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GENERAL PARAMETRIC DUALITY MODELS FOR DISCRETE MINMAX FRACTIONAL PROGRAMMING BASED ON HIGHER ORDER INVEXITY

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Abstract. In this paper we investigate the significant role of higher-order parametric duality models for a discrete minmax fractional programming problem regarding higher-order necessary and sufficient optimality conditions. Several higher-order duality models are formulated and investigated along with weak, strong, and strict converse duality theorems by applying some new classes of higher-order invex functions. To the best of our knowledge, the obtained results are new and have a wide range of applications to other parametric duality models, including interdisciplinary research in nature.

Keywords: Discrete minmax fractional programming; generalized higher-order $(\phi, \eta, \omega, \rho, \theta, m)$-invex functions; second-order duality models; duality theorems

2020 AMS Subject Classification: 90C26, 90C30, 90C32, 90C46, 90C47

1. Introduction

Here in this paper, we introduce and formulate a number of higher-order/generalized second-order parametric duality models and establish some duality models for the following discrete minmax fractional programming problem:

$$(P) \qquad \text{Minimize} \max_{1\le i\le p} \frac{f_i(x)}{g_i(x)}$$

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subject to $\quad G_j(x) \leq 0,\ j \in \underline{q},\ \ H_k(x) = 0,\ k \in \underline{r},\ \ x \in X,$

where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), f_i, g_i, $i \in \underline{p} \equiv \{1,2,\ldots,p\}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are real-valued functions defined on X, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P).

As the invexity function theory has been generalized and investigated mostly related to mathematical programming and its applications in several publications, including [19, 33, 34, 37, 45, 46, 56, 74, 75, 76, 78, 79, 80, 81, 83, 84, 86], mostly concentrated to the minmax fractional programming, where some new classes of generalized second-order invex functions are defined, a set of second-order necessary optimality conditions is established, and numerous sets of second-order sufficient optimality conditions are discussed using various generalized second order invexity assumptions. In this paper, we intend to introduce and investigate new classes of generalized second-order invex functions (referred to as sonvex functions), to formulate a set of second-order necessary optimality conditions, and numerous sets of second-order sufficient optimality conditions using various generalized $(\phi,\eta,\omega,\rho,\theta,m)$-sonvexity assumptions. Furthermore, we construct four second-order parametric duality models and prove a class of weak, strong, and strict converse duality theorems utilizing various $(\phi,\eta,\omega,\rho,\theta,m)$-sonvexity hypotheses. The paper is organized as follows: In Section 2, we introduce a class of basic definitions and auxiliary results that will be used for the problem on hand. Section 3 deals with two second-order parametric duality models for (P) with relatively simple constraint structures and proving weak, strong, and strict converse duality theorems using various generalized second-order $(\phi,\eta,\omega,\rho,\theta,m)$-invexity assumptions. In Section 4, we formulate another pair of second-order parametric duality models with more flexible constraint structures that allow for a greater variety of conditions under which duality can be achieved, and we discuss a multitude of second-order duality results under a great variety of generalized second-order $(\phi,\eta,\omega,\rho,\theta,m)$-invexity conditions. Section 5 is concerned with concluding remarks, especially a future vision for our main results in the sense of further research opportunities arising from certain modifications of the principal minmax model investigated in the present work.

It seems that all the duality results obtained for (P) are also applicable, when appropriately specialized, to discrete max, fractional, and conventional objective functions, which are particular cases of (P):

2. Preliminaries

In this section we introduce the generalized concept of $(\phi,\eta,\omega,\rho,\theta,m)$-invex functions (which are referred to as sonvex functions). Let $f : X \to \mathbb{R}$ be a twice differentiable function.

Definition 2.1. *The function f is said to be* (strictly) $(\phi,\eta,\omega,\rho,\theta,m)$-sonvex *at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta,\omega : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,*

$$\phi\big(f(x) - f(x^*)\big)(>) \geq \frac{1}{2}\langle \nabla f(x^*), \eta(x,x^*)\rangle + \frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle + \frac{1}{2}\langle \nabla f(x^*), z\rangle$$

$$+\rho(x,x^*)\|\theta(x,x^*)\|^m,$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

The function f is said to be (strictly) $(\phi,\eta,\omega,\rho,\theta,m)$-sonvex *on X if it is* (strictly) $(\phi,\eta,\omega,\rho,\theta,m)$-sonvex *at each $x^* \in X$.*

Definition 2.2. *The function f is said to be* (strictly) $(\phi,\eta,\omega,\rho,\theta,m)$-pseudosonvex *at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta,\omega : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,*

$$\frac{1}{2}\langle \nabla f(x^*), \eta(x,x^*)\rangle + \frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle$$

$$+\frac{1}{2}\langle \nabla f(x^*), z\rangle \geq -\rho(x,x^*)\|\theta(x,x^*)\|^m$$

$$\Rightarrow \ \phi\big(f(x)-f(x^*)\big)(>) \geq 0.$$

The function f is said to be (strictly) $(\phi,\eta,\omega,\rho,\theta,m)$-pseudosonvex *on X if it is* (strictly)
$(\phi,\eta,\omega,\rho,\theta,m)$-pseudosonvex *at each $x^* \in X$.*

Definition 2.3. *The function f is said to be* (prestrictly) $(\phi,\eta,\omega,\rho,\theta,m)$-quasisonvex *at x^* if there exist functions $\phi:\mathbb{R}\to\mathbb{R}$, $\eta,\omega: X\times X\to\mathbb{R}^n$, $\rho: X\times X\to\mathbb{R}$, and $\theta: X\times X\to\mathbb{R}^n$, and a positive integer m such that for each $x\in X$ and $z\in\mathbb{R}^n$,*

$$\phi\big(f(x)-f(x^*)\big)(<) \leq 0 \ \Rightarrow$$

$$\frac{1}{2}\langle \nabla f(x^*), \eta(x,x^*)\rangle + \frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle + \frac{1}{2}\langle \nabla f(x^*), z\rangle$$

$$\leq -\rho(x,x^*)\|\theta(x,x^*)\|^m.$$

The function f is said to be (prestrictly) $(\phi,\eta,\omega,\rho,\theta,m)$-quasisonvex *on X if it is* (prestrictly) $(\phi,\eta,\omega,\rho,\theta,m)$-quasisonvex *at each $x^* \in X$.*

From the above definitions it is clear that if f is $(\phi,\eta,\omega,\rho,\theta,m)$-sonvex at x^*, then it is both $(\phi,\eta,\omega,\rho,\theta,m)$-pseudosonvex and $(\phi,\eta,\omega,\rho,\theta,m)$-quasisonvex at x^*, if f is $(\phi,\eta,\omega,\rho,\theta,m)$-quasisonvex at x^*, then it is prestrictly $(\phi,\eta,\omega,\rho,\theta,m)$-quasisonvex at x^*, and if f is strictly $(\phi,\eta,\omega,\rho,\theta,m)$-pseudosonvex at x^*, then it is $(\phi,\eta,\omega,\rho,\theta,m)$-quasisonvex at x^*.

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\phi,\eta,\omega,\rho,\theta,m)$-quasisonvexity can be defined in the following equivalent way:

The function f is said to be $(\phi,\eta,\omega,\rho,\theta,m)$-*quasisonvex* at x^* if there exist functions $\phi:\mathbb{R}\to\mathbb{R}$, $\eta,\omega:X\times X\to\mathbb{R}^n$, $\rho:X\times X\to\mathbb{R}$, and $\theta:X\times X\to\mathbb{R}^n$, and a positive integer m such that for each $x\in X$ and $z\in\mathbb{R}^n$,

$$\frac{1}{2}\langle\nabla f(x^*),z\rangle > \frac{1}{2}\langle\nabla f(x^*),\eta(x,x^*)\rangle+\frac{1}{2}\langle\omega(x,x^*),\nabla^2 f(x^*)z\rangle + -\rho(x,x^*)\|\theta(x,x^*)\|^m$$

$$\Rightarrow\ \phi\big(f(x)-f(x^*)\big)>0.$$

We observe that the new classes of generalized convex functions specified in Definitions 2.1 - 2.3 contain a variety of special cases that can easily be identified by appropriate choices of ϕ, η, ω, ρ, θ, and m.

We next recall a set of second-order necessary optimality conditions for (P). This result will be needed for proving strong and strict converse duality theorems.

Theorem 2.1. [80] *Let x^* be an optimal solution of (P), let $\lambda^*=\varphi(x^*)\equiv$ $\max_{1\leq i\leq p} f_i(x^*)/g_i(x^*)\}$, and assume that the functions f_i, g_i, $i\in\underline{p}$, G_j, $j\in\underline{q}$, and H_k, $k\in\underline{r}$, are twice continuously differentiable at x^*, and that the second-order Guignard constraint qualification holds at x^*. Then for each $z^*\in C(x^*)$, there exist*

$$u^*\in U\equiv\{u\in\mathbb{R}^p: u\geq 0,\ \sum_{i=1}^p u_i=1\},$$

$v^*\in\mathbb{R}^q_+\equiv\{v\in\mathbb{R}^q: v\geq 0\}$, *and $w^*\in\mathbb{R}^r$ such that*

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0,$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*) \Big\} z^* \Big\rangle \geq 0,$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \;\; i \in \underline{p},$$

$$v_j^* G_j(x^*) = 0, \;\; j \in \underline{q},$$

where $C(x^*)$ is the set of all critical directions of (P) at x^*, that is,

$$C(x^*) = \{z \in \mathbb{R}^n : \langle \nabla f_i(x^*) - \lambda \nabla g_i(x^*), z \rangle = 0, \;\; i \in A(x^*), \;\; \langle \nabla G_j(x^*), z \rangle \leq 0, \;\; j \in B(x^*), \\ \langle \nabla H_k(x^*), z \rangle = 0, \;\; k \in \underline{r}\},$$

$A(x^*) = \{j \in \underline{p} : f_j(x^*)/g_j(x^*) = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)\}$, *and* $B(x^*) = \{j \in \underline{q} : G_j(x^*) = 0\}$.

For brevity, we shall henceforth refer to x^* as a *normal* optimal solution of (P) if it is an optimal solution and satisfies the second-order Guignard constraint qualification.

In the remainder of this paper, we shall assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are twice continuously differentiable on the open set X. Moreover, we shall assume, without loss of generality, that $g_i(x) > 0$, $i \in \underline{p}$, and $\varphi(x) \geq 0$ for all $x \in X$.

3. DUALITY MODEL I

In this section, we consider two duality models with relatively simple constraint structures and prove weak, strong, and strict converse duality theorems.

Consider the following two problems:

(DI) Maximize λ

subject to

$$\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \sum_{k=1}^{r} w_k \nabla H_k(y) = 0, \tag{3.1}$$

$$\left\langle z, \left\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^{q} v_j \nabla^2 G_j(y) + \sum_{k=1}^{r} w_k \nabla^2 H_k(y) \right\} z \right\rangle \geq 0, \tag{3.2}$$

$$f_i(y) - \lambda g_i(y) + \sum_{j=1}^{q} v_j G_j(y) + \sum_{k=1}^{r} w_k H_k(y) \geq 0, \;\; i \in \underline{p}, \tag{3.3}$$

$$y \in X,\ z \in C(y),\ u \in U,\ v \in \mathbb{R}^q_+,\ w \in \mathbb{R}^r,\ \lambda \in \mathbb{R}_+; \tag{3.4}$$

$(\tilde{D}I)$ Maximize λ

subject to (3.2) - (3.4) and

$$\left\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x,y) \right\rangle \geq 0 \text{ for all } x \in \mathbb{F}, \tag{3.5}$$

where η is a function from $X \times X$ to $\mathbb{R}^n$.

Comparing (DI) and $(\tilde{D}I)$, we see that $(\tilde{D}I)$ is relatively more general than (DI) in the sense that any feasible solution of (DI) is also feasible for $(\tilde{D}I)$, but the converse is not necessarily true. Furthermore, we observe that (3.1) is a system of n equations, whereas (3.5) is a single inequality. Clearly, from a computational point of view, (DI) is preferable to $(\tilde{D}I)$ because of the dependence of (3.5) on the feasible set of (P).

Despite these apparent differences, it turns out that the statements and proofs of all the duality theorems for $(P)-(DI)$ and $(P)-(\tilde{D}I)$ are almost identical and, therefore, we shall consider only the pair $(P)-(DI)$.

For the sake of economy of space and expression, we shall use the following list of symbols in the statements and proofs of our duality theorems:

$$\begin{aligned}
\mathscr{C}(x,v) &= \sum_{j=1}^{q} v_j G_j(x),\\
\mathscr{D}_k(x,w) &= w_k H_k(x),\ \ k \in \underline{r},\\
\mathscr{D}(x,w) &= \sum_{k=1}^{r} w_k H_k(x),\\
\mathscr{E}_i(x,\lambda) &= f_i(x) - \lambda g_i(x),\ \ i \in \underline{p},\\
\mathscr{E}(x,u,\lambda) &= \sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)],\\
\mathscr{G}(x,v,w) &= \sum_{j=1}^{q} v_j G_j(x) + \sum_{k=1}^{r} w_k H_k(x),
\end{aligned}$$

$$I_+(u) = \{i \in \underline{p} : u_i > 0\},\ \ J_+(v) = \{j \in \underline{q} : v_j > 0\},\ \ K_*(w) = \{k \in \underline{r} : w_k \neq 0\}.$$

In the proofs of our duality theorems, we shall make frequent use of the following auxiliary result which provides an alternative expression for the objective function of (P).

Lemma 3.1. [83] *For each $x \in X$,*

$$\varphi(x) = \max_{1\le i\le p} \frac{f_i(x)}{g_i(x)} = \max_{u\in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

The next two theorems show that (DI) is a dual problem for (P).

Theorem 3.1. *(Weak Duality) Let x and $\mathscr{S} \equiv (y,z,u,v,w,\lambda)$ be arbitrary feasible solutions of (P) and (DI), respectively, and assume that either one of the following two sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u)$, f_i is $(\phi,\eta,\omega,\bar{\rho}_i,\theta,m)$-sonvex and $-g_i$ is $(\phi,\eta,\omega,\tilde{\rho}_i,\theta,m)$-sonvex at y,*

(ii) *for each $j \in J_+ \equiv J_+(v)$, G_j is $(\phi,\eta,\omega,\hat{\rho}_j,\theta,m)$-sonvex at y;*

(iii) *for each $k \in K_* \equiv K_*(w)$, $w_k H_k$ is $(\phi,\eta,\omega,\breve{\rho}_k,\theta,m)$-sonvex at y;*

(iv) *ϕ is superlinear and $\phi(a) \geq 0 \Rightarrow a \geq 0$;*

(v) $\sum_{i\in I_+} u_i[\bar{\rho}_i(x,y)+\lambda\tilde{\rho}_i(x,y)] + \sum_{j\in J_+} v_j\hat{\rho}_j(x,y) + \sum_{k\in K_*}\breve{\rho}_k(x,y) \geq 0$;

(b) *the Lagrangian-type function*

$$\xi \to L(\xi,u,v,w,\lambda) = \sum_{i=1}^{p} u_i[f_i(\xi) - \lambda g_i(\xi)] + \sum_{j=1}^{q} v_j G_j(\xi) + \sum_{k=1}^{r} w_k H_k(\xi)$$

is $(\phi,\eta,\omega,\rho,\theta,m)$-pseudosonvex at y, $\rho(x,y) \geq 0$, and $\phi(a) \geq 0 \Rightarrow a \geq 0$.

Then $\varphi(x) \geq \lambda$.

Proof. Using the hypotheses specified in (i) - (iii), we have

$$\phi\big(f_i(x) - f_i(y)\big) \geq \frac{1}{2}\langle \nabla f_i(y), \eta(x,y)\rangle + \frac{1}{2}\langle \omega(x,y), \nabla^2 f_i(y) z\rangle$$

$$+\frac{1}{2}\langle \nabla f_i(y), z\rangle + \bar{\rho}_i(x,y)\|\theta(x,y)\|^m, \;\; i \in I_+, \tag{3.6}$$

$$\phi\big(-g_i(x) + g_i(y)\big) \geq \frac{1}{2}\langle -\nabla g_i(y), \eta(x,y)\rangle - \frac{1}{2}\langle \omega(x,y), \nabla^2 g_i(y) z\rangle$$

$$+\frac{1}{2}\langle -\nabla g_i(y), z\rangle + \tilde{\rho}_i(x,y)\|\theta(x,y)\|^m, \;\; i \in I_+, \tag{3.7}$$

$$\phi\big(G_j(x)-G_j(y)\big) \geq \frac{1}{2}\langle \nabla G_j(y), \eta(x,y)\rangle + \frac{1}{2}\langle \omega(x,y), \nabla^2 G_j(y)z\rangle + \frac{1}{2}\langle \nabla G_j(y), z\rangle$$

$$+\hat{\rho}_j(x,y)\|\theta(x,y)\|^m, \;\; j\in J_+, \tag{3.8}$$

$$\phi\big(H_k(x)-H_k(y)\big) \geq \frac{1}{2}\langle \nabla H_k(y), \eta(x,y)\rangle + \frac{1}{2}\langle \omega(x,y), \nabla^2 H_k(y)z\rangle$$

$$+\frac{1}{2}\langle \nabla H_k(y), z\rangle, \; k\in K_*. \tag{3.9}$$

Now, multiplying (3.6) by u_i and then summing over $i \in \underline{p}$, (3.7) by λu_i and then summing $i \in \underline{p}$, (3.8) by v_j and then summing over $j \in \underline{q}$, summing (3.9) over $k \in \underline{r}$, adding the resulting inequalities, using the superlinearity of ϕ, and setting $u_i = 0,\ i \notin I_+,\ v_j = 0,\ j \notin J_+$, and $w_k = 0,\ k \notin K_*$, we obtain

$$\begin{aligned}
&\phi\frac{1}{2}\Big(\sum_{i=1}^{p} u_i[f_i(x)-\lambda g_i(x)] + \sum_{j=1}^{q} v_j G_j(x) + \sum_{k=1}^{r} w_k H_k(x) - \Big\{\sum_{i=1}^{p} u_i[f_i(y)-\lambda g_i(y)] + \\
&\sum_{j=1}^{q} v_j G_j(y) + \sum_{k=1}^{r} w_k H_k(y)\Big\}\Big) \geq \frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y)-\lambda\nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \\
&\sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \Big\{\sum_{i=1}^{p} u_i[\nabla^2 f_i(y)-\lambda\nabla^2 g_i(y)] + \sum_{j=1}^{q} v_j \nabla^2 G_j(y) \\
&+ \sum_{k=1}^{r} w_k \nabla^2 H_k(y)\Big\}z\Big\rangle + \frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y)-\lambda\nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \\
&\sum_{k=1}^{r} w_k \nabla H_k(y), z\Big\rangle \\
&+\Big\{\sum_{i\in I_+} u_i[\bar{\rho}_i(x,y)+\lambda\tilde{\rho}_i(x,y)] + \sum_{j\in J_+} v_j\hat{\rho}_j(x,y) \\
&+ \sum_{k\in K_*} w_k\breve{\rho}_k(x,y)\Big\}\|\theta(x,y)\|^m.
\end{aligned}$$

Because of the dual feasibility of $\mathscr{S}$, (3.1), (3.2), and (v), the above inequality becomes

$$\phi\Big(\sum_{i=1}^{p} u_i[f_i(x)-\lambda g_i(x)]+\sum_{j=1}^{q} v_jG_j(x) + \sum_{k=1}^{r} w_kH_k(x)-\Big\{\sum_{i=1}^{p} u_i[f_i(y)-\lambda g_i(y)]+\sum_{j=1}^{q} v_jG_j(y)+\sum_{k=1}^{r} w_kH_k(y)\Big\}\Big)\geq 0.$$

But $\phi(a)\geq 0 \Rightarrow a\geq 0$, and hence

$$\sum_{i=1}^{p} u_i[f_i(x)-\lambda g_i(x)]+\sum_{j=1}^{q} v_jG_j(x) + \sum_{k=1}^{r} w_kH_k(x)-\Big\{\sum_{i=1}^{p} u_i[f_i(y)-\lambda g_i(y)]+\sum_{j=1}^{q} v_jG_j(y)+\sum_{k=1}^{r} w_kH_k(y)\Big\}\geq 0.$$

In view of the primal feasibility of x and (3.3), this inequality further reduces to

$$\sum_{i=1}^{p} u_i[f_i(x)-\lambda g_i(x)]\geq 0. \tag{3.10}$$

Now using (3.10) and Lemma 3.1, we obtain the weak duality inequality as follows:

$$\varphi(x)=\max_{a\in U}\frac{\sum_{i=1}^{p} a_if_i(x)}{\sum_{i=1}^{p} a_ig_i(x)}\geq\frac{\sum_{i=1}^{p} u_if_i(x)}{\sum_{i=1}^{p} u_ig_i(x)}\geq\lambda.$$

(b): Using the dual feasibility of $\mathscr{S}$, nonnegativity of $\rho(x,y)$, (3.1), and (3.2), we obtain the following inequality:

$$\frac{1}{2}\langle\nabla L(y,u,v,w,\lambda),\eta(x,y)\rangle+\frac{1}{2}\big\langle\omega(x,y),\nabla^2L(y,u,v,w,\lambda)z\big\rangle+\frac{1}{2}\langle\nabla L(y,u,v,w,\lambda),z\rangle$$

$$\geq\ 0\ \geq\ -\rho(x,y)\|\theta(x,y)\|^m,$$

which in view of our $(\phi,\eta,\omega,\rho,\theta,m)$-pseudosonvexity assumption implies that

$$\phi\big(L(x,u,v,w,\lambda) - L(y,u,v,w,\lambda)\big) \geq 0.$$

Since $\phi(a) \geq 0 \Rightarrow a \geq 0$, we have

$$L(x,u,v,w,\lambda) - L(y,u,v,w,\lambda) \geq 0.$$

Because $x \in \mathbb{F}$, $v \geq 0$, and (3.3) holds, we get

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \geq 0,$$

which is precisely (3.10). As seen in the proof of part (a), this inequality leads to the desired conclusion that $\varphi(x) \geq \lambda$. □

Theorem 3.2. *(Strong Duality) Let x^* be a normal optimal solution of (P), let $\lambda^* = \varphi(x^*)$, and assume that either one of the two sets of conditions specified in Theorem 3.1 is satisfied for all feasible solutions of (DI). Then for each $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that $\mathscr{S}^* \equiv (x^*, z^*, u^*, v^*, w^*, \lambda^*)$ is an optimal solution of (DI) and $\varphi(x^*) = \lambda^*$.*

Proof. Since x^* is a normal optimal solution of (P), by Theorem 2.1, for each $z^* \in C(x^*)$, there exist u^*, v^*, w^*, and $\lambda^*(= \varphi(x^*))$, as specified above, such that $\mathscr{S}^*$ is a feasible solution of (DI). If $\mathscr{S}^*$ were not optimal, then there would exist a feasible solution $(\tilde{y}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda})$ of (DI)such that $\tilde{\lambda} > \lambda^* = \varphi(x^*)$ contradicting Theorem 3.1. Therefore, $\mathscr{S}^*$ is an optimal solution of (DI). □

We also have the following converse duality result for (P) and (DI).

Theorem 3.3. *(Strict Converse Duality) Let x^* be a normal optimal solution of (P), let $\tilde{\mathscr{S}} \equiv (\tilde{x}, \tilde{z}, \tilde{\lambda}, \tilde{u}, \tilde{v}, \tilde{w})$ be an optimal solution of (DI), and assume that either one of the following two sets of hypotheses is satisfied:*

(a) *The assumptions of part (a) of Theorem 3.1 are satisfied for the feasible solution $\tilde{\mathscr{S}}$ of (DI), $\phi(a) > 0 \Rightarrow a > 0$, and f_i is strictly $(\phi, \eta, \omega, \bar{\rho}_i, \theta, m)$-sonvex* *at $\tilde{x}$ for at least one index $i \in I_+(\tilde{u})$, or $-g_i$ is strictly $(\phi,\eta,\omega,\tilde{\rho}_i,\theta,m)$-sonvex at $\tilde{x}$ for at least one index $i \in I_+(\tilde{u})$ (and $\tilde{\lambda} > 0$), or G_j is strictly $(\phi,\eta,\omega,\hat{\rho}_j,\theta,m)$-sonvex at $\tilde{x}$ for at least one index $j \in J_+(\tilde{v})$, or $\mathscr{D}_k(\cdot,\tilde{w})$ is strictly $(\phi,\eta,\breve{\rho}_k,\theta,m)$-sonvex at $\tilde{x}$ for at least one index $k \in K_*(\tilde{w})$, or* $\sum_{i\in I_+(\tilde{u})} \tilde{u}_i[\bar{\rho}_i(x^*,\tilde{x}) + \tilde{\lambda}\tilde{\rho}_i(x^*,\tilde{x})] + \sum_{j\in J_+(\tilde{v})} \tilde{v}_j\hat{\rho}_j(x^*,\tilde{x}) + \sum_{k\in K_*(\tilde{w})} \breve{\rho}_k(x^*,\tilde{x}) > 0$.

(b) *The Lagrangian-type function $\zeta \to L(\zeta,\tilde{u},\tilde{v},\tilde{w},\tilde{\lambda})$ is strictly $(\phi,\eta,\omega,\rho,\theta,m)$-pseudosonvex at $\tilde{x}$, $\rho(x^*,\tilde{x}) \geq 0$, and $\phi(a) > 0 \Rightarrow a > 0$.*

Then $\tilde{x} = x^*$, that is, $\tilde{x}$ is an optimal solution of (P), and $\varphi(x^*) = \tilde{\lambda}$.

Proof. (a): Since x^* is a normal optimal solution of (P), by Theorem 2.1, there exist $z^* \in \mathbb{R}^n$, $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, and $\lambda^*(=\varphi(x^*))$ such that $\mathscr{S}^* \equiv (x^*,z^*,u^*,v^*,w^*,\lambda^*)$ is an optimal solution of (DI) and $\varphi(x^*) = \lambda^*$. Suppose to the contrary that $\tilde{x} \neq x^*$. Now proceeding as in the proof of Theorem 3.1 (with x replaced by x^* and $\mathscr{S}$ by $\tilde{\mathscr{S}}$) and using any of the conditions set forth above, we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i[f_i(x^*) - \tilde{\lambda} g_i(x^*)] > 0.$$

Now using this inequality in conjunction with Lemma 3.1, as in the proof of part (a) of Theorem 3.1, we arrive at the strict inequality $\varphi(x^*) > \tilde{\lambda}$ which contradicts the fact that $\varphi(x^*) = \lambda^* = \tilde{\lambda}$. Therefore, we conclude that $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.

(b): The proof is similar to that of part (a). □

4. Duality Model II

In this section, we consider another pair of duality models for (P) with more flexible constraint structures which allow for a greater variety of generalized $(\phi,\eta,\omega,\rho,\theta,m)$-sonvexity hypotheses under which duality can be established.

Consider the following two problems:

(DII) Maximize λ

subject to

$$\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \sum_{k=1}^{r} w_k \nabla H_k(y) = 0, \tag{4.1}$$

$$\Big\langle z, \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^{q} v_j \nabla^2 G_j(y) + \sum_{k=1}^{r} w_k \nabla^2 H_k(y) \Big\} z \Big\rangle \geq 0, \tag{4.2}$$

$$u_i[f_i(y) - \lambda g_i(y)] \geq 0, \;\; i \in \underline{p}, \tag{4.3}$$

$$v_j G_j(y) \geq 0, \;\; j \in \underline{q}, \tag{4.4}$$

$$w_k H_k(y) \geq 0, \;\; k \in \underline{r}, \tag{4.5}$$

$$y \in X, \; z \in C(y), \; u \in U, \; v \in \mathbb{R}^q_+, \; w \in \mathbb{R}^r, \; \lambda \in \mathbb{R}_+; \tag{4.6}$$

$(\tilde{D}II)$ Maximize λ

subject to (3.5) and (4.2) - (4.6).

The remarks made earlier about the relationships between (DI) and $(\tilde{D}I)$ are, of course, also valid for (DII) and $(\tilde{D}II)$.

The next two theorems show that (DII) is a dual problem for (P).

Theorem 4.1. *(Weak Duality) Let x and $\mathscr{S} \equiv (y,z,u,v,w,\lambda)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u)$, f_i is $(\bar{\phi}, \eta, \omega, \bar{\rho}_i, \theta, m)$-sonvex and $-g_i$ is $(\bar{\phi}, \eta, \omega, \tilde{\rho}_i, \theta, m)$-sonvex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *for each* $j \in J_+(v) \equiv J_+$, G_j *is* $(\hat{\phi}_j, \eta, \omega, \hat{\rho}_j, \theta, m)$*-quasisonvex at* y, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) *for each* $k \in K_*(w) \equiv K_*$, $\xi \to \mathscr{D}_k(\cdot, w)$ *is* $(\breve{\phi}_k, \eta, \breve{\rho}_k, \theta, m)$*-quasisonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\rho^*(x,y) + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k \in K_*} \breve{\rho}_k(x,y) \geq 0$, *where* $\rho^*(x,y) = \sum_{i \in I_+} u_i[\bar{\rho}_i(x,y) + \lambda \tilde{\rho}_i(x,y)$;

(b) (i) *for each* $i \in I_+$, f_i *is* $(\bar{\phi}, \eta, \omega, \bar{\rho}_i, \theta, m)$*-sonvex and* $-g_i$ *is* $(\bar{\phi}, \eta, \omega, \tilde{\rho}_i, \theta, m)$*-sonvex at* y, $\bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathscr{C}(\xi, v)$ *is* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-quasisonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) *for each* $k \in K_*$, $\xi \to \mathscr{D}_k(\xi, w)$ *is* $(\breve{\phi}_k, \eta, \breve{\rho}_k, \theta, m)$*-quasisonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\rho^*(x,y) + \hat{\rho}(x,y) + \sum_{k \in K_*} \breve{\rho}_k(x,y) \geq 0$;

(c) (i) *for each* $i \in I_+$, f_i *is* $(\bar{\phi}, \eta, \omega, \bar{\rho}_i, \theta, m)$*-sonvex and* $-g_i$ *is* $(\bar{\phi}, \eta, \omega, \tilde{\rho}_i, \theta, m)$*-sonvex at* y, $\bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $j \in J_+$, G_j *is* $(\hat{\phi}_j, \eta, \omega, \omega, \hat{\rho}_j, \theta, m)$*-quasisonvex at* y, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-quasisonvex at* y, $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$;

(iv) $\rho^*(x,y) + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \breve{\rho}(x,y) \geq 0$;

(d) (i) *for each* $i \in I_+$, f_i *is* $(\bar{\phi}, \eta, \omega, \bar{\rho}_i, \theta, m)$*-sonvex and* $-g_i$ *is* $(\bar{\phi}, \eta, \omega, \tilde{\rho}_i, \theta, m)$*-sonvex at* y, $\bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathscr{C}(\xi, v)$ *is* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-quasisonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-quasisonvex at* y, $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$;

(iv) $\rho^*(x,y)+\hat{\rho}(x,y)+\breve{\rho}(x,y)\geq 0$;

(e) (i) *for each* $i\in I_+$, f_i *is* $(\bar{\phi},\eta,\omega,\bar{\rho}_i,\theta,m)$*-sonvex and* $-g_i$ *is* $(\bar{\phi},\eta,\omega,\tilde{\rho}_i,\theta,m)$*-sonvex at* y, $\bar{\phi}$ *is superlinear, and* $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\xi\to\mathscr{G}(\xi,v,w)$ *is* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-quasisonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$;

(iii) $\rho^*(x,y)+\hat{\rho}(x,y)\geq 0$.

Then $\varphi(x)\geq\lambda$.

Proof. (a): In view of the assumptions in (i), (3.6) and (3.7) hold. Combining these inequalities, we get

$$\bar{\phi}\Big(\sum_{i=1}^{p}u_i[f_i(x)-\lambda g_i(x)]-\sum_{i=1}^{p}u_i[f_i(y)-\lambda g_i(y)]\Big)\geq\frac{1}{2}\Big\langle\sum_{i=1}^{p}u_i[\nabla f_i(y)-\lambda\nabla g_i(y),\eta(x,y)\Big\rangle+$$
$$\frac{1}{2}\Big\langle\omega(x,y),\sum_{i=1}^{p}u_i[\nabla^2 f_i(y)-\lambda\nabla^2 g_i(y)]z\Big\rangle+\frac{1}{2}\Big\langle\sum_{i=1}^{p}u_i[\nabla f_i(y)-\lambda\nabla g_i(y),z\Big\rangle$$
$$+\sum_{i\in I_+}u_i[\bar{\rho}_i(x,y)+\lambda\tilde{\rho}_i(x,y)]\|\theta(x,y)\|^m. \quad (4.7)$$

Since $x\in\mathbb{F}$ and (4.4) holds, it follows from the properties of the functions $\hat{\phi}_j$ that for each $j\in J_+$, $\hat{\phi}_j\big(G_j(x)-G_j(y)\big)\leq 0$ which in view of (ii) implies that

$$\frac{1}{2}\langle\nabla G_j(y),\eta(x,y)\rangle+\frac{1}{2}\langle\omega(x,y),\nabla^2 G_j(y)z\rangle+\frac{1}{2}\langle\nabla G_j(y),z\rangle$$
$$\leq-\hat{\rho}_j(x,y)\|\theta(x,y)\|^m.$$

As $v_j\geq 0$ for each $j\in\underline{q}$ and $v_j=0$ for each $j\in\underline{q}\backslash J_+$ (complement of J_+ relative to $\underline{q}$), the above inequalities yield

$$\frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j \nabla G_j(y), \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{j=1}^{q} v_j \nabla^2 G_j(y) z\Big\rangle + \frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j \nabla G_j(y), z\Big\rangle$$

$$\leq - \sum_{j\in J_+} v_j \hat{\rho}_j(x,y) \|\theta(x,y)\|^m. \tag{4.8}$$

In a similar manner, we can show that (iii) leads to the following inequality:

$$\frac{1}{2}\Big\langle \sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{k=1}^{r} w_k \nabla^2 H_k(y) z\Big\rangle + \frac{1}{2}\Big\langle \sum_{k=1}^{r} w_k \nabla H_k(y), z\Big\rangle$$

$$\leq - \sum_{k\in K_*} w_k \breve{\rho}_k(x,y) \|\theta(x,y)\|^m. \tag{4.9}$$

Using (4.1), (4.2), and (4.7) - (4.9), we see that

$$\begin{aligned}
&\bar{\phi}\Big(\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] - \sum_{i=1}^{p} u_i[f_i(y) - \lambda g_i(y)]\Big)\\
&\geq -\Big[\frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j \nabla G_j(y), \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{j=1}^{q} v_j \nabla^2 G_j(y) z\Big\rangle + \frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j \nabla G_j(y), z\Big\rangle\\
&\quad + \frac{1}{2}\Big\langle \sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{k=1}^{r} w_k \nabla^2 H_k(y) z\Big\rangle\Big] + \frac{1}{2}\Big\langle \sum_{k=1}^{r} w_k \nabla H_k(y), z\Big\rangle\\
&\quad \sum_{i\in I_+} u_i[\bar{\rho}_i(x,y) + \lambda \tilde{\rho}_i(x,y)] \|\theta(x,y)\|^m\\
&\geq \Big\{\sum_{i\in I_+} u_i[\bar{\rho}_i(x,y) + \lambda \tilde{\rho}_i(x,y)] + \sum_{j\in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k\in K_*} w_k \breve{\rho}_k(x,y)\Big\} \|\theta(x,y)\|^m \qquad (4.10)\\
&\geq 0 \ \text{(by (iv))}
\end{aligned}$$

But $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$, and hence we have

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \geq \sum_{i=1}^{p} u_i[f_i(y) - \lambda g_i(y)] \geq 0,$$

where the second inequality follows from the dual feasibility of $\mathscr{S}$ and (4.3). As shown in the proof of part (a) of Theorem 3.1, this inequality leads to the conclusion that $\varphi(x) \geq \lambda$.

(b) : As shown in part (a), for each $j \in J_+$, we have $G_j(x) - G_j(y) \leq 0$, and hence using the properties of the function $\hat{\phi}$, we get

$$\hat{\phi}\Big(\sum_{j=1}^{q} v_j G_j(x) - \sum_{j=1}^{q} v_j G_j(y)\Big) \leq 0,$$

which in view of (ii) implies that

$$\frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j \nabla G_j(y), \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{j=1}^{q} v_j \nabla^2 G_j(y) z\Big\rangle + \frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j \nabla G_j(y), z\Big\rangle$$
$$\leq -\hat{\rho}(x,y)\|\theta(x,y)\|^m. \tag{4.11}$$

Now proceeding as in the proof of part (a) and using this inequality instead of (4.8), we arrive at (3.10), which leads to the desired conclusion that $\varphi(x) \geq \lambda$.

(c) - (e) : The proofs are similar to those of parts (a) and (b). □

Theorem 4.2. *(Strong Duality) Let x^* be a normal optimal solution of (P) and assume that any one of the five sets of conditions specified in Theorem 4.1 is satisfied for all feasible solutions of (DII). Then for each $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, and $\lambda^*(= \varphi(x^*)) \in \mathbb{R}_+$ such that $\mathscr{S}^* \equiv (x^*, z^*, u^*, v^*, w^*, \lambda^*)$ is an optimal solution of (DII) and $\varphi(x^*) = \lambda^*$.*

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 4.3. *(Strict Converse Duality) Let x^* be a normal optimal solution of (P), let $\tilde{\mathscr{S}} \equiv (\tilde{x}, \tilde{z}, \tilde{\lambda}, \tilde{u}, \tilde{v}, \tilde{w})$ be an optimal solution of (DII), and assume that any one of the following five sets of hypotheses is satisfied:*

(a) *The assumptions of part (a) of Theorem 4.1 are satisfied for the feasible solution $\tilde{\mathscr{S}}$ of (DII), $\bar{\phi}(a) > 0 \Rightarrow a > 0$, and f_i is strictly $(\bar{\phi}, \eta, \omega, \bar{\rho}_i, \theta, m)$-sonvex at $\tilde{x}$ for at least one index $i \in I_+(\tilde{u})$, or $-g_i$ is strictly $(\phi, \eta, \omega, \tilde{\rho}_i, \theta, m)$-sonvex at $\tilde{x}$ for at least one index $i \in I_+(\tilde{u})$ (and $\tilde{\lambda} > 0$), or G_j is strictly $(\hat{\phi}_j, \eta, \omega, \hat{\rho}_j, \theta, m)$-pseudosonvex at $\tilde{x}$ for at least one $j \in J_+(\tilde{v})$, or $\xi \to \mathscr{D}_k(\xi, \tilde{w})$ is strictly $(\breve{\phi}_k, \eta, \omega, \breve{\rho}_k, \theta, m)$-pseudosonvex at $\tilde{x}$ for at least one*

$k \in K_*(\tilde{w})$, *or* $\rho^*(x^*,\tilde{x}) + \sum_{j\in J_+(\tilde{v})} \tilde{v}_j \hat{\rho}_j(x^*,\tilde{x}) + \sum_{k\in K_*(\tilde{w})} \breve{\rho}_k(x^*,\tilde{x}) > 0$, *where* $\rho^*(x^*,\tilde{x}) = \sum_{i\in I_+(\tilde{u})} \tilde{u}_i[\bar{\rho}_i(x^*,\tilde{x}) + \tilde{\lambda}\tilde{\rho}_i(x^*,\tilde{x})]$.

(b) *The assumptions of part (b) of Theorem 4.1 are satisfied for the feasible solution* $\tilde{\mathscr{S}}$ *of (DII),* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, *and* f_i *is strictly* $(\bar{\phi},\eta,\omega,\bar{\rho}_i,\theta,m)$*-sonvex at* $\tilde{x}$ *for at least one index* $i \in I_+(\tilde{u})$, *or* $-g_i$ *is strictly* $(\phi,\eta,\omega,\tilde{\rho}_i,\theta,m)$*-sonvex at* $\tilde{x}$ *for at least one index* $i \in I_+(\tilde{u})$ *(and* $\tilde{\lambda} > 0$*), or* $\xi \to \mathscr{C}(\xi,\tilde{v})$ *is strictly* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-pseudosonvex at* $\tilde{x}$, *or* $\xi \to \mathscr{D}_k(\xi,\tilde{w})$ *is strictly* $(\breve{\phi}_k,\eta,\omega,\breve{\rho}_k,\theta,m)$*-pseudosonvex at* $\tilde{x}$ *for at least one* $k \in K_*(\tilde{w})$, *or* $\rho^*(x^*,\tilde{x}) + \hat{\rho}(x^*,\tilde{x}) + \sum_{k\in K_*(\tilde{w})} \breve{\rho}_k(x^*,\tilde{x}) > 0$.

(c) *The assumptions of part (c) of Theorem 4.1 are satisfied for the feasible solution* $\tilde{\mathscr{S}}$ *of (DII),* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, *and* f_i *is strictly* $(\bar{\phi},\eta,\omega,\bar{\rho}_i,\theta,m)$*-sonvex at* $\tilde{x}$ *for at least one index* $i \in I_+(\tilde{u})$, *or* $-g_i$ *is strictly* $(\phi,\eta,\omega,\tilde{\rho}_i,\theta,m)$*-sonvex at* $\tilde{x}$ *for at least one index* $i \in I_+(\tilde{u})$ *(and* $\tilde{\lambda} > 0$*), or* G_j *is strictly* $(\hat{\phi}_j,\eta,\omega,\hat{\rho}_j,\theta,m)$*-pseudosonvex at* $\tilde{x}$ *for at least one* $j \in J_+(\tilde{v})$, *or* $\xi \to \mathscr{D}(\xi,\tilde{w})$ *is strictly* $(\breve{\phi},\eta,\omega,\breve{\rho},\theta,m)$*-pseudosonvex at* $\tilde{x}$, *or* $\rho^*(x^*,\tilde{x}) + \sum_{j\in J_+(\tilde{v})} \tilde{v}_j\hat{\rho}_j(x^*,\tilde{x}) + \breve{\rho}(x^*,\tilde{x}) > 0$.

(d) *The assumptions of part (d) of Theorem 4.1 are satisfied for the feasible solution* $\tilde{\mathscr{S}}$ *of (DII),* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, *and* f_i *is strictly* $(\bar{\phi},\eta,\omega,\bar{\rho}_i,\theta,m)$*-sonvex at* $\tilde{x}$ *for at least one index* $i \in I_+(\tilde{u})$, *or* $-g_i$ *is strictly* $(\phi,\eta,\omega,\tilde{\rho}_i,\theta,m)$*-sonvex at* $\tilde{x}$ *for at least one index* $i \in I_+(\tilde{u})$ *(and* $\tilde{\lambda} > 0$*), or* $\xi \to \mathscr{C}(\xi,\tilde{v})$ *is strictly* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-pseudosonvex at* $\tilde{x}$, *or* $\xi \to \mathscr{D}(\xi,\tilde{w})$ *is strictly* $(\breve{\phi},\eta,\omega,\breve{\rho},\theta,m)$*-pseudosonvex at* $\tilde{x}$, *or* $\rho^*(x^*,\tilde{x}) + \hat{\rho}(x^*,\tilde{x}) + \breve{\rho}(x^*,\tilde{x}) > 0$.

(e) *The assumptions of part (e) of Theorem 4.1 are satisfied for the feasible solution* $\tilde{\mathscr{S}}$ *of (DII),* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, *and* f_i *is strictly* $(\bar{\phi},\eta,\omega,\bar{\rho}_i,\theta,m)$*-sonvex at* $\tilde{x}$ *for at least one index* $i \in I_+(\tilde{u})$, *or* $-g_i$ *is strictly* $(\phi,\eta,\omega,\tilde{\rho}_i,\theta,m)$*-sonvex at* $\tilde{x}$ *for at least one index* $i \in I_+(\tilde{u})$ *(and* $\tilde{\lambda} > 0$*), or* $\xi \to \mathscr{G}(\xi,\tilde{v},\tilde{w})$ *is strictly* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-pseudosonvex at* $\tilde{x}$, *or* $\rho^*(x^*,\tilde{x}) + \hat{\rho}(x^*,\tilde{x}) > 0$.

Then $\tilde{x} = x^*$ *and* $\varphi(x^*) = \tilde{\lambda}$.

Proof. The proof is similar to that of Theorem 3.3. □

In Theorem 4.1, separate $(\phi,\eta,\omega,\rho,\theta,m)$-sonvexity assumptions were imposed on the functions f_i and $-g_i$, $i \in I_+$. It is possible to establish a great variety of additional duality results in which various generalized $(\phi,\eta,\omega,\rho,\theta,m)$-sonvexity requirements are placed on certain combinations of these functions. In the remainder of this paper, we shall discuss a series of duality theorems in which appropriate generalized $(\phi,\eta,\omega,\omega,\rho,\theta,m)$-sonvexity assumptions will be imposed on the functions $\xi \to \mathscr{E}_i(\xi,\lambda)$, $i \in \underline{p}$, $\xi \to \mathscr{E}(\xi,u,\lambda)$, G_j, $j \in \underline{q}$, $\xi \to \mathscr{C}(\xi,v)$, $\xi \to \mathscr{D}_k(\xi,w)$, $k \in \underline{r}$, $\xi \to \mathscr{D}(\xi,w)$, and $\xi \to \mathscr{G}(\xi,v,w)$.

Theorem 4.4. *(Weak Duality) Let x and $\mathscr{S} \equiv (y,z,u,v,w,\lambda)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) $\xi \to \mathscr{E}(\xi,u,\lambda)$ *is* $(\bar{\phi},\eta,\omega,\bar{\rho},\theta,m)$*-pseudosonvex at* y*, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $j \in J_+ \equiv J(v)$, G_j *is* $(\hat{\phi}_j,\eta,\omega,\hat{\rho}_j,\theta,m)$*-quasisonvex at* y, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) *for each* $k \in K_* \equiv K(w)$, $\xi \to \mathscr{D}_k(\xi,w)$ *is* $(\breve{\phi}_k,\eta,\omega,\breve{\rho}_k,\theta,m)$*-quasisonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\bar{\rho}(x,y) + \sum_{j\in J_+} v_j\hat{\rho}_j(x,y) + \sum_{k\in K_*} \breve{\rho}_k(x,y) \geq 0$;

(b) (i) $\xi \to \mathscr{E}(\xi,u,\lambda)$ *is* $(\bar{\phi},\eta,\omega,\bar{\rho},\theta,m)$*-pseudosonvex at* y*, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathscr{C}(\xi,v)$ *is* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-quasisonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) *for each* $k \in K_*$, $\xi \to \mathscr{D}_k(\xi,w)$ *is* $(\breve{\phi}_k,\eta,\omega,\breve{\rho}_k,\theta,m)$*-quasisonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\bar{\rho}(x,y)+\hat{\rho}(x,y)+\sum_{k\in K_*}\breve{\rho}_k(x,y)\geq 0$;

(c) (i) $\xi\to\mathscr{E}(\xi,u,\lambda)$ *is* $(\bar{\phi},\eta,\omega,\bar{\rho},\theta,m)$*-pseudosonvex at y, and* $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) *for each* $j\in J_+$, G_j *is* $(\hat{\phi}_m,\eta,\omega,\hat{\rho}_j,\theta,m)$*-quasisonvex at y,* $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0)=0$;

(iii) $\xi\to\mathscr{D}(\xi,w)$ *is* $(\breve{\phi},\eta,\omega,\breve{\rho},\theta,m)$*-quasisonvex at y,* $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0)=0$;

(iv) $\bar{\rho}(x,y)+\sum_{j\in J_+}v_j\hat{\rho}_j(x,y)+\breve{\rho}(x,y)\geq 0$;

(d) (i) $\xi\to\mathscr{E}(\xi,u,\lambda)$ *is* $(\bar{\phi},\eta,\omega,\bar{\rho},\theta,m)$*-pseudosonvex at y, and* $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\xi\to\mathscr{C}(\xi,v)$ *is* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-quasisonvex at y,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$;

(iii) $\xi\to\mathscr{D}(\xi,w)$ *is* $(\breve{\phi},\eta,\omega,\breve{\rho},\theta,m)$*-quasisonvex at y,* $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0)=0$;

(iv) $\bar{\rho}(x,y)+\hat{\rho}(x,x^*)+\breve{\rho}(x,y)\geq 0$;

(e) (i) $\xi\to\mathscr{E}(\xi,u,\lambda)$ *is* $(\bar{\phi},\eta,\omega,\bar{\rho},\theta,m)$*-pseudosonvex at y, and* $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\xi\to\mathscr{G}(\xi,v,w)$ *is* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-quasisonvex at y,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$;

(iii) $\bar{\rho}(x,y)+\hat{\rho}(x,y)\geq 0$.

Then $\varphi(x)\geq\lambda$.

Proof. (a): In view of our assumptions specified in (ii) and (iii), (4.8) and (4.9) remain valid for the present case. From (4.1), (4.2), (4.8), (4.9), and (iv) we deduce that

$$\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)]z\Big\rangle$$
$$+ \frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], z\Big\rangle$$
$$\geq -\Big[\frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j \nabla G_j(y), \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{j=1}^{q} v_j \nabla^2 G_j(y) z\Big\rangle +$$
$$\frac{1}{2}\Big\langle \sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x,y)\Big\rangle$$
$$+ \frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j \nabla G_j(y), z\Big\rangle + \frac{1}{2}\Big\langle \sum_{k=1}^{r} w_k \nabla H_k(y), z\Big\rangle$$
$$+ \frac{1}{2}\Big\langle \omega(x,y), \sum_{k=1}^{r} w_k \nabla^2 H_k(y) z\Big\rangle\Big] \text{ (by (4.1) and (4.2))}$$
$$\geq \Big[\sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k \in K_*} \breve{\rho}_k(x,y)\Big] \|\theta(x,y)\|^m \text{ (by (4.8) and (4.9))}$$
$$\geq -\bar{\rho}(x,y)\|\theta(x,y)\|^m \text{ (by (iv))},$$

which in view of (i) implies that

$$\bar{\phi}\big(\mathscr{E}(x,u,\lambda) - \mathscr{E}(y,u,\lambda)\big) \geq 0.$$

Because of the properties of the function $\bar{\phi}$, the last inequality yields

$$\mathscr{E}(x,u,\lambda) \geq \mathscr{E}(y,u,\lambda) \geq 0,$$

where the inequality follows from the dual feasibility of $\mathscr{S}$ and (4.3). As shown in the proof of Theorem 3.1, this inequality leads to the conclusion that $\varphi(x) \geq \lambda$.

(b) - (e) : The proofs are similar to that of part (a). □

Theorem 4.5. *(Strong Duality) Let x^* be a normal optimal solution of (P) and assume that any one of the five sets of conditions specified in Theorem 4.4 is satisfied for all feasible solutions of (DII). Then for any $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in$*

$\mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, *and* $\lambda^*(=\varphi(x^*)) \in \mathbb{R}_+$ *such that* $\mathscr{S}^* \equiv (x^*, z^*, u^*, v^*, w^*, \lambda^*)$ *is an optimal solution of (DII) and* $\varphi(x^*) = \lambda^*$.

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 4.6. *(Strict Converse Duality) Let* x^* *be a normal optimal solution of (P), let* $\tilde{\mathscr{S}} \equiv (\tilde{x}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda})$ *be an optimal solution of (DII), and assume that any one of the five sets of hypotheses specified in Theorem 4.4 is satisfied, and that the function* $\xi \to \mathscr{E}(\xi, \tilde{u}, \tilde{\lambda})$ *is strictly* $(\bar{\phi}, \eta, \omega, \bar{\rho}, \theta, m)$*-pseudosonvex at* $\tilde{x}$ *and* $\bar{\phi}(a) > 0 \Rightarrow a > 0$. *Then* $\tilde{x} = x^*$ *and* $\varphi(x^*) = \tilde{\lambda}$.

Proof. (a): Since x^* is a normal optimal solution of (P), by Theorem 2.1, for any $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, and $\lambda^* \in \mathbb{R}_+$ such that $\mathscr{S}^* \equiv (x^*, z^*, u^*, v^*, w^*, \lambda^*)$ is an optimal solution of (DII) and $\varphi(x^*) = \lambda^*$. Suppose to the contrary that $\tilde{x} \neq x^*$. Now proceeding as in the proof of Theorem 4.4 (with x replaced by x^* and $\mathscr{S}$ by $\tilde{\mathscr{S}}$), we arrive at the inequality

$$\frac{1}{2}\Big\langle \sum_{i=1}^{p} \tilde{u}_i[\nabla f_i(\tilde{x}) - \tilde{\lambda}\nabla g_i(\tilde{x}), \eta(x^*, \tilde{x})\Big\rangle + \frac{1}{2}\Big\langle \tilde{z}, \sum_{i=1}^{p} \tilde{u}_i[\nabla^2 f_i(\tilde{x}) - \tilde{\lambda}\nabla^2 g_i(\tilde{x})]\tilde{z}\Big\rangle$$
$$+\frac{1}{2}\Big\langle \sum_{i=1}^{p} \tilde{u}_i[\nabla f_i(\tilde{x}) - \tilde{\lambda}\nabla g_i(\tilde{x}))\Big\rangle$$
$$\geq \Big[\sum_{j\in J_+} \tilde{v}_j\hat{\rho}_j(x^*, \tilde{x}) + \sum_{k\in K_*} \breve{\rho}_k(x^*, \tilde{x})\Big]\|\theta(x^*, \tilde{x})\|^m \geq -\bar{\rho}(x^*, \tilde{x})\|\theta(x^*, \tilde{x})\|^m,$$

which in view of our strict $(\bar{\phi}, \eta, \pi, \omega, \bar{\rho}, \theta, m)$-pseudosonvexity hypothesis implies that

$$\bar{\phi}\big(\mathscr{E}(x^*, \tilde{u}, \tilde{\lambda}) - \mathscr{E}(\tilde{x}, \tilde{u}, \tilde{\lambda})\big) > 0.$$

Because of the properties of the function $\bar{\phi}$, the last inequality yields

$$\mathscr{E}(x^*, \tilde{u}, \tilde{\lambda}) > \mathscr{E}(\tilde{x}, \tilde{u}, \tilde{\lambda}) \geq 0,$$

where the second inequality follows from the dual feasibility of $\tilde{\mathscr{S}}$ and (4.3). Now, using this inequality and invoking Lemma 3.1, we see that

$$\varphi(x^*) = \max_{a\in U} \frac{\sum_{i=1}^p a_i f_i(x^*)}{\sum_{i=1}^p a_i g_i(x^*)} \geq \frac{\sum_{i=1}^p \tilde{u}_i f_i(x^*)}{\sum_{i=1}^p \tilde{u}_i g_i(x^*)} > \tilde{\lambda},$$

which contradicts the fact that $\varphi(x^*) = \lambda^* = \tilde{\lambda}$. Therefore, we conclude that $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.

(b) - (e) : The proofs are similar to that of part (a). □

Theorem 4.7. *(Weak Duality) Let x and $\mathscr{S} \equiv (y,z,u,v,w,\lambda)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *$\xi \to \mathscr{E}(\xi,u,\lambda)$ is prestrictly $(\bar{\phi},\eta,\omega,\bar{\rho},\theta,m)$-quasisonvex at y, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *for each $j \in J_+ \equiv J_+(v)$, G_j is $(\hat{\phi}_j,\eta,\omega,\hat{\rho}_j,\theta,m)$-quasisonvex at y, $\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0) = 0$;*

(iii) *for each $k \in K_* \equiv K(w)$, $\xi \to \mathscr{D}_k(\xi,w), H_k$ is $(\breve{\phi}_k,\eta,\omega,\breve{\rho}_k,\theta,m)$-quasisonvex at y, $\breve{\phi}_k$ is increasing, and $\breve{\phi}_k(0) = 0$;*

(iv) *$\bar{\rho}(x,y) + \sum_{j\in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k\in K_*} w_k \breve{\rho}_k(x,y) > 0$;*

(b) (i) *$\xi \to \mathscr{E}(\xi,u,\lambda)$ is prestrictly $(\bar{\phi},\eta,\omega,\bar{\rho},\theta,m)$-quasisonvex at y, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *$\xi \to \mathscr{C}(\xi,v)$ is $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$-quasisonvex at y, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;*

(iii) *for each $k \in K_*$, $\xi \to \mathscr{D}_k(\xi,w)$ is $(\breve{\phi}_k,\eta,\omega,\breve{\rho}_k,\theta,m)$-quasisonvex at y, $\breve{\phi}_k$ is increasing, and $\breve{\phi}_k(0) = 0$;*

(iv) *$\bar{\rho}(x,y) + \hat{\rho}(x,y) + \sum_{k\in K_*} \breve{\rho}_k(x,y) > 0$;*

(c) (i) *$\xi \to \mathscr{E}(\xi,u,\lambda)$ is prestrictly $(\bar{\phi},\eta,\omega,\bar{\rho},\theta,m)$-quasisonvex at y, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *for each $j \in J_+$, G_j is $(\hat{\phi}_j,\eta,\omega,\hat{\rho}_j,\theta,m)$-quasisonvex at y, $\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0) = 0$;*

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-quasisonvex at* y*,* $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$*;*

(iv) $\bar{\rho}(x,y) + \sum_{j\in J_+} v_j\hat{\rho}_j(x,y) + \breve{\rho}(x,y) > 0$*;*

(d) (i) $\xi \to \mathscr{E}(\xi, u, \lambda)$ *is prestrictly* $(\bar{\phi}, \eta, \omega, \bar{\rho}, \theta, m)$*-quasisonvex at* y*, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) $\xi \to \mathscr{C}(\xi, w)$ *is* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-quasisonvex at* y*,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-quasisonvex at* y*,* $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$*;*

(iv) $\bar{\rho}(x,y) + \hat{\rho}(x,y) + \breve{\rho}(x,y) > 0$*;*

(e) (i) $\xi \to \mathscr{E}(\xi, u, \lambda)$ *is prestrictly* $(\bar{\phi}, \eta, \omega, \bar{\rho}, \theta, m)$*-quasisonvex at* y*, and* $\bar{\phi}(a) \geqq 0 \Rightarrow a \geq 0$*;*

(ii) $\xi \to \mathscr{G}(\xi, v, w)$ *is* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-quasisonvex at* y*,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\bar{\rho}(x,y) + \hat{\rho}(x,y) > 0$.

Then $\varphi(x) \geq \lambda$*.*

Proof. (a) : Because of our assumptions specified in (ii) and (iii), (4.8) and (4.9) remain valid for the present case. From (4.1), (4.2), (4.8), (4.9), and (iv) we deduce that

$$\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)]z\Big\rangle$$
$$+\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], z\Big\rangle$$
$$\geq \Big[\sum_{j\in J_+} v_j\hat{\rho}_j(x,y) + \sum_{k\in K_*} w_k\breve{\rho}_k(x,y)\Big]\|\theta(x,y)\|^m > -\bar{\rho}(x,y)\|\theta(x,y)\|^m,$$

which in view of (i) implies that

$$\bar{\phi}\big(\mathscr{E}(x,u,\lambda) - \mathscr{E}(y,u,\lambda)\big) \geq 0.$$

Because of the properties of the function $\bar{\phi}$, the last inequality yields

$$\mathscr{E}(x,u,\lambda) \geq \mathscr{E}(y,u,\lambda) \geq 0,$$

where the second inequality follows from the dual feasibility of $\mathscr{S}$ and (4.3). As shown in the proof of Theorem 3.1, this inequality leads to the conclusion that $\varphi(x) \geq \lambda$.

(b) - (e) : The proofs are similar to that of part (a). □

Theorem 4.8. *(Strong Duality) Let x^* be a normal optimal solution of (P) and assume that any one of the five sets of conditions specified in Theorem 4.7 is satisfied for all feasible solutions of (DII). Then for any $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, and $\lambda^*(= \varphi(x^*)) \in \mathbb{R}_+$ such that $(x^*,z^*,u^*,v^*,w^*,\lambda^*)$ is an optimal solution of (DII) and $\varphi(x^*) = \lambda^*$.*

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 4.9. *(Strict Converse Duality) Let x^* be a normal optimal solution of (P), let $(\tilde{x},\tilde{z},\tilde{u},\tilde{v},\tilde{w},\tilde{\lambda})$ be an optimal solution of (DII), and assume that any one of the five sets of hypotheses specified in Theorem 4.7 is satisfied, and that the function $\xi \to \mathscr{E}(\xi,\tilde{u},\tilde{\lambda})$ is $(\bar{\phi},\eta,\bar{\rho},\theta,m)$-quasisonvex at $\tilde{x}$ and $\bar{\phi}(a) > 0 \Rightarrow a > 0$. Then $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.*

Proof. The proof is similar to that of Theorem 4.6. □

Theorem 4.10. *(Weak Duality) Let x and $\mathscr{S} \equiv (y,z,u,v,w,\lambda)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following seven sets of hypotheses is satisfied:*

(a) (i) *$\xi \to \mathscr{E}(\xi,u,\lambda)$ is prestrictly $(\bar{\phi},\eta,\omega,\bar{\rho},\theta,m)$-quasisonvex at y and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *for each $j \in J_+ \equiv J_+(v)$, G_j is strictly $(\hat{\phi}_j,\eta,\omega,\hat{\rho}_j,\theta,m)$-pseudosonvex at y, $\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0) = 0$;*

(iii) *for each* $k \in K_* \equiv K_*(w)$, $\xi \to \mathscr{D}_k(\xi, w)$ *is* $(\breve{\phi}_k, \eta, \omega, \breve{\rho}_k, \theta, m)$*-quasisonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\bar{\rho}(x,y) + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k \in K_*} \breve{\rho}_k(x,y) \geq 0$;

(b) (i) $\xi \to \mathscr{E}(\xi, u, \lambda)$ *is prestrictly* $(\bar{\phi}, \eta, \omega, \bar{\rho}, \theta, m)$*-quasisonvex at* y, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathscr{C}(\xi, v)$ *is strictly* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-pseudosonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) *for each* $k \in K_*$, $\xi \to \mathscr{D}_k(\xi, w)$ *is* $(\breve{\phi}_k, \eta, \omega, \breve{\rho}_k, \theta, m)$*-quasisonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\bar{\rho}(x,y) + \hat{\rho}(x,y) + \sum_{k \in K_*} \breve{\rho}_k(x,y) \geq 0$;

(c) (i) $\xi \to \mathscr{E}(\xi, u, \lambda)$ *is prestrictly* $(\bar{\phi}, \eta, \omega, \bar{\rho}, \theta, m)$*-quasisonvex at* y *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $j \in J_+$, G_j *is* $(\hat{\phi}_j, \eta, \omega, \hat{\rho}_j, \theta, m)$*-quasisonvex at* y, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) *for each* $k \in K_*$, $\xi \to \mathscr{D}_k(\xi, w)$ *is strictly* $(\breve{\phi}_k, \eta, \omega, \breve{\rho}_k, \theta, m)$*-pseudosonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\bar{\rho}(x,y) + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k \in K_*} w_k \breve{\rho}_m(x,y) \geq 0$;

(d) (i) $\xi \to \mathscr{E}(\xi, u, \lambda)$ *is prestrictly* $(\bar{\phi}, \eta, \omega, \bar{\rho}, \theta, m)$*-quasisonvex at* y *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $j \in J_+$, $\xi \to G_j$ *is* $(\hat{\phi}_j, \eta, \omega, \hat{\rho}_j, \theta, m)$*-quasisonvex at* y, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is strictly* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-pseudosonvex at* y, $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$;

(iv) $\bar{\rho}(x,y) + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \breve{\rho}(x,y) \geq 0$;

(e) (i) $\xi \to \mathscr{E}(\xi, u, \lambda)$ *is prestrictly* $(\bar{\phi}, \eta, \omega, \bar{\rho}, \theta, m)$*-quasisonvex at* y *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathscr{C}(\xi, v)$ *is strictly* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-pseudosonvex at* y*,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-quasisonvex at* y*,* $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$*;*

(iv) $\bar{\rho}(x,y) + \hat{\rho}(x,y) + \breve{\rho}(x,y) \geq 0$*;*

(f) (i) $\xi \to \mathscr{E}(\xi, u, \lambda)$ *is prestrictly* $(\bar{\phi}, \eta, \omega, \bar{\rho}, \theta, m)$*-quasisonvex at* y *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) $\xi \to \mathscr{C}(\xi, v)$ *is* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-quasisonvex at* y*,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is strictly* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-pseudosonvex at* y*,* $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}(0) = 0$*;*

(iv) $\bar{\rho}(x,y) + \hat{\rho}(x,y) + \breve{\rho}(x,y) \geq 0$*;*

(g) (i) $\xi \to \mathscr{E}(\xi, u, \lambda)$ *is prestrictly* $(\bar{\phi}, \eta, \omega, \bar{\rho}, \theta, m)$*-quasisonvex at* y *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) $\xi \to \mathscr{G}(\xi, v, w)$ *is strictly* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-pseudosonvex at* y*,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\bar{\rho}(x,y) + \hat{\rho}(x,y) \geq 0$.

Then $\varphi(x) \geq \lambda$.

Proof. (a) : Since for each $j \in J_+$, $G_j(x) - G_j(y) \leq 0$ and hence $\hat{\phi}_j\big(G_j(x) - G_j(y)\big) \leq 0$, (ii) implies that

$$\frac{1}{2}\langle \nabla G_j(y), \eta(x,y)\rangle + \frac{1}{2}\langle \omega(x,y), \nabla^2 G_j(y) z\rangle + \frac{1}{2}\langle \nabla G_j(y), z\rangle$$
$$< -\hat{\rho}_j(x,y)\|\theta(x,y)\|^m. \tag{4.12}$$

As $v_j \geq 0$ for each $j \in \underline{q}$ and $v_j = 0$ for each $j \in \underline{q} \backslash J_+$, the above inequalities yield

$$\frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j[\nabla G_j(y), \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{j=1}^{q} v_j \nabla^2 G_j(y) z\Big\rangle] + \frac{1}{2}\Big\langle \sum_{j=1}^{q} v_j[\nabla G_j(y), z\Big\rangle$$
$$< - \sum_{j\in J_+} v_j \hat{\rho}_j(x,y)\|\theta(x,y)\|^m. \tag{4.13}$$

Now combining this inequality with (4.1), (4.2), and (4.9) (which is valid for the present case because of (iii)), and using (iv), we obtain

$$\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] z\Big\rangle$$
$$+ \frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], z\Big\rangle$$
$$\geq \Big[\sum_{j\in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k\in K_*} w_k \breve{\rho}_k(x,y)\Big] \|\theta(x,y)\|^m > -\bar{\rho}(x,y)\|\theta(x,y)\|^m,$$

which in view of (i) implies that

$$\bar{\phi}\big(\mathscr{E}(x,u,\lambda) - \mathscr{E}(y,u,\lambda)\big) \geq 0.$$

The rest of the proof is identical to that of Theorem 4.1.

(b) - (g) : The proofs are similar to that of part (a). □

Theorem 4.11. *(Strong Duality) Let x^* be a normal optimal solution of (P) and assume that any one of the seven sets of conditions specified in Theorem 4.10 is satisfied for all feasible solutions of (DII). Then for each $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, and $\lambda^* \in \mathbb{R}_+$ such that $(x^*, z^*, u^*, v^*, w^*, \lambda^*)$ is an optimal solution of (DII) and $\varphi(x^*) = \lambda^*$.*

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 4.12. *(Strict Converse Duality) Let x^* be a normal optimal solution of (P), let $(\tilde{x}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda})$ be an optimal solution of (DII), and assume that any one of the seven sets of hypotheses specified in Theorem 4.10 is satisfied, and that the function $\xi \to \mathscr{E}(\xi, \tilde{u}, \tilde{\lambda})$ is $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at $\tilde{x}$. Then $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.*

Proof. The proof is similar to that of Theorem 4.6. □

In Theorems 4.4 - 4.12, various generalized $(\phi,\eta,\omega,\rho,\theta,m)$-sonvexity conditions were imposed on the function $\xi \to \mathscr{E}(\xi,u,\lambda)$, which is the weighted sum of the functions $\xi \to \mathscr{E}_i(\xi,\lambda)$, $i \in \underline{p}$. In the next few theorems, we shall assume that the individual functions $\xi \to \mathscr{E}_i(\xi,\lambda)$, $i \in \underline{p}$, satisfy appropriate generalized $(\phi,\eta,\rho,\theta,m)$-sonvexity hypotheses.

Theorem 4.13. *(Weak Duality) Let x and $\mathscr{S} \equiv (y,z,u,v,w,\lambda)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u)$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ is prestrictly $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$-quasisonvex at y, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $j \in J_+ \equiv J_+(v)$, G_j is $(\hat{\phi}_j,\eta,\omega,\hat{\rho}_j,\theta,m)$-quasisonvex at y, $\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0)=0$;*

(iii) *for each $k \in K_* \equiv K_*(w)$, $\xi \to \mathscr{D}_k(\xi,w)$ is $(\breve{\phi}_k,\eta,\omega,\breve{\rho}_k,\theta,m)$-quasisonvex at y, $\breve{\phi}_k$ is increasing, and $\breve{\phi}_k(0)=0$;*

(iv) *$\rho^\circ(x,y)+\sum_{j\in J_+} v_j\hat{\rho}_j(x,y)+\sum_{k\in K_*}\breve{\rho}_k(x,y)>0$, where $\rho^\circ(x,y)=\sum_{i\in I_+} u_i\bar{\rho}_i(x,y)$;*

(b) (i) *for each $i \in I_+$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ is prestrictly $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$-quasisonvex at y, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *$\xi \to \mathscr{C}(\xi,v)$ is $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$-quasisonvex at y, $\hat{\phi}$ is increasing, and $\hat{\phi}(0)=0$;*

(iii) *for each $k \in K_*$, $\xi \to \mathscr{D}_k(\xi)$ is $(\breve{\phi}_k,\eta,\omega,\breve{\rho}_k,\theta,m)$-quasisonvex at y, $\breve{\phi}_k$ is increasing, and $\breve{\phi}_k(0)=0$;*

(iv) *$\rho^\circ(x,y)+\hat{\rho}(x,y)+\sum_{k\in K_*}\breve{\rho}_m(x,y)>0$;*

(c) (i) *for each $i \in I_+$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ is prestrictly $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$-quasisonvex at y, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each* $j \in J_+$, G_j *is* $(\hat{\phi}_j, \eta, \omega, \pi, \hat{\rho}_j, \theta, m)$*-quasisonvex at* y, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$*;*

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-quasisonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}(0) = 0$*;*

(iv) $\rho^\circ(x,y) + \sum_{j\in J_+} v_j \hat{\rho}_j(x,y) + \breve{\rho}(x,y) > 0$*;*

(d) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi, \lambda)$ *is prestrictly* $(\bar{\phi}_i, \eta, \omega, \bar{\rho}_i, \theta, m)$*-quasisonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) $\xi \to \mathscr{C}(\xi, v)$ *is* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-quasisonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-quasisonvex at* y, $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$*;*

(iv) $\rho^\circ(x,y) + \hat{\rho}(x,y) + \breve{\rho}(x,y) > 0$*;*

(e) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi, \lambda)$ *is prestrictly* $(\bar{\phi}_i, \eta, \omega, \bar{\rho}_i, \theta, m)$*-quasisonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) $\xi \to \mathscr{G}(\xi, v, w)$ *is* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-quasisonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\rho^\circ(x,y) + \hat{\rho}(x,y) > 0$.

Then $\varphi(x) \geq \lambda$.

Proof. Suppose that $\varphi(x) < \lambda$. This implies that for each $i \in \underline{p}$, $\mathscr{E}_i(x,\lambda) < 0$. Since $\mathscr{E}_i(y,\lambda) \geq 0$ by the dual feasibility of $\mathscr{S}$ and (4.3), it follows that $\mathscr{E}_i(x,\lambda) < \mathscr{E}_i(y,\lambda)$, and hence for each $i \in I_+$, $\bar{\phi}_i\big(\mathscr{E}_i(x,\lambda) - \mathscr{E}_i(y,\lambda)\big) < 0$, which by virtue of (i) implies that

$$\begin{aligned}
&\frac{1}{2}\langle \nabla f_i(y) - \lambda \nabla g_i(y), \eta(x,y)\rangle + \frac{1}{2}\langle \omega(x,y), [\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)]z\rangle \\
&+\frac{1}{2}\langle \nabla f_i(y) - \lambda \nabla g_i(y), \pi(x,y)\rangle \\
&\leq -\bar{\rho}_i(x,y)\|\theta(x,y)\|^m.
\end{aligned}$$

Since $u \geq 0$ and $\sum_{i=1}^{p} u_i = 1$, the above inequalities yield

$$\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)]z\Big\rangle$$
$$+\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], z\Big\rangle$$
$$\leq -\sum_{i\in I_+} u_i \bar{\rho}_i(x,y)\|\theta(x,y)\|^m.$$

Next, combining (4.1) and (4.2) with (4.8) and (4.9), which are valid for the present case because of the assumptions set forth in (ii) and (iii), and using (iv), we get

$$\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)]z\Big\rangle$$
$$+\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], z\Big\rangle$$
$$\geq \Big[\sum_{j\in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k\in K_*} \breve{\rho}_k(x,y)\Big]\|\theta(x,y)\|^m > -\sum_{i\in I_+} u_i \bar{\rho}_i(x,y)\|\theta(x,y)\|^m,$$

which contradicts (4.10). Therefore, we conclude that $\varphi(x) \geq \lambda$.

(b) - (e) : The proofs are similar to that of part (a). □

Theorem 4.14. *(Strong Duality) Let x^* be a normal optimal solution of (P) and assume that any one of the five sets of conditions specified in Theorem 4.13 is satisfied for all feasible solutions of (DII). Then for each $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, and $\lambda^* \in \mathbb{R}_+$ such that $(x^*, z^*, u^*, v^*, w^*, \lambda^*)$ is an optimal solution of (DII) and $\varphi(x^*) = \lambda^*$.*

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 4.15. *(Weak Duality) Let x and (y,z,u,v,w,λ) be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following seven sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u)$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ is prestrictly $(\bar{\phi}_i, \eta, \omega, \bar{\rho}_i, \theta, m)$-quasisonvex at y, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each* $j \in J_+ \equiv J_+(v)$, G_j *is strictly* $(\hat{\phi}_j, \eta, \omega, \hat{\rho}_j, \theta, m)$*-pseudosonvex at* y, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) *for each* $k \in K_* \equiv K_*(w)$, $\xi \to \mathscr{D}_k(\xi, w)$ *is* $(\breve{\phi}_k, \eta, \omega, \breve{\rho}_k, \theta, m)$*-quasisonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\rho^\circ(x,y) + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k \in K_*} \breve{\rho}_k(x,y) \geq 0$, *where* $\rho^\circ(x,y) = \sum_{i \in I_+} u_i \bar{\rho}_i(x,y)$;

(b) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi, \lambda)$ *is prestrictly* $(\bar{\phi}_i, \eta, \omega, \bar{\rho}_i, \theta, m)$*-quasisonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0) = 0$;

(ii) *for each* $j \in J_+$, G_j *is* $(\hat{\phi}_j, \eta, \omega, \hat{\rho}_j, \theta, m)$*-quasisonvex at* y, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) *for each* $k \in K_*$, $\xi \to \mathscr{D}_k(\xi, w)$ *is strictly* $(\breve{\phi}_k, \eta, \omega, \breve{\rho}_k, \theta, m)$*-pseudosonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\rho^\circ(x,y) + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k \in K_*} \breve{\rho}_k(x,y) \geq 0$;

(c) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi, \lambda)$ *is prestrictly* $(\bar{\phi}_i, \eta, \omega, \bar{\rho}_i, \theta, m)$*-quasisonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0) = 0$;

(ii) $\xi \to \mathscr{C}(\xi, v)$ *is strictly* $(\hat{\phi}, \eta, \omega, \hat{\rho}, \theta, m)$*-pseudosonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) *for each* $k \in K_*$, $\xi \to \mathscr{D}_k(\xi, w)$ *is strictly* $(\breve{\phi}_k, \eta, \omega, \breve{\rho}_k, \theta, m)$*-pseudosonvex at* y, $\breve{\phi}_k$ *is increasing, and* $\breve{\phi}_k(0) = 0$;

(iv) $\rho^\circ(x,y) + \hat{\rho}(x,y) + \sum_{k \in K_*} \breve{\rho}_k(x,y) \geq 0$;

(d) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi, \lambda)$ *is prestrictly* $(\bar{\phi}_i, \eta, \omega, \bar{\rho}_i, \theta, m)$*-quasisonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0) = 0$;

(ii) *for each* $j \in J_+$, G_j *is* $(\hat{\phi}_j, \eta, \omega, \hat{\rho}_j, \theta, m)$*-quasisonvex at* y, $\hat{\phi}_j$ *is increasing, and* $\hat{\phi}_j(0) = 0$;

(iii) $\xi \to \mathscr{D}(\xi, w)$ *is strictly* $(\breve{\phi}, \eta, \omega, \breve{\rho}, \theta, m)$*-pseudosonvex at* y, $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$;

(iv) $\rho^\circ(x,y) + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \breve{\rho}(x,y) \geq 0$;

(e) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ *is prestrictly* $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$*-quasisonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0)=0$;

(ii) $\xi \to \mathscr{C}(\xi,v)$ *is strictly* $(\hat{\phi},\omega,\eta,\hat{\rho},\theta,m)$*-pseudosonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$;

(iii) $\xi \to \mathscr{D}(\xi,w)$ *is* $(\breve{\phi},\eta,\omega,\breve{\rho},\theta,m)$*-quasisonvex at* y, $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0)=0$;

(iv) $\rho^\circ(x,y)+\hat{\rho}(x,y)+\breve{\rho}(x,y)\geq 0$;

(f) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ *is prestrictly* $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$*-quasisonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0)=0$;

(ii) $\xi \to \mathscr{C}(\xi,v)$ *is* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-quasisonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$;

(iii) $\xi \to \mathscr{D}(\xi,w)$ *is strictly* $(\breve{\phi},\eta,\omega,\breve{\rho},\theta,m)$*-pseudosonvex at* y, $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0)=0$;

(iv) $\rho^\circ(x,y)+\hat{\rho}(x,y)+\breve{\rho}(x,y)\geq 0$;

(g) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ *is prestrictly* $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$*-quasisonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0)=0$;

(ii) $\xi \to \mathscr{G}(\xi,v,w)$ *is strictly* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-pseudosonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$;

(iii) $\rho^\circ(x,y)+\hat{\rho}(x,y)\geq 0$.

Then $\varphi(x)\geq\lambda$.

Proof. The proof is similar to that of Theorem 4.10. □

Theorem 4.16. *(Strong Duality) Let* x^* *be a normal optimal solution of (P) and assume that any one of the seven sets of conditions specified in Theorem 4.15 is satisfied for all feasible solutions of (DII). Then for each* $z^* \in C(x^*)$*, there exist* $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$*, and* $\lambda^* \in \mathbb{R}_+$ *such that* $(x^*,z^*,u^*,v^*,w^*,\lambda^*)$ *is an optimal solution of (DII) and* $\varphi(x^*)=\lambda^*$.

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 4.17. *(Weak Duality) Let x and $\mathscr{S} \equiv (y,z,u,v,w,\lambda)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u)$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ is $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$-pseudosonvex at y, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $j \in J_+ \equiv J_+(v)$, G_j is $(\hat{\phi}_j,\eta,\omega,\hat{\rho}_j,\theta,m)$-quasisonvex at y, $\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0)=0$;*

(iii) *for each $k \in K_* \equiv K_*(w)$, $\xi \to \mathscr{D}_k(\xi,w)$ is $(\breve{\phi}_k,\eta,\omega,\breve{\rho}_k,\theta,m)$-quasisonvex at y, $\breve{\phi}_k$ is increasing, and $\breve{\phi}_k(0)=0$;*

(iv) $\rho^\circ(x,y)+\sum_{j\in J_+} v_j\hat{\rho}_j(x,y)+\sum_{k\in K_*}\breve{\rho}_k(x,y)\geq 0$, *where* $\rho^\circ(x,y)=\sum_{i\in I_+} u_i\bar{\rho}_i(x,y)$;

(b) (i) *for each $i \in I_+$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ is $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$-pseudosonvex at y, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *$\xi \to \mathscr{C}(\xi,v)$ is $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$-quasisonvex at y, $\hat{\phi}$ is increasing, and $\hat{\phi}(0)=0$;*

(iii) *for each $k \in K_*$, $\xi \to \mathscr{D}_k(\xi,w)$ is $(\breve{\phi}_k,\eta,\omega,\breve{\rho}_k,\theta,m)$-quasisonvex at y, $\breve{\phi}_k$ is increasing, and $\breve{\phi}_k(0)=0$;*

(iv) $\rho^\circ(x,y)+\hat{\rho}(x,y)+\sum_{k\in K_*}\breve{\rho}_k(x,y)\geq 0$;

(c) (i) *for each $i \in I_+$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ is $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$-pseudosonvex at y, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $j \in J_+$, G_j is $(\hat{\phi}_j,\eta,\omega,\hat{\rho}_j,\theta,m)$-quasisonvex at y, $\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0)=0$;*

(iii) *$\xi \to \mathscr{D}(\xi,w)$ is $(\breve{\phi},\eta,\omega,\breve{\rho},\theta,m)$-quasisonvex at y, $\breve{\phi}$ is increasing, and $\breve{\phi}(0)=0$;*

(iv) $\rho^\circ(x,y)+\sum_{j\in J_+} v_j\hat{\rho}_j(x,y)+\breve{\rho}(x,y)\geq 0$;

(d) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ *is* $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$*-pseudosonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0)=0$*;*

(ii) $\xi \to \mathscr{C}(\xi,v)$ *is* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-quasisonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$*;*

(iii) $\xi \to \mathscr{D}(\xi,w)$ *is* $(\breve{\phi},\eta,\omega,\breve{\rho},\theta,m)$*-quasisonvex at* y, $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0)=0$*;*

(iv) $\rho^\circ(x,y)+\hat{\rho}(x,y)+\breve{\rho}(x,y)\geq 0$*;*

(e) (i) *for each* $i \in I_+$, $\xi \to \mathscr{E}_i(\xi,\lambda)$ *is* $(\bar{\phi}_i,\eta,\omega,\bar{\rho}_i,\theta,m)$*-pseudosonvex at* y, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0)=0$*;*

(ii) $\xi \to \mathscr{G}(\xi,v,w)$ *is* $(\hat{\phi},\eta,\omega,\hat{\rho},\theta,m)$*-quasisonvex at* y, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$*;*

(iii) $\rho^\circ(x,y)+\hat{\rho}(x,y)\geq 0.$

Then $\varphi(x)\geq\lambda$.

Proof. (a) : Suppose that $\varphi(x)<\lambda$. This implies that for each $i\in \underline{p}$, $\mathscr{E}_i(x,\lambda)<0$. Since $\mathscr{E}_i(y,\lambda)\geq 0$ by the dual feasibility of $\mathscr{S}$ and (4.3), it follows that $\mathscr{E}_i(x,\lambda)<\mathscr{E}_i(y,\lambda)$, and hence for each $i\in I_+$, $\bar{\phi}_i\big(\mathscr{E}_i(x,\lambda)-\mathscr{E}_i(y,\lambda)\big)<0$, which by virtue of (i) implies that

$$\frac{1}{2}\langle\nabla f_i(y)-\lambda\nabla g_i(y),\eta(x,y)\rangle+\frac{1}{2}\langle\omega(x,y),[\nabla^2 f_i(y)-\lambda\nabla^2 g_i(y)]z\rangle$$
$$+\frac{1}{2}\langle\nabla f_i(y)-\lambda\nabla g_i(y),z\rangle$$
$$<-\bar{\rho}_i(x,y)\|\theta(x,y)\|^m.$$

Since $u\geq 0$ and $\sum_{i=1}^{p} u_i=1$, the above inequalities yield

$$\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)]z\Big\rangle$$
$$+\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], z\Big\rangle$$
$$< -\sum_{i\in I_+} u_i \bar{\rho}_i(x,y)\|\theta(x,y)\|^m. \tag{4.14}$$

Next, combining (4.1) and (4.2) with (4.8) and (4.9), which are valid for the present case because of the assumptions set forth in (ii) and (iii), and using (iv), we get

$$\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], \eta(x,y)\Big\rangle + \frac{1}{2}\Big\langle \omega(x,y), \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)]z\Big\rangle$$
$$+\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)], z\Big\rangle$$
$$\geq \Big[\sum_{j\in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k\in K_*} \breve{\rho}_k(x,y)\Big]\|\theta(x,y)\|^m \geq -\sum_{i\in I_+} u_i \bar{\rho}_i(x,y)\|\theta(x,y)\|^m,$$

which contradicts (4.10). Therefore, we conclude that $\varphi(x) \geq \lambda$.

(b) - (e) : The proofs are similar to that of part (a). □

5. Concluding Remarks

In this paper, we have introduced and formulated a number of second-order parametric duality models for a discrete minmax fractional programming problem and established a multiplicity of duality theorems using the several classes of the generalized $(\phi, \eta, \omega, \rho, \theta, m)$-sonvexity assumptions. Furthermore, our approach for main results may prove useful for applications to other related branches of nonlinear programming problems based on other similar generalized invexity concepts. It may be interesting to observe that employing similar techniques, one can investigate (and establish) the sufficient optimality and duality models of the following semiinfinite minmax fractional programming problem:

$$\text{Minimize } \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)}$$

subject to

$$G_j(x,t) \leq 0 \text{ for all } t \in T_j, \ \ j \in \underline{q},$$

$$H_k(x,s) = 0 \text{ for all } s \in S_k, \ \ k \in \underline{r},$$

$$x \in X,$$

where X, f_i, and g_i, $i \in \underline{p}$, are as defined in the description of (P), for each $j \in \underline{q}$ and $k \in \underline{r}$, T_j and S_k are compact subsets of complete metric spaces, for each $j \in \underline{q}$, $\xi \to G_j(\xi,t)$ is a real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r}$, $\xi \to H_k(\xi,s)$ is a real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $t \to G_j(x,t)$ and $s \to H_k(x,s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$.

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