S-stationary point of PMMPEC(1.1)*

Proof. Let z^* be a strong efficient S-stationary point, then there exist multipliers $(\lambda^f, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ such that the following conditions hold

$$\sum_{i=1}^{p} \lambda_i^f \nabla f_i(z^*) + \sum_{i=1}^{q} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{r} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} \lambda_i^G \nabla G_i(z^*) - \sum_{i=1}^{m} \lambda_i^H \nabla H_i(z^*) = 0,$$

$$\lambda_i^f > 0,\ \lambda_i^g \geqq 0,\ g(z^*)^T \lambda^g = 0, \lambda_i^G = 0\ (i \in \gamma(z^*)), \lambda_i^H = 0\ (i \in \alpha(z^*)),$$

$$\lambda_i^G \geqq 0\ (i \in \alpha(z^*)),\ \lambda_i^H \geqq 0\ (i \in \gamma(z^*)), \lambda_i^G \geqq 0,\ \lambda_i^H \geqq 0\ (i \in \beta(z^*)).$$

Assume z^* is not an efficient solution. Then, there exist a feasible point $z \neq z^*$ such that $f_i(z) \leqq f_i(z^*)$ for all i except at least one k such that $f_k(z) < f_k(z^*)$. Now, from pseudolinearity we have

$$f_i(z) - f_i(z^*) = p_i^f(z, z^*)\langle \nabla f_i(z^*), z - z^* \rangle \leqq 0 \ \forall\, i \in \{1, ..., p\} \setminus \{k\} \tag{3.1}$$

$$f_k(z) - f_k(z^*) = p_k^f(z, z^*)\langle \nabla f_k(z^*), z - z^* \rangle < 0, \tag{3.2}$$

$$g_i(z) - g_i(z^*) = p_i^g(z, z^*)\langle \nabla g_i(z^*), z - z^* \rangle \leqq 0,\ i \in I_g(z^*), \tag{3.3}$$

$$h_i(z) - h_i(z^*) = p_i^h(z, z^*)\langle \nabla h_i(z^*), z - z^* \rangle = 0,\ i \in I_h(z^*), \tag{3.4}$$

$$-G_i(z) + G_i(z^*) = p_i^G(z, z^*)\langle -\nabla G_i(z^*), z - z^* \rangle \leqq 0,\ i \in \alpha(z^*) \cup \beta(z^*), \tag{3.5}$$

$$-H_i(z) + H_i(z^*) = p_i^H(z, z^*)\langle -\nabla H_i(z^*), z - z^* \rangle \leqq 0,\ i \in \gamma(z^*) \cup \beta(z^*). \tag{3.6}$$

Multiplying (3.1)-(3.6) by $\lambda_i^f > 0\ (i \in I_f), \lambda_i^g \geqq 0\ (i \in I_g(z^*)), \lambda_l^h\ (i \in I_h(z^*)),\ \lambda_i^G = 0\ (i \in \gamma(z^*)),\ \lambda_i^G \geqq 0\ (i \in \alpha(z^*) \cup \beta(z^*)),\ \lambda_i^H = 0\ (i \in \alpha(z^*)),\ \lambda_i^H \geqq 0\ (i \in \beta(z^*) \cup$

$\gamma(z^*)$), respectively and using the fact that each $p_i > 0$, we get

$$\left\langle \sum_{i=1}^{p} \lambda_i^f \nabla f_i(z^*) + \sum_{i=1}^{q} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{r} \lambda_i^h \nabla h_i(z^*) \right.$$
$$\left. - \sum_{i=1}^{m} \lambda_i^G \nabla G_i(z^*) - \sum_{i=1}^{m} \lambda_i^H \nabla H_i(z^*),\ z - z^* \right\rangle < 0,$$

which contradicts the stationarity of z^* as $z - z^* \neq 0$. Hence the result.

Conversely, suppose z^* is an efficient solution of PMMPEC(1.1), then from pseudolinearity of the functions there does not exist any feasible point z different from z^* such that the following system has solution.

$$\begin{aligned}
&\langle \nabla f_i(z^*), z - z^* \rangle \leqq 0,\ i = \{1, ..., p\} \setminus \{k\}, \\
&\langle \nabla f_i(z^*), z - z^* \rangle < 0,\ i = k, \\
&\langle \nabla g_i(z^*), z - z^* \rangle \leqq 0,\ i \in I_g(z^*), \\
&\langle \nabla h_i(z^*), z - z^* \rangle = 0 \\
&\langle -\nabla G_i(z^*), z - z^* \rangle \leqq 0\ (i \in \alpha(z^*) \cup \beta(z^*)), \\
&\langle -\nabla H_i(z^*), z - z^* \rangle \leqq 0\ (i \in \gamma(z^*) \cup \beta(z^*)).
\end{aligned} \tag{3.7}$$

That is, the inequality system (3.7) has no solution. Therefore, from Tucker theorem [33] and setting

$$\lambda_i^f > 0,\ \ i \in I = \{1, ...p\},\ \lambda_i^g \geqq 0,\ g(z^*)^T \lambda^g = 0, \lambda_i^G = 0\ (i \in \gamma(z^*)),$$
$$\lambda_i^H = 0\ (i \in \alpha(z^*)),\ \lambda_i^G \geqq 0\ (i \in \alpha(z^*)),\ \lambda_i^H \geqq 0\ (i \in \gamma(z^*)),$$
$$\lambda_i^G \geqq 0,\ \lambda_i^H \geqq 0\ (i \in \beta(z^*)),$$

we find $\lambda = (\lambda^f, \lambda^g, \lambda^h, \lambda^H, \lambda^G) \in \mathbb{R}_+^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$, such that

$$\sum_{i=1}^{p} \lambda_i^f \nabla f_i(z^*) + \sum_{i=1}^{q} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{r} \lambda_i^h \nabla h_i(z^*)$$
$$- \sum_{i=1}^{m} \lambda_i^G \nabla G_i(z^*) - \sum_{i=1}^{m} \lambda_i^H \nabla H_i(z^*) = 0, \tag{3.8}$$

Hence, we get the required result. □

Example 3.1. Consider the problem

$$\min(f_1(z),\ f_2(z)), \text{where } f_1(z) = z_1 + z_1^3,\ f_2(z) = z_2 + z_2^3$$
$$\text{subject to } G(z) = z_1 \geqq 0,\ H(z) = z_2 \geqq 0,\ G(z)H(z) = z_1 z_2 = 0.$$

Since,

$$\begin{aligned}&\lambda_1^f \nabla f_1(z) + \lambda_2^f \nabla f_2(z) - \lambda^G \nabla G(z) - \lambda^H \nabla H(z)\\ &= \lambda_1^f \begin{bmatrix} 1+3z_1^2 \\ 0 \end{bmatrix} + \lambda_2^f \begin{bmatrix} 0 \\ 1+3z_2^2 \end{bmatrix} - \lambda^G \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \lambda^H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\end{aligned}$$

for $\lambda_1^f > 0,\ \lambda_2^f > 0, \lambda^G = \lambda_1^f(1+3z_1^2),\ \lambda^H = \lambda_2^f(1+3z_2^2)$ at point $z = (0,0)$. Hence, point $z = (0,0)$ is an efficient solution. Since, except point $z = (0,0)$ no other point satisfies above conditions (strong efficient S-stationary conditions). That is, there is no efficient solution except $z = (0,0)$.

Remark 3.1. In Definition 3.1 if we set $\bar{\lambda}_i^f = \frac{\lambda_i^f}{\sum_{i=1}^p \lambda_i^f}$, $\bar{\lambda}_i^g = \frac{\lambda_i^g}{\sum_{i=1}^p \lambda_i^f}$, $\bar{\lambda}_i^h = \frac{\lambda_i^h}{\sum_{i=1}^p \lambda_i^f}$, $\bar{\lambda}_i^G = \frac{\lambda_i^G}{\sum_{i=1}^p \lambda_i^f}$ and $\bar{\lambda}_i^H = \frac{\lambda_i^H}{\sum_{i=1}^p \lambda_i^f}$. Then, Definition 3.1 is reformulated as:

$$\sum_{i=1}^p \bar{\lambda}_i^f \nabla f_i(z^*) + \sum_{i=1}^q \bar{\lambda}_i^g \nabla g_i(z^*) + \sum_{i=1}^r \bar{\lambda}_i^h \nabla h_i(z^*) - \sum_{i=1}^m \bar{\lambda}_i^G \nabla G_i(z^*) - \sum_{i=1}^m \bar{\lambda}_i^H \nabla H_i(z^*) = 0,$$

$$\bar{\lambda}_i^f > 0,\ \sum_{i=1}^p \bar{\lambda}_i^f = 1,\ \bar{\lambda}_i^g \geqq 0,\ g(z^*)^T \bar{\lambda}^g = 0,\ \lambda_i^G = 0\ (i \in \gamma(z^*)),\ \lambda_i^H = 0\ (i \in \alpha(z^*)),$$
$$\lambda_i^G \geqq 0\ (i \in \alpha(z^*)),\ \lambda_i^H \geqq 0\ (i \in \gamma(z^*)),\ \lambda_i^G \geqq 0,\ \lambda_i^H \geqq 0\ (i \in \beta(z^*)).$$

This is an alternate form of strong efficient S-stationary conditions.

4. **Duality**

In this section, we propose Mond-Weir type dual model to a pseudolinear MMPEC(1.1), which is motivated by [31] and establish weak and strong duality results. Mond-Weir type dual for pseudolinear multiobjective mathematical programs with equilibrium constraints (MWDPMMPEC) is formulated as follows:

$$\max f(u) \tag{4.1}$$

subject to $S_{MWD} = \{(u,\lambda^f,\lambda^g,\lambda^h,\lambda^G,\lambda^H) : \nabla\Phi(u,\lambda^f,\lambda^g,\lambda^h,\lambda^G,\lambda^H) = 0,$

$$\lambda_i^f > 0,\ \lambda_i^g \geqq 0,\ g(u)^T\lambda^g \geqq 0,\ h(u) = 0,\ \lambda^G \geqq 0\ (i \in \alpha(u) \cup \beta(u)),$$

$$G(u)^T\lambda^G \leqq 0,\ \lambda^H \geqq 0\ (i \in \gamma(u) \cup \beta(u)),\ \lambda_i^H = 0\ (i \in \alpha(u)),$$

$$\lambda_i^G = 0\ (i \in \gamma(u)),\ H(u)^T\lambda^H \leqq 0\},$$

where $\Phi(u,\lambda^f,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$

$$= \sum_{i=1}^{p}\lambda_i^f f_i(u) + \sum_{i=1}^{q}\lambda_i^g g_i(u) + \sum_{i=1}^{r}\lambda_i^h h_i(u) - \sum_{i=1}^{m}\lambda_i^G G_i(u) - \sum_{i=1}^{m}\lambda_i^H H_i(u),$$

Consider the set,

$$S_{MWD}^u = \{u : (u,\lambda^f,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in S_{MWD}\}.$$

Theorem 4.1. *(Weak Duality) Let z be a feasible point of the PMMPEC (1.1) and $(u,\lambda^f,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$ be a feasible point of the MWDPMMPEC(4.1). If all given functions are pseudolinear at $u \in S \cup S_{MWD}^u$ and any of the following holds:*

(a) $\lambda_i^f > 0$ and $f_i (\forall i \in I_f)$ are pseudolinear at $u \in S \cup S_{MWD}^u$;

(b) $\lambda_i^f > 0\ (\forall i \in I_f)$ and $\sum_{i=1}^{p}\lambda_i^f f_i(\cdot)$ is pseudolinear at $u \in S \cup S_{MWD}^u$.

Then,

$$f(z) \nleq f(u). \tag{4.2}$$

Proof. Assume that

$$f(z) \leq f(u),$$

Then,

$$f_i(z) \leqq f_i(u),\ \forall\ i \in I_f,\ \text{ except at least one } k,\ \text{such that}$$

$$f_k(z) < f_k(u).$$

Multiplying by $\lambda_i^f > 0$ and adding, we get

$$(\lambda^f)^T f(z) < (\lambda^f)^T f(u). \tag{4.3}$$

Using the pseudolinearity assumptions, we get

$$\langle\sum_{i=1}^{p}\lambda_i^f\nabla f_i(u),z-u\rangle<0, \tag{4.4}$$

$$\sum_{i=1}^{q}\lambda_i^g g_i(z)\leqq\sum_{i=1}^{q}\lambda_i^g g_i(u)\implies\langle\sum_{i=1}^{q}\lambda_i^g\nabla g_i(u),z-u\rangle\leqq 0,\ i\in I_g(u), \tag{4.5}$$

$$\sum\lambda_i^h h_i(z)=\sum\lambda_i^h h_i(u)\implies\langle\sum\lambda_i^h\nabla h_i(u),z-u\rangle=0,\ i\in I_h, \tag{4.6}$$

$$-\sum\lambda_i^G G_i(z)\leqq-\sum\lambda_i^G G_i(u)\implies\langle-\sum\lambda_i^G\nabla G_i(u),z-u\rangle\leqq 0,\ \forall\ i, \tag{4.7}$$

$$-\sum\lambda_i^H H_i(z)\leqq-\sum\lambda_i^H H_i(u)\implies\langle-\sum\lambda_i^H\nabla H_i(u),z-u\rangle\leqq 0,\ \forall\ i, \tag{4.8}$$

by adding (4.4)-(4.8), we get

$$\left\langle\sum_{i=1}^{p}\lambda_i^f\nabla f_i(u)+\sum_{i=1}^{q}\lambda_i^g\nabla g_j(u)+\sum_{i=1}^{r}\lambda_i^h\nabla h_i(u)\right.$$
$$\left.-\sum_{i=1}^{m}\lambda_i^G\nabla G_i(u)-\sum_{i=1}^{m}\lambda_i^H\nabla H_i(u),\ z-u\right\rangle<0,$$

which contradicts the feasibility of u. Hence the theorem. □

Theorem 4.2. *(Strong Duality) Let a feasible solution z^* be efficient solution of the PMMPEC (1.1). If assumptions of weak duality Theorem 4.1 holds. Then, there exist $(\bar{\lambda}^f,\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H)\in\mathbb{R}_+^p\times\mathbb{R}^q\times\mathbb{R}^r\times\mathbb{R}^m\times\mathbb{R}^m$ such that $(z^*,\bar{\lambda}^f,\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H)$ is an efficient solution of the MWDPMMPEC(4.1) and respective objective values are equal.*

Proof. As feasible point z^* is an efficient solution of the pseudolinear MMPEC (1.1). Then, from Theorem 3.1, there exist $(\bar{\lambda}^f,\bar{\lambda}^g,\bar{\lambda}^h,\bar{\lambda}^G,\bar{\lambda}^H)\in\mathbb{R}_+^p\times\mathbb{R}^q\times\mathbb{R}^r\times\mathbb{R}^m\times\mathbb{R}^m$ such that strong efficient S-stationary conditions are satisfied. That is,

$$\sum_{i=1}^{p}\bar{\lambda}_i^f\nabla f_i(z^*)+\sum_{i=1}^{q}\bar{\lambda}_i^g\nabla g_i(z^*)+\sum_{i=1}^{r}\bar{\lambda}_i^h\nabla h_i(z^*)-\sum_{i=1}^{m}\bar{\lambda}_i^G\nabla G_i(z^*)-\sum_{i=1}^{m}\bar{\lambda}_i^H\nabla H_i(z^*)=0.$$

$\bar{\lambda}_i^f>0,\ \bar{\lambda}_i^g\geqq 0,\ g(z^*)^T\bar{\lambda}^g=0,\ \bar{\lambda}_i^G=0\ (i\in\gamma(z^*)),\ \bar{\lambda}_i^H=0\ (i\in\alpha(z^*)),\ \bar{\lambda}_i^G\geqq 0,\ \bar{\lambda}_i^H\geqq 0\ (i\in\beta(z^*)),\ \lambda_i^G\geqq 0\ (i\in\alpha(z^*)),\ \lambda_i^H\geqq 0\ (i\in\gamma(z^*))$. Since z^* is feasible point then $h_i(z^*) = 0$ $(i \in I_h)$, $(\bar{\lambda}^G)^T G(z^*) = 0$, and $(\bar{\lambda}^H)^T H(z^*) = 0$. Therefore, $(z^*, \bar{\lambda}^f, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is feasible of the MWDPMMPEC(4.1). Then, from feasibility and weak duality Theorem 4.1, we have

$$f(z^*) \geqq f(u),$$

for any feasible solution $(u, \lambda^f, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^n \times \mathbb{R}^p_+ \times \mathbb{R}^q_+ \times \mathbb{R}^{r+2m}$ of pseudolinear MWDPMMPEC (4.1). Hence $(z^*, \bar{\lambda}^f, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is an efficient solution of MWDPMMPEC(4.1) and their respective values are equal. □

Following example illustrate the Theorem 4.1 and Theorem 4.2.

Example 4.1. Consider the following PMMPEC problem:

$$\min \; f(z) = (f_1(z), f_2(z)), \text{ where } f_1(z) = z_1 + z_2, \; f_2(z) = z_2 + z_2^3,$$
$$\text{subject to } \; G(z) = z_1 \geqq 0, \; H(z) = z_2 \geqq 0, \; G(z)^T H(z) = z_1 z_2 = 0, \; z \in \mathbb{R}^2.$$

Feasible region for PMMPEC is $S = \{(z_1, z_2) \in \mathbb{R}^2 : z_1 \geqq 0, \; z_2 \geqq 0, \; z_1 z_2 = 0\}$. Now, we formulate MWDPMMPEC dual model according as above discussion.

$$\begin{aligned} \max \; & f(u) = (u_1 + u_2, \; u_2 + u_2^3), \; u \in \mathbb{R}^2, \\ \text{subject to } \; & \lambda_1^f \nabla f_1(u) + \lambda_2^f \nabla f_2(u) - \lambda^G \nabla G(u) - \lambda^H \nabla H(u) \\ & = \lambda_1^f \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_2^f \begin{bmatrix} 0 \\ 1 + 3u_2^2 \end{bmatrix} - \lambda^G \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \lambda^H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ & \lambda_1^f > 0, \; \lambda_2^f > 0, \; \lambda^G \geqq 0, \; \lambda^H \geqq 0, \; \lambda^G G(u) = \lambda^G u_1 \leqq 0, \\ & \lambda^H H(u) = \lambda^H u_2 \leqq 0. \end{aligned}$$

Solving above, we get

$$\lambda^H = \lambda_1^f + \lambda_2^f (1 + 3u_2^2) > 0, \; \lambda^G = \lambda_1^f > 0 \implies u_1 \leqq 0, \; u_2 \leqq 0.$$

It is clear from the feasibility

$$f(z) \nleq f(u).$$

Hence the weak duality Theorem 4.1 is verified. Now, at point $z^* = (0, \; 0)$ strong duality Theorem 4.2 can be easily verified.

Conclusions

Kruk and Wolkowicz [34] concluded from Mathis and Mathis [35], that the hospital management problem described by Mathis and Mathis [35] comes under the class of linear constrained pseudolinear optimization problems. The paper of Kruk and Wolkowicz justifies the importance of the class of pseudolinear functions and related programming problems. Since in this paper, we have considered the pseudolinear multiobjective mathematical programs with equilibrium constraints and established necessary and sufficient optimality conditions without any constraint qualifications, proposed Mond-Weir type dual models for pseudolinear multiobjective mathematical programs with equilibrium constraints and established weak and strong duality results with suitable examples. Therefore, application point of view study of pseudolinear multiobjective mathematical programs with equilibrium constraints is useful. Wolfe dual model can be formulated also by taking motivation from [33]. Since wolfe duality results required convexity assumptions, but here all used functions are pseudolinear, which is bigger class. We tried to establish Wolfe duality results, but pseudolinearity arises issues during the proof of weak and strong duality theorems.

Funding

The second author is financially supported by "Research Grant for Faculty" (IoE Scheme) under Dev. Scheme NO. 6031 and Department of Science and Technology, SERB, New Delhi, India with grant no. MTR/2018/000121.

Acknowledgements

We indebted to the anonymous Reviewers and Editors for many suggestions and stimulating comments to improve the original manuscript.

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability statement

I confirm, data will be made available on reasonable request.

REFERENCES

[1] Raghunathan, A.U., Biegler, L.T.: Mathematical programs with equilibrium constraints (MPECs) in process engineering. *Comput. Chem. Eng.* **27**, (2003), 1381–1392.

[2] Britz, W., Ferris, M., Khun, A.: Modeling water allocating institutions based on multiple optimization problems with equilibrium constraints. *Envior. Model Softw.* **46** , (2013), 196–207.

[3] Ralph, D.: Mathematical programs with complementarity constraints in traffic and telecommunications networks. *Phil. Trans. R. Sco.* **366**, (2008), 1973–1987.

[4] Luo, Z.Q., Pang, J.S., Ralph, D.: *Mathematical Programs with Equilibrium Constraints.* Cambridge University Press, Cambridge 1996.

[5] Fletcher, R., Leyffer, S., Ralph, D., Scholtes, S.: Local convergence of SQP methods for mathematical programs with equilibrium constraints. *SIAM J. Optim.* **17 (1)**, (2006), 259–286.

[6] Ye, J.J.: Necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints. *J. Math. Anal. Appl.* **307**, (2005), 350–369.

[7] Scheel, H., Scholtes, S.: Mathematical programs with complementarity constraints, stationarity optimality and sensitivity. *Math. Oper. Res.* 25, (2000), 1–22.

[8] Pandey, Y., Mishra, S. K.: On strong KKT type sufficient optimality conditions for nonsmooth multiobjective semi-infinite mathematical programming problems with equilibrium constraints. *Operations Research Letters*, **44**(1), (2016), 148–151.

[9] Mishra, S.K., Singh, V., Laha, V.: On duality for mathematical programs with vanishing constraints. *Annals of Operations Research*, **355**(1), (2016), 249-272.

[10] Guu, S. M., Singh, Y., Mishra, S. K.: On strong KKT type sufficient optimality conditions for multiobjective semi-infinite programming problems with vanishing constraints. *Journal of inequalities and applications*, **2017**(1), (2017), 1-9.

[11] Pandey, Y., Mishra, S. K.: Optimality conditions and duality for semi-infinite mathematical programming problems with equilibrium constraints, using convexificators. *Annals of Operations Research*, 2**69**(1), (2018), 549-564.

[12] Singh, Y., Pandey, Y., Mishra, S. K.: Saddle point optimality criteria for mathematical programming problems with equilibrium constraints. *Operations Research Letters*, **45**(3), (2017), 254-258.

[13] Pandey, Y., Mishra, S. K.: Duality of mathematical programming problems with equilibrium constraints. *Pacific Journal of Optimization*, **13**(1), (2017), 105-122.

[14] Pandey, Y., Mishra, S. K.: Duality for nonsmooth optimization problems with equilibrium constraints, using convexificators. *Journal of Optimization Theory and Applications*, **171**(2), (2016), 694-707.

[15] Lai, K.K.: Hassan, M.; Singh, S.K.: Maurya, J.K.; Mishra, S.K. Semidefinite Multiobjective Mathematical Programming Problems with Vanishing Constraints Using Convexificators. *Fractal Fract.* **6**(3), (2022) https://doi.org/10.3390/fractalfract6010003

[16] Flegel, M.L., Kanzow, C.: On the Guignard constraint qualification for mathematical programs with equilibrium constraints. *Optimization*, **54**, (2005), 517–534.

[17] Flegel, M.L., Kanzow, C.: Abadie type constraint qualification for mathematical programs with equilibrium constraints. *J. Optim. Theory Appl.* **12**, (2005), 595–614.

[18] Mordukhovich, B.S.: Multiobjective optimization problems with equilibrium constraints. *Math. Program.* **117**(12), (2009), 331–354.
[19] Bao, T.Q., Gupta, P., Mordukhovich, B.S.: Necessary conditions in multiobjective optimization with equilibrium constraints. *J. Optim. Theory Appl.* **135**, (2007), 179–203.
[20] Zhang, P., Jhang, J., Lin, G.H., Yang, X.: Constraint qualifications and proper Pareto optimality conditions for multiobjective problems with equilibrium constraints. *J. Optim. Theory. Appl.* **176**, (2018), 763–782.
[21] Zhang, P., Jhang, J., Lin, G.H., Yang, X.: Some kind of Pareto stationarity for multiobjective. Optimization. **doi.org/10.1080/02331934.2019.1591406**
[22] Kortanek , K.O., Evans, J.P.: Pseudo-concave programming and Lagrange regularity. *Oper. Res.* **15**, (1967), 882–891.
[23] Chew, K.L., Choo, E. U.: Pseudolinearity and efficiency. *Math. Program.* **28**, (1984), 226–239.
[24] Mishra, S.K., Upadhyay, B.B., An, L.T.H.: Lagrange multiplier characterizations of solution sets of constrained nonsmooth pseudolinear optimization problems. J. Optim. *Theory Appl.* **160**(3), (2014), 763–777.
[25] Mishra, S.K., Upadhyay, B.B.: *Pseudolinear functions and optimization.* Boca Raton, FL: CRC Press, 2015.
[26] Lin, G.H., Zhang, D.L., Liang, Y.C.: Stochastic multiobjective problems with complementarity constraints and applications in healthcare management. *Eur. J. Oper. Res.* **226**(3), (2013), 461–470.
[27] Siddiqui, S., Christensen, A.: Determining energy and climate market policy using multiobjective programs with equilibrium constraints. *Energy.* **94**, (2016), 316–325.
[28] Egudo, R.R.: Efficiency and generalized convex duality for multiobjective programs. *J. Math. Anal. Appl.* **138**, (1989), 84–94.
[29] Kaul, R.N., Suneja, S.K., Srivastava, M.K.: Optimality criteria and duality in multiple objective optimization involving generalized invexity. *J. Optim. Theory Appl.* **80**, (1994), 465–482.
[30] Kuk, H., Tanino, T.: Optimality and duality in nonsmooth multiobjective optimization involving generalized type I functions. *Comput. Math. Appl.* **45**, (2003), 1497–1506.
[31] Pandey, Y., Mishra, S.K.: Duality for Nonsmooth Optimization Problems with Equilibrium Constraints, Using Convexificators. *J. Optim. Theory Appl.* **171**, (2016), 694–707.
[32] Miettinen, K.M.: Nonlinear Multiobjective Optimization. Kluwer Academic Publishers, Boston, 1999
[33] Mangasarian, O.L.: *Nonlinear Programming.* McGraw-Hill, New York, 1969.
[34] Kruk, S., Wolkowicz, H.: Pseudolinear programming. *SIAM Rev.* **41**, (1999), 795–805.
[35] Mathis, F.H., Mathis, L.J.: A nonlinear programming algorithm for hospital management. *SIAM Rev.* **37**, (1995), 230–234.