Journal of the Orissa Mathematical Society ISSN:0975-2323
Volume 40, No.01-02, 2021, 17-56

EQUILIBRIUM PROBLEMS AND VARIATIONAL INEQUALITIES: A SURVEY OF EXISTENCE RESULTS

O. CHADLI[1], R. N. MOHAPATRA[2,*] AND B. K. SAHU[3]

[1]*Department of Economics, Ibn Zohr University, B.P. 8658, Poste Dakhla, Agadir, Morocco*
[2]*Department of Mathematics, University of Central Florida, USA*
[3]*Department of Mathematics, Chandbali College, Chandbali, Under the Dept. of Higher Education, Govt. of Odisha, Bhadrak, Odisha, India.*

Abstract. Equilibrium problem takes a leading role over a variety of mathematical problems including but not limited to variational inequalities, optimizations, complementarity problems, fixed point theory in last the four decades. It provides an unified framework for the solution of many problems with numerous applications. This paper presents a state-of-the-art survey of equilibrium problems, variational inequalities and hemivariational inequalities.

Keywords: Hemivariational inequality; Star-shaped set; Equilibrium Problems; Maximal monotone bifunction; Pseudomonotone bifunction; Clarke subgradient; Distance function

2020 AMS Subject Classification: 90C33 · 49J40 · 47J20 · 47H05 · 49J53

1. Introduction

In the last three decades equilibrium problem plays a key role for the study of many problems in nonlinear analysis. The initial incitement for research on computation of equilibria come from the need of analyzing the general equilibrium theory and to apply this theory for the study problems like taxation and unemployment. Consequently in 1955, Nikaido and Isoda [1] considered implicitly an equilibrium problem

*Corresponding author.
E-mail addresses: o.chadli@uiz.ac.ma, ram.mohapatra@ucf.edu, sahubk1987@gmail.com

formulation with an aim to characterize the Nash equilibrium. The first results on the existence of solutions for equilibrium problems were given in the paper by Ky Fan [2] where he proved a minimax principle by using his own generalized version (Ky Fan [3]) of Knaster-Kuratowski-Mazurkiewicz's theorem. The problem itself was called *minimax inequality* by Ky Fan, but nowadays it is widely known within the literature as *equilibrium problem*.

In 1976, Mosco [4] introduced the notion of monotonicity for bifunctions and obtained some existence results under weaker assumptions than those obtained by Ky Fan [2]. The appellation *equilibrium problem* was introduced in the paper by Blum and Oettli [5] in 1994, where it has been shown that equilibrium problems include variational inequalities, fixed point, Nash equilibrium and game theory as special cases. Equilibrium problems, for short (EP), have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. We particularly refer our readers to [6, 7] and the bibliographies therein for excellent surveys concerning a large spectrum of optimization and equilibrium models that can be written in the form (EP).

Like equilibrium problem, *variational inequality problem* is a general problem formulation that encompasses a surfeit of mathematical problems like, complementarity problems, partial differential equations, optimization problems and fixed point problems. The concept of variational inequality was first appeared in an elastostatic problem, posed by Antonio Signorini in 1953, known as Signorini problem. Historically, the variational inequality problem was introduced by Philip Hartman and Guido Stampacchia in the seminal paper [8]. Variational inequality problems have successively been applied in the following major areas: fluid flow through porous media, lubrication problem, Nash equilibrium problems, traffic network equilibrium problems, spatial price equilibrium problems, financial equilibrium problems and nonlinear programing, see e.g. [9, 10, 11, 12].

Convexity of functions plays a key role in different areas of applied mathematics, for instance in extremum problems. That is, some necessary conditions for the existence of a minimum become also sufficient in the presence of convexity. Another important application of convexity is that in the presence of convexity, the lower semicontinuity property of an operator becomes weakly lower semicontinuity [13]. Whereas, many real-life problems can be described in terms of nonconvex functions. However, these functions, in spite of being nonconvex, retain some of the nice properties and characteristics of convex functions. For instance, their presence may ensure that necessary conditions for a minimum are also sufficient or that a local minimum is also a global one. This led to the introduction of several generalizations of the classical concept of a convex function. Another well-known and useful property of a convex function is that its sublevel sets are convex. Many simple nonconvex functions also have this property. If we consider the class of all functions whose sublevel sets are convex, we obtain what is called the class of quasiconvex functions that was initiated by Finetti [14] in 1949. A significant generalization of convex functions is the class of invex functions introduced by Hanson [15] in the year of 1981, where he has further weaken the weakest conditions speculated by Mangasarian [16] for the Kuhn-Tucker conditions of minimization problem. The concepts of invex functions lead to important applications in convex programming. Application of the invexity concept widens the scope of both variational inequalities and equilibrium problems. Study of equilibrium problems with invexity is carried out in [17, 18, 19, 20] and the references therein.

The monotonicity of operators takes a key role for the description of the equilibrium problems and variational inequalities. The notion of monotonicity was introduced by Minty [21] and Browder [22]. When the underlying space X of the variational inequalities is finite dimensional, then the continuity properties of the associated operators guarantees the existence of solutions for the variational inequalities [8]. But, when X is infinite dimensional, then continuity properties is not sufficient for the existence of solutions, whereas monotonicity properties of the operator guarantees the existence of solutions. A variant of monotonicity known as strict monotonicity provides the uniqueness of solutions for variational inequalities, see e.g. [11]. The key feature of monotonicity theory is that it does not require continuity assumptions or the single-valued properties of the operators that needed in contraction mappings and fixed point approaches. In order to deal with the existence of solutions for nonlinear partial differential equations of elliptic and parabolic type in the context of monotone operators, various generalizations of monotone operators are needed. The convexity of a proper lower semicontinuous function can be characterized by monotonicity of its subdifferential. More about the convexity and monotonicity can be found in the handbook [23] and the references therein. By the time, the concept of monotonicity has been generalized into pseudomonotonicity, quasimonotonicity, relaxed monotonicity; see e.g. [24, 25, 26, 27].

In 2003, Fang and Huang [28] have defined relaxed $\eta-\alpha$ monotone mappings and established certain results of variational-like inequalities. Bai et al. [29] in 2006, have defined relaxed $\eta-\alpha$ pseudomonotone operators and generalized the results of Fang and Huang [28]. Inspired by Fang and Huang [28] and Bai et al. [29], Liu and Zeng [19] in 2016, studied rigorously the equilibrium problem in the framework of invexity assumptions with relaxed $\alpha-\eta$ pseudomonotonicity for bifunctions. They called it as invex equilibrium problem. The work of Liu and Zeng [19] on invex equilibrium problem was then generalized by Pany et al. [30] under generalized $\alpha-\eta$ pseudomonotone mappings and then by Sahu and Pani [31] under the setting of generalized relaxed $\eta - \alpha$ pseudomonotone mappings and strictly η-quasimonotone mappings. In 2020, Sahu et al. [32] generalized the invex equilibrium problems to mixed invex equilibrium problems under the setting of generalized relaxed monotone mappings.

Another most general property of operators that can be used to manipulate equilibrium problems as well as variational inequalities is pseudomonotonicity in topological sense. It was initiated by H. Brézis [33] in the year of 1968 and later it is known as pseudomonotonicity in the sense of Brézis. The unique property of Brézis pseudomonotonicity that it is the hybridization of both monotonicity as well as continuity, distinguishes it from the others. For instance, Kien et al. [34] proves that there exists an operator which is pseudomonotone in the sense of Brézis but not pseudomonotone in the sense of Karamardian. The class of pseudomonotone operators in the sense of Brézis is quite large and rich in applications, see the very important paper by Browder and Hess [35]. A very interesting and useful application of Brézis pseudomonotonicity was given by Chadli et al. [36] in 2016, for finding the anti-periodic solutions of nonlinear evolution equations. Another very important use of Brézis pseudomonotonicity in the study of coercive and noncoercive hemivariational inequalities can be found in [37, 38].

The concept of hemivariational inequalities was introduced by Panagiotopoulos [39] in 1983 as a variational formulation for several classes of mechanical problems with non-smooth and non-convex energy function. Various existence results for hemivariational inequalities in a coercive and semicoercive framework can be found in [40, 41], where the authors have used the well known Browder technique based on a Galerkin approximaiton and a fixed point theorem. The case of noncoercive hemivariational inequalities was then analyzed by Adly et al. [42] in 1995 using the concept of recession analysis. The use of Equilibrium Problems Theory in the study of coercive and noncoercive hemivariational inequalities has been initiated in [43, 44] by means of monotonicity assumptions and arguments from the recession analysis. In

[37], a Tikhonov type regularization procedure has been introduced for equilibrium problems and used to derive some existence results for non coercive hemivariational inequalities.

The paper is organized as follows. In Section 2, we provide some preliminary concepts and results that are needed in the sequel. Section 3 deals with the existence of solutions for the equilibrium problems with various kinds of generalized monotone bifunctions. In section 4, we focus on invex equilibrium problems and mixed invex equilibrium problems, where we meet various generalizations of Karamardian pseudomonotone operators. In Section 5, we give various results on the existence of solutions for Brézis pseudomonotone mixed equilibrium problems involving a set-valued mapping in a general setting vector spaces in duality. We present also some recent results on the existence of solutions for quasi-hemivariational inequalities in Banach spaces.

2. Preliminaries

In the sequel, for a subset M of a topological vector space X, we shall denote by $co(M)$ the convex hull of M, by $cl(M)$ the closure of M in X, by $int(M)$ the interior of M and by $\mathscr{F}(M)$ the family of all finite subsets of M. For X be a topological vector space over $\mathbb{R}$, Y be a real vector space and $F : X \rightrightarrows Y$ is a set-valued mapping, we shall denote by $\mathscr{D}(F)$ the domain of F, i.e. $\mathscr{D}(F) = \{x \in X : F(x) \neq \emptyset\}$, and by $\mathscr{G}(F)$ the graph of F, i.e. $\mathscr{G}(F) = \{(x, \varpi) \in X \times Y : \varpi \in F(x)\}$.

Definition 2.1. Let $\mathbb{B}$ be a real Banach space, K be a nonempty closed subset of $\mathbb{B}$ and $\eta : K \times K \to \mathbb{B}$ be a mapping. The set K is said to be invex at a point $u \in K$, with respect to η if $u + t\eta(v,u) \in K$, for all $v \in K$ and $t \in [0,1]$. The set K is said to be invex with respect to η if it is invex at each point $u \in K$.

Definition 2.2. Let X be a metric space and $f : X \to \mathbb{R}$ be a real-valued function. The function f is said to be

(i) lower semicontinuous at $x \in X$ if, for any sequence $\{x_n\} \in X$ converging to x, we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n);$$

(ii) weakly lower semicontinuous at $x \in X$ if, for any sequence $\{x_n\} \in X$ converging weakly to x, we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n);$$

The function f is upper semicontinuous (weakly upper semicontinuous) at $x \in X$ if $-f$ is lower semicontinuous (weakly lower semicontinuous) at $x \in X$. We say that f is lower semicontinuous (weakly lower semicontinuous) if f is lower semicontinuous (weakly lower semicontinuous) for every $x \in X$.

Lemma 2.1. [13] *Let X be a normed space. If a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and convex, then it is weakly lower semicontinuous.*

Lemma 2.2. [13] *Let X be a reflexive Banach space, X^* be its dual and $K \subset X$ be bounded, closed and convex subset of X. Then K is compact in the weak topology $\sigma(X, X^*)$.*

Definition 2.3. Let X and Z be two Hausdorff topological vector spaces. A set-valued mapping $F : X \rightrightarrows Z$ is said to be

(i) lower semicontinuous at a point $x_0 \in X$ (for short, l.s.c. at x_0), if and only if, for any open set $\mathscr{O} \subset Z$ such that $F(x_0) \cap \mathscr{O} \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap \mathscr{O} \neq \emptyset$ for every $x \in U$. We say that F is lower semicontinuous (for short, l.s.c.) if F is l.s.c. for every $x_0 \in X$;

(ii) upper semicontinuous at a point $x_0 \in X$ (for short, u.s.c. at x_0), if and only if, for any open set $\mathscr{O} \subset Z$ such that $F(x_0) \subset \mathscr{O}$, there exists a neighborhood U of x_0 such that $F(x) \subset \mathscr{O}$ for every $x \in U$. We say that F is upper semicontinuous (for short, u.s.c.) if F is u.s.c. for every $x_0 \in X$;

(iii) closed, if and only if, for every net $\{x_\alpha\}_{\alpha\in I} \subset X$ converging to x and any $\{y_\alpha\}_{\alpha\in I} \subset Y$ converging to y such that $y_\alpha \in F(x_\alpha)$ for all $\alpha \in I$, we have $y \in F(x)$, i.e. $\mathscr{G}(F)$ is closed.

The following concepts and properties can be recalled from Aubin et al. [45].

Proposition 2.1. [45] *Let X and Z be two Hausdorff topological vector spaces and $F : X \rightrightarrows Z$ be a set-valued mapping. Let M be a subset of X such that $F(x)$ is closed for all $x \in M$. If F is u.s.c. and M is closed, then $\mathscr{G}(F)$ is closed. If M is closed and $cl(F(M))$ is compact, then F is u.s.c. if and only if $\mathscr{G}(F)$ is closed.*

Proposition 2.2. [45] *Let X and Z be two Hausdorff topological spaces, and $F : K \rightrightarrows Z$ be a set-valued mapping where K is a compact subset of X. If F is upper semicontinuous with compact values, then $F(K) := \bigcup_{x\in K} F(x)$ is compact.*

Proposition 2.3. [45] *Let X and Z be two Hausdorff topological vector spaces and $F : X \rightrightarrows Z$ a set-valued mapping. Then, F is l.s.c. if and only if, for any pair $(x,y) \in \mathscr{G}(F)$ and any net $\{x_\alpha\}_{\alpha\in I} \subset X$ converging to x, we can determine, for each $\alpha \in I$, an element $y_\alpha \in F(x_\alpha)$ such that $y_\alpha \to y$;*

Definition 2.4. A real-valued function f defined on a convex subset K of a real reflexive Banach space $\mathbb{B}$ is said to be hemicontinuous if $\lim\limits_{t\to 0^+} f(tx+(1-t)y) = f(y)$ for all $x, y \in K$.

Definition 2.5. Let X be a topological vector space over $\mathbb{R}$, Y be a real vector space and $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{R}$ be a bilinear form. A single-valued mapping $F : X \to Y$ is said to be

(i) hemicontinuous if, for all $x, y, z \in X$, the functional $t \mapsto \langle F(x+ty), z\rangle$ is continuous on $[0,1]$;

(ii) upper hemicontinuous if, for all $x, y, z \in X$, the functional $t \mapsto \langle F(x+ty), z\rangle$ is upper semicontinuous on $[0,1]$.

Definition 2.6. Let X be a topological vector space. A real-valued bifunction $\Phi : X \times X \to \mathbb{R}$ is said to be

(i) hemicontinuous if, for all $x, y, z \in X$, the functional $t \mapsto \Phi(tx + (1-t)y, z)$ is continuous on $[0,1]$;

(ii) upper hemicontinuous if, for all $x, y, z \in X$, the functional $t \mapsto \Phi(tx + (1-t)y, z)$ is upper semicontinuous on $[0,1]$.

Definition 2.7. Let K be a nonempty closed and convex subset of a real Banach space $\mathbb{B}$ and $\mathbb{B}^*$ be its dual. Let $\langle \cdot, \cdot \rangle : \mathbb{B}^* \times \mathbb{B} \to \mathbb{R}$ be a duality pairing between $\mathbb{B}^*$ and $\mathbb{B}$. A mapping $F : K \to \mathbb{B}^*$ is said to be,

(i) monotone if, for all $x, y \in K$,

$$\langle F(x) - F(y), x - y \rangle \geq 0;$$

(ii) pseudomonotone (in the sense of Karamardian) if, for all $x, y \in K$,

$$\langle F(x), y - x \rangle \geq 0 \implies \langle F(y), y - x \rangle \geq 0.$$

Definition 2.8. Let X be a topological vector space over $\mathbb{R}$, Y be a real vector space and $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ be a bilinear form. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(i) monotone if, for any $x, z \in \mathscr{D}(F)$,

$$\langle \varpi - \vartheta, x - z \rangle \geq 0, \quad \text{for all } \varpi \in F(x) \text{ and } \vartheta \in F(z);$$

(ii) maximal monotone if, $\langle \varpi - \vartheta, x - z \rangle \geq 0$ for all $(z, \vartheta) \in \mathscr{G}(F)$ implies $x \in \mathscr{D}(F)$ and $\varpi \in F(x)$.

Now, we recall some concepts for real-valued bifunctions inspired from similar concepts defined for operators acting from a topological vector space to its dual space [5, 6].

Definition 2.9. Let X be a topological vector space and K be a nonempty subset of X. A real-valued bifunction $\Phi : K \times K \to \mathbb{R}$ is said to be

(i) monotone if, for all $x, z \in K$

$$\Phi(x,z) + \Phi(z,x) \leq 0; \tag{2.1}$$

(ii) pseudomonotone (in the sense of Karamardian) if, for all $x, z \in K$

$$\Phi(x,z) \geq 0 \text{ implies } \Phi(z,x) \leq 0.$$

If we take $\Phi(x,y) = \langle F(x), y - x\rangle$, where F is an operator from $\mathbb{B}$ to $\mathbb{B}^*$, the dual of real Banach space $\mathbb{B}$, then the monotonicity (2.1) of the bifunction Φ reduces to the monotonicity of the operator F as described earlier.

The following notion of maximal monotonicity was initiated by Blum and Oettli [5] in an aim to extend the concept maximal monotone operators to bifunctions describing an equilibrium problem.

Definition 2.10. [5] Let K be a nonempty closed and convex subset of a real reflexive Banach space $\mathbb{B}$ and $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Φ is said to be maximal monotone in the sense of Blum and Oettli if and only if for every $x \in K$ and for every convex function $\phi : K \to \mathbb{R}$ with $\phi(x) = 0$, we have

$$\Phi(y,x) \leq \phi(y),\ \forall y \in K \text{ implies } 0 \leq \Phi(x,y) + \phi(y),\ \forall y \in K.$$

Definition 2.11. Let K be a nonempty closed and convex subset of a real reflexive Banach space $\mathbb{B}$ and $\Phi : K \times K \to \mathbb{R}$ be a bifunction.

(i) For each $z \in K$, $\Phi(\cdot, z)$ is said to be upper sign continuous if

$$\Phi(x_t, z) \geq 0,\ \forall t \in (0,1) \text{ implies } \Phi(x,z) \geq 0,$$

for every $x \in K$ and $x_t = (1-t)x + tz$.

(ii) For each $z \in K$, $\Phi(z, \cdot)$ is said to be quasiconvex if, for every $x, y \in K$ and $x_t = (1-t)x + ty$, $t \in [0,1]$, we have

$$\Phi(z, x_t) \leq \max\{\Phi(z,x), \Phi(z,y)\}.$$

The finite dimensional version of KKM theorem given by Knaster et al. [46] was extended by Fan [3] in 1961 and again by himself [2] in 1972. For a nonempty subset

K of a Hausdorff topological vector space X, a set-valued mapping $F : K \rightrightarrows X$ is said to be a *KKM mapping* if for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of K, the convex hull $co(\{y_1, y_2, \cdots, y_n\}) \subset \bigcup_{i=1}^{n} F(y_i)$.

Lemma 2.3. [3] *Let K be a nonempty subset of a Hausdorff topological vector space X and $F : K \rightrightarrows X$ be a KKM mapping. If $F(y)$ is closed in X for all $y \in K$ and compact for some $y_0 \in K$, then*

$$\bigcap_{y \in K} F(y) \neq \emptyset.$$

3. Equilibrium Problems with Generalized Monotone Operators

Let K be a nonempty subset of a topological vector space X and $\Phi : K \times K \to \mathbb{R}$ be a real-valued bifunction. By equilibrium problem, we mean the following problem: Find $\bar{x} \in K$, such that

$$\Phi(\bar{x}, y) \geq 0, \text{ for all } y \in K. \tag{3.1}$$

The bifunction $\Phi : K \times K \to \mathbb{R}$ is said to be an *equilibrium bifunction* if it satisfies $\Phi(x,x) = 0$ for all $x \in K$.

The first existence results for the problem (3.1) was given by Ky Fan [2]. The equilibrium problem (3.1) was then studied extensively by various authors, see e.g. [5, 9, 25, 47, 48, 49, 50]. In particular, Noor and Oettli [51] extended the problem (3.1) in the year of 1995 to quasi equilibrium problems and pointed out that this is an extension of Nash equilibria. A significant generalization of problem (3.1) was given by Bianchi and Schaible [47] in the year of 1996 using quasimonotone and pseudomonotone bifunctions in topological vector spaces, where they have also extended the work of Karamardian [52] from variational inequality and complementarity problems to equilibrium problems.

When the bifunction Φ is vector-valued, then a general formulation of the problem (3.1) leads to what is called *vector equilibrium problem*. Vector equilibrium problems can be viewed as further and natural extension of vector variational inequalities introduced by Giannessi [53] in 1980. It is a unified model of several known problems, namely, vector variational inequality problems, vector optimization problems, vector saddle point problems and Nash equilibrium problems for vector-valued functions. Many authors have contributed to the study of vector equilibrium problems and their applications, we refer our readers to [54, 55, 56, 57, 58, 59, 60] and to the recent book by Ansari, Köbis and Yao [61].

The dual of an equilibrium problem has been considered in the following references [48, 62, 63]. It is defined as the following:

Definition 3.1. Let X be a real topological vector space, K be a nonempty subset of X and $\Phi : K \times K \to \mathbb{R}$ be a real-valued bifunction. The dual equilibrium problem consists to find an $x \in K$ such that

$$\Phi(y,x) \leq 0, \text{ for all } y \in K. \tag{3.2}$$

Remark 3.1. Martinez-Legaz and Sosa [64] used a diffrent approach to define the dual of an equilibrium problem. They introduced an optimization problem as a dual to (EP), relying on a gap function. On the contrary, the definition of a dual equilibrium problem given above allows to introduce a dual equilibrium problem with no need of a gap function and without formulating it as an optimization problem. In this way also the duality approach for variational inequalities developed by Mosco [65] can be easily recovered within this framework.

Below, we give some results on the existence of solutions for the dual equilibrium problem (3.2). For more results, we refer to [62, 63, 64, 66].

Theorem 3.1. [48] *Let X be a Hausdorff topological vector space and K be a nonempty closed convex subset of X. Consider two real-valued bifunctions Φ and Ψ defined on $K \times K$ such that:*

(H1) For each $x,y \in K$, if $\Psi(x,y) \leq 0$, then $\Phi(x,y) \leq 0$;

(H2) For each fixed $x \in X$, the function $\Phi(x,\cdot)$ is lower semicontinuous on every compact subset of K;

(H3) For each finite subset A of K, one has

$$\sup_{y\in co(A)} \min_{x\in A} \psi(x,y) \leq 0.$$

(H4) (Coercivity condition) There exists a compact convex subset C of K such that either (i) or (ii) below holds:

(i) for all $y \in K \setminus C$, there exists $x \in C$ such that $\Phi(x,y) \geq 0$;

(ii) there exists $x_0 \in C$ such that for all $y \in K \setminus C$, $\Psi(x_0,y) \geq 0$.

Then, the dual equilibrium problem (3.2) has a solution. Furthermore, the set of solutions is compact.

Theorem 3.2. [48] *Let X be a Hausdorff topological vector space, K be a nonempty closed convex subset of X and $\Phi, \Psi : K \times K \to \mathbb{R}$ be two real-valued bifunctions. Assume that*

(H2) For each fixed $x \in X$, the function $\Phi(x,\cdot)$ is lower semicontinuous on every compact subset of K.

(H4) (Coercivity condition) There exists a compact convex subset C of K such that either (i) or (ii) below holds:

(i) for all $y \in K \setminus C$, there exists $x \in C$ such that $\Phi(x,y) \geq 0$;

(ii) there exists $x_0 \in C$ such that for all $y \in K \setminus C$, $\Psi(x_0,y) \geq 0$.

(H5) For each $x,y \in K$, if $\Psi(x,y) < 0$, then $\Phi(x,y) \leq 0$.

(H6) For each $x \in K$, $\Psi(x,x) \leq 0$ and $\Phi(x,x) \leq 0$.

(H7) For each finite subset $\{x_i, x_2, \cdots, x_n\}$ of K with $n \geq 2$, for each $y \in co(\{x_i, x_2, \cdots, x_n\})$, with $y \neq x_i$, for all $i = 1, \cdots, n$, one has

$$\min_{1\leq i\leq n} \Psi(x_i,y) < 0.$$

Then, the dual equilibrium problem (3.2) has a solution. Furthermore, the set of solutions is compact.

Coercivity conditions for equilibrium problems have been studied by many authors. The results of Flores-Bazán [49] was then extended by Bianchi and Pini [66] in 2005, where they have provided the coercivity conditions as weak as possible, exploiting the generalized monotonicity properties of the function Φ defining the equilibrium problem. The following existence result for the equilibrium problem (3.1) proved by Bianchi and Pini [66] improves Theorem 3.7 of Flores-Bazán [49].

Theorem 3.3. [66] *Let K be a nonempty closed convex subset of a real reflexive Banach space $\mathbb{B}$ and $\Phi : K \times K \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

(i) *Φ is pseudomonotone;*

(ii) *$\Phi(\cdot, y)$ is upper sign continuous, for every $y \in K$;*

(iii) *If $\Phi(x,y) = 0$ and $\Phi(x,z) < 0$, then $\Phi(x,(1-t)y+tz) < 0$, for every $t \in (0,1)$;*

(iv) *$\Phi(x,\cdot)$ is quasiconvex, for every $x \in K$;*

(v) *$\Phi(x,x) \geq 0$, for every $x \in K$.*

Then, if the solution set S_K of (3.1) is nonempty and bounded, the following coercivity condition (C) holds.

$$(C)\ \exists r > 0 :\ \forall x \in K \setminus K_r,\ \exists y \in K_r,\ \Phi(x,y) < 0,$$

where K_r is the weakly compact subset of K defined as

$$K_r = \{x \in K : \|x\| \leq r\}.$$

Moreover, if we add the following condition:

(vi) *$lev_{\leq 0}\Phi(x,\cdot)$ is closed, for every $x \in K$,*

then condition (C) implies that S_K is nonempty and bounded.

Bai et al. [29] have introduced the notion of relaxed $\eta - \alpha$ pseudomonotone operators, they have generalized the results of Fang [28] for variational-like inequalities associated to relaxed $\eta - \alpha$ monotone mappings. The notion of relaxed $\eta - \alpha$ pseudomonotone operators introduced by Bai et al. [29] is defined as the following:

Definition 3.2. [29] Let $\mathbb{B}$ be a Banach space with dual $\mathbb{B}^*$ and K be a nonempty subset of $\mathbb{B}$. A mapping $F : K \to \mathbb{B}^*$ is said to be relaxed $\eta - \alpha$ pseudomonotone if there exist a mapping $\eta : K \times K \to \mathbb{B}$ and a function $\alpha : \mathbb{B} \to \mathbb{R}$ with $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in \mathbb{B}$, where $p > 0$ is a constant, such that, for any $x, y \in K$, we have

$$\langle F(y), \eta(x,y)\rangle \geq 0 \text{ implies } \langle F(x), \eta(x,y)\rangle \geq \alpha(x-y). \tag{3.3}$$

The variational-like inequality problem considered by Bai et al. [29] is as the following: Find $x \in K$ such that

$$\langle F(x), \eta(y,x)\rangle \geq 0, \ \forall y \in K. \tag{3.4}$$

They have used KKM techniques and proved the following existence result for the variational-like inequality (3.4) in Banach spaces.

Theorem 3.4. [29] *Let K be a closed, convex and bounded subset of the real reflexive Banach space $\mathbb{B}$ and $F : K \to \mathbb{B}^*$ be an η-hemicontinuous relaxed $\eta - \alpha$ pseudomonotone. Assume that*

(i) $\eta(x,y) + \eta(y,x) = 0$, *for all x, y in K;*

(ii) For any fixed y, z in K, the mapping $x \mapsto \langle F(z), \eta(x,y)\rangle$ is convex and lower semicontinuous;

(iii) $\alpha : \mathbb{B} \to \mathbb{R}$ *is weakly lower semicontinuous.*

Then there exists at least one solution for the problem (3.4).

The relaxed $\eta - \alpha$ pseudomonotonicity of Bai et al. [29] was then extended by Arunchai et al. [67] in 2014. They have used a KKM techniques and proved some existence results for generalized variational-like inequalities with relaxed $\eta - \alpha$ pseudomonotone mappings and strictly η-quasimonotone mappings which extend some results of Kien et al. [68] given in 2008. In 2014, Mahato and Nahak [18] have defined generalized relaxed α-monotonicity and generalized relaxed α-pseudomonotonicity for bifunctions and proved some results on mixed equilibrium problems which in fact generalized the results of Fang [28] for the variational-like inequalities.

Definition 3.3. Let K be a nonempty closed convex subset of a real reflexive Banach space $\mathbb{B}$ and $\alpha : K \times K \to \mathbb{R}$ be a real-valued function such that $\lim_{t\to 0} \frac{\alpha(ty+(1-t)x,x)}{t} = 0$. The bifunction $\Phi : K \times K \to \mathbb{R}$ is said to be generalized relaxed α-monotone if

$$\Phi(x,y) + \Phi(y,x) \leq \alpha(y,x), \ \forall x,y \in K.$$

The mixed equilibrium problem (MEP) considered by Mahato and Nahak [18] is to find a vector $\bar{x} \in K$ such that

$$\Phi(\bar{x},y) + \varphi(y) - \varphi(\bar{x}) \geq 0, \ \forall y \in K, \tag{3.5}$$

where $\varphi : K \to \mathbb{R}$ is a real-valued function and $\Phi : K \times K \to \mathbb{R}$ is an equilibrium bifunction.

The following existence results for mixed equilibrium problem (3.5) was established by Mahato and Nahak [18].

Theorem 3.5. [18] *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space $\mathbb{B}$ and $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is generalized relaxed α-monotone and hemicontinuous in the first argument. Also assume that:*

(i) For fixed $z \in K$, the mapping $x \mapsto \Phi(z,x)$ is convex and lower semicontinuous;
(ii) $\alpha : X \times X \to \mathbb{R}$ is weakly upper semicontinuous in the second argument.

Then the mixed equilibrium problem (3.5) has a solution.

Theorem 3.6. [18] *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space $\mathbb{B}$ and $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is generalized relaxed α-monotone and hemicontinuous in the first argument. Furthermore, assume that:*

(i) For fixed $z \in K$, the mapping $x \mapsto \Phi(z,x)$ is convex and lower semicontinuous;

(ii) $\alpha : X \times X \to \mathbb{R}$ *is weakly upper semicontinuous in the second argument;*

(iii) (Coercivity condition) There exists a point $x_0 \in K$ *and* $R > 0$ *such that* $\Phi(x, x_0) + \varphi(x_0) - \varphi(x) < 0$*, whenever* $\|x\| > R$.

Then the mixed equilibrium problem (3.5) has a solution.

4. Invex Equilibrium Problems with Generalized Monotone Operators

Noor [69, 70] introduced and studied the following invex equilibrium problem: Find $x \in K$ such that

$$\Phi(x, \eta(y,x)) \geq 0, \ \forall y \in K, \tag{4.1}$$

where $\Phi : K \times K \to \mathbb{R}$ is a bifunction and $\eta : K \times K \to K$ is a mapping and K is an invex set with respect to η.

In 2014, Chen et al. [17] have introduced the η-pseudomonotone bifunction and gave some existence results for the problem (4.1) in a finite dimensional setting. The η-pseudomonotonicity notion introduced by Chen et al. [17] is defined as the following:

Definition 4.1. Let $K \subset \mathbb{R}^n$ be an invex set with respect to η. A bifunction $\Phi : K \times K \to \mathbb{R}$ is said to be η-pseudomonotone on K if, for any $x, y \in K$

$$\Phi(x, \eta(y,x)) \geq 0 \implies \Phi(y, -\eta(y,x)) \leq 0.$$

Inspired by Chen et al. [17], Liu and Zeng [19] have introduced the relaxed $\alpha - \eta$ pseudomonotonicity notion for bifunctions and proved many existence results for the invex equilibrium problem (4.1). The relaxed $\alpha - \eta$ pseudomonotonicity notion introduced by Liu and Zeng [19] is defined as the following:

Definition 4.2. [19] Let K be a nonempty subset of a real Banach space $\mathbb{B}$, the equilibrium bifunction $\Phi : K \times \mathbb{B} \to \mathbb{R}$ is said to be relaxed $\alpha - \eta$ pseudomonotone if, for any $x, y \in K$

$$\Phi(x, \eta(y,x)) \geq 0 \implies \Phi(y, \eta(y,x)) \geq \alpha(y - x),$$

where $\eta : K \times K \to \mathbb{B}$ and $\alpha : \mathbb{B} \to \mathbb{R}$.

Using Schauders fixed point theorem as well as KKM technique, Liu and Zeng [19] have proved the following existence result for the invex equilibrium problem (4.1).

Theorem 4.1. [19] *Let K be a nonempty closed and invex set with respect η with $\eta(x,x) = 0$ for all $x \in K$. Let $\Phi : K \times \mathbb{B} \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Furthermore, suppose that*

(i) Φ is positively homogeneous with respect to the second argument and a relaxed $\alpha - \eta$ pseudomonotone;

(ii) Φ is hemicontinuous with respect to the first argument;

(iii) For all $x,y,z \in K$, the function Φ satisfies

$$\Phi(y, \eta(x + t\eta(z,u),x)) \leq t\Phi(y,\eta(z,x)) + (1-t)\Phi(y,\eta(x,x));$$

(iv) $\alpha : \mathbb{B} \to \mathbb{R}$ satisfies

$$\lim_{t \to o^+} \frac{\alpha(tx)}{t} = 0, \ \forall x \in K;$$

(v) For all $x \in K$, the function $\Phi(x, \eta(\cdot,x))$ is convex;

(vi) For each $y \in K$, the function $\Phi(y, \eta(y,\cdot))$ is upper semicontinuous.

(vii) $\limsup_{n\to\infty} \alpha(x_n) \geq \alpha(x)$ whenever $x_n \to x$;

(viii) There exists $y_0 \in K$, such that $V_0 := \{x \in K : \Phi(y_0, \eta(y_0,x)) \geq \alpha(y_0 - x)\}$ is a compact subset of $\mathbb{B}$.

Then, the invex equilibrium problem (4.1) has at least one solution.

In 2016, Pany et al. [27] extended the results of Bai et al. [29] and Mahato and Nahak [18] to nonlinear mixed variational-like inequality problem under generalized weakly relaxed $\eta - \alpha$ monotonicity. They have also introduced the concept generalized weakly relaxed α-monotonicity for a trifunction. It is defined as the following: Given a nonempty convex subset K of a real Banach space $\mathbb{B}$ and a real-valued function $\alpha : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$, the trifunction $\Phi : K \times K \times K \to \mathbb{R}$ is said to be generalized weakly relaxed α-monotone if, for all $x,y,z \in K$ one has

$$\Phi(z,x,y)+\Phi(z,y,x) \leq \alpha(x,y),$$

with $\lim_{t\to 0} \frac{d}{dt}\alpha(tx+(1-t)y,y)=0$. Using the generalized weakly relaxed α-monotonicity notion, Pany et al. [27] proved many existence results for the following mixed equilibrium problem: Find $x \in K$ such that

$$\Phi(z,y,x)+\varphi(x,y)-\varphi(x,x) \geq 0, \ \forall y \in K,$$

where K is a nonempty convex subset of a Banach $\mathbb{B}$ with dual space $\mathbb{B}^*$, $\varphi : \mathbb{B}\times\mathbb{B} \to \mathbb{R}$ is a real-valued bifunction, $\eta : K\times K \to B$ is a mapping and $\Phi : K\times K\times K \to \mathbb{R}$ is a trifunction defined by $\Phi(z,y,x) = \langle N(y,z),\eta(x,y)\rangle$, with $N : \mathbb{B}\times\mathbb{B} \to \mathbb{B}^*$ is an operator.

Inspired by Mahato and Nahak [18], Liu and Zeng [19] and Pany et al. [27], Sahu and Pani [31] have defined generalized relaxed $\eta-\alpha$ pseudomonotone mapping and strictly η-quasimonotone mapping for bifunctions and proved some existence results of invex equilibrium problem in reflexive Banach spaces. Let K be a nonempty subset of a real reflexive Banach space $\mathbb{B}$, $\Phi : K\times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$ and $\eta : K\times K \to \mathbb{B}$ be a mapping. Then the invex equilibrium problems considered by Sahu and Pani [31] is to find $x \in K$ such that

$$\Phi(x,\eta(y,x)) \geq 0, \ \forall y \in K. \tag{4.2}$$

Definition 4.3. [31] A bifunction $\Phi : K\times K \to \mathbb{R}$ is said to be generalized relaxed $\eta-\alpha$ pseudomonotone if there exists a function $\eta : K\times K \to \mathbb{B}$ and a function $\alpha : K\times K \to \mathbb{R}$ with $\lim\limits_{t\to 0^+} \dfrac{\alpha(t\eta(y,x),x)}{t} = 0$ for all $(y,x) \in K\times K$ such that for any $x,y \in K$, we have

$$\Phi(x,\eta(y,x)) \geq 0 \text{ implies } \Phi(y,\eta(y,x)) \geq \alpha(\eta(y,x),x). \tag{4.3}$$

With the above generalized relaxed $\eta-\alpha$ pseudomonotone mapping, Sahu and Pani [31] have proved the following existence results for the invex equilibrium problem (4.2) in the setting of reflexive Banach spaces.

Theorem 4.2. [31] *Let K be a closed, bounded and convex subset of a real reflexive Banach space $\mathbb{B}$. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is generalized relaxed $\eta - \alpha$ pseudomonotone, hemicontinuous in the first argument and positively homogeneous in the second argument. In addition, assume that*

(i) for any fixed y, z, the mapping $x \mapsto \Phi(z, \eta(x,y))$ is convex;
(ii) for any fixed y, z, the mapping $x \mapsto \Phi(z, \eta(y,x))$ is upper semicontinuous;
(iii) for each $y \in \mathbb{B}$, the function $\alpha(\eta(y,\cdot),\cdot)$ is weakly lower semicontinuous;
(iv) $\eta(x,x) = 0, \forall x \in K$;
(v) $\eta(tx + (1-t)y, z) = t\eta(x,z) + (1-t)\eta(y,z), \forall x,y,z \in K,\ t \in [0,1]$.

Then the invex equilibrium problem (4.2) has a solution.

Theorem 4.3. [31] *Let K be a closed, unbounded and convex subset of a real reflexive Banach space $\mathbb{B}$. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is generalized relaxed $\eta - \alpha$ pseudomonotone, hemicontinuous in the first argument and positively homogeneous in the second argument. In addition, assume that*

(i) for any fixed y, z, the mapping $x \mapsto \Phi(z, \eta(x,y))$ is convex;
(ii) for any fixed y, z, the mapping $x \mapsto \Phi(z, \eta(y,x))$ is upper semicontinuous;
(iii) for each $y \in \mathbb{B}$, the function $\alpha(\eta(y,\cdot),\cdot)$ is weakly lower semicontinuous;
(iv) $\eta(x,x) = 0, \forall x \in K$;
(v) $\eta(tx + (1-t)y, z) = t\eta(x,z) + (1-t)\eta(y,z), \forall x,y,z \in K,\ t \in [0,1]$;
(vi) (Coercivity condition) there exists $x_0 \in K$ and $R > 0$ such that $\Phi(x, \eta(x,x_0)) > 0$, whenever $\|x\| > R$ and $x \in K$.

Then the invex equilibrium problem (4.2) has a solution.

Theorem 4.4. [31] *Let K be a nonempty closed and convex subset of a real reflexive Banach space $\mathbb{B}$. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is generalized relaxed $\eta - \alpha$ pseudomonotone, hemicontinuous in* *the first argument and positively homogeneous in the second argument. In addition, assume that*

(i) *for any fixed y, z, the mapping* $x \mapsto \Phi(z, \eta(x,y))$ *is convex;*

(ii) *for any fixed y, z, the mapping* $x \mapsto \Phi(z, \eta(y,x))$ *is upper semicontinuous;*

(iii) *for each* $y \in \mathbb{B}$, *the function* $\alpha(\eta(y,\cdot),\cdot)$ *is weakly lower semicontinuous;*

(iv) $\eta(x,x) = 0, \forall x \in K$;

(v) $\eta(tx + (1-t)y, z) = t\eta(x,z) + (1-t)\eta(y,z), \forall x,y,z \in K,\ t \in [0,1]$.

Then the invex equilibrium problem (4.2) *and the following problem* (4.4) *are equivalent:*

There exists a point $x \in K$ *such that the set*

$$B_x^0 = \{y \in K : \Phi(y, \eta(y,x)) < \alpha(\eta(y,x),x)\} \tag{4.4}$$

is bounded.

The following notion of strictly η-quasimonotonicity for bifunction was defined by Sahu and Pani [31].

Definition 4.4. [31] A bifunction $\Phi : K \times K \to \mathbb{R}$ is said to be strictly η-quasimonotone if there exists a function $\eta : K \times K \to \mathbb{B}$ such that, for any $x, y \in K$,

$$\Phi(x, \eta(y,x)) > 0 \text{ implies } \Phi(y, \eta(y,x)) > 0. \tag{4.5}$$

Sahu and Pani [31] then proved the following existence results for the invex equilibrium problem (4.2) using strictly η-quasimonotone mappings.

Theorem 4.5. [31] *Let K be a closed, bounded and convex subset of a real reflexive Banach space* $\mathbb{B}$. *Let* $\Phi : K \times \mathbb{B} \to \mathbb{R}$ *be a bifunction such that* $\Phi(x,x) = 0$ *for all* $x \in K$. *Suppose that* Φ *is strictly* η*-quasimonotone, hemicontinuous in the first argument and positively homogeneous in the second argument. In addition, assume that*

(i) *for any fixed y, z, the mapping* $x \mapsto \Phi(z, \eta(x,y))$ *is convex;*

(ii) *for any fixed y, z, the mapping* $x \mapsto \Phi(z, \eta(y,x))$ *is upper semicontinuous;*

(iii) $\Phi(x, -y) = -\Phi(x,y)$;

(iv) $\eta(x,y)+\eta(y,x)=0,\ \forall x,y\in K.$

Then the invex equilibrium problem (4.2) has a solution.

Theorem 4.6. [31] *Let K be a closed, unbounded and convex subset of a real reflexive Banach space $\mathbb{B}$. Let $\Phi : K\times\mathbb{B}\to\mathbb{R}$ be a bifunction such that $\Phi(x,x)=0$ for all $x\in K$. Suppose that Φ is strictly η-quasimonotone, hemicontinuous in the first argument and positively homogeneous in the second argument. In addition, assume that*

(i) *for any fixed y, z, the mapping $x\mapsto\Phi(z,\eta(x,y))$ is convex;*
(ii) *for any fixed y, z, the mapping $x\mapsto\Phi(z,\eta(y,x))$ is upper semicontinuous;*
(iii) $\Phi(x,-y)=-\Phi(x,y)$;
(iv) $\eta(x,y)+\eta(y,x)=0,\forall x,y\in K$;
(v) *(Coercivity condition) there exists $x_0\in K$ and $R>0$ such that $\Phi(x,\eta(x,x_0))>0$, whenever $\|x\|>R$ and $x\in K$.*

Then the invex equilibrium problem (4.2) has a solution.

Theorem 4.7. [31] *Let K be a nonempty closed and convex subset of a real reflexive Banach space $\mathbb{B}$. Let $\Phi : K\times\mathbb{B}\to\mathbb{R}$ be a bifunction such that $\Phi(x,x)=0$ for all $x\in K$. Suppose that Φ is strictly η-quasimonotone, hemicontinuous in the first argument and positively homogeneous in the second argument. In addition, assume that*

(i) *for any fixed y, z, the mapping $x\mapsto\Phi(z,\eta(x,y))$ is convex;*
(ii) *for any fixed y, z, the mapping $x\mapsto\Phi(z,\eta(y,x))$ is upper semicontinuous;*
(iii) $\Phi(x,-y)=-\Phi(x,y)$;
(iv) $\eta(x,y)+\eta(y,x)=0,\forall x,y\in K.$

Then invex equilibrium problem (4.2) and the following problem (4.6) are equivalent: There exists a point $x\in K$ such that the set

$$B_x^0=\{y\in K:\Phi(y,\eta(y,x))<0\}\tag{4.6}$$

is bounded.

Theorem 4.8. [31] *Let K be a nonempty closed and convex subset of a real reflexive Banach space $\mathbb{B}$. Let $\Phi : K \times \mathbb{B} \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is strictly η-quasimonotone, hemicontinuous in the first argument and positively homogeneous in the second argument. In addition, assume that*

(i) for any fixed y, z, the mapping $x \mapsto \Phi(z, \eta(x,y))$ is convex;
(ii) for any fixed y, z, the mapping $x \mapsto \Phi(z, \eta(y,x))$ is upper semicontinuous;
(iii) $\Phi(x,-y) = -\Phi(x,y)$;
(iv) $\eta(x,y) + \eta(y,x) = 0, \forall x,y \in K$.

If there exists a point $x \in K$ such that the set

$$B_x = \{y \in K : \Phi(y, \eta(y,x)) \leq 0\} \tag{4.7}$$

is bounded, then the solution set $S(\Phi,\eta)$ of the invex equilibrium problem (4.2) is nonempty and bounded.

The results of Sahu and Pani [31] was then generalized by Sahu et al. [32] in 2020, where they have also generalized the the results of Liu and Zeng [19]. They have introduced the generalized relaxed $\eta - \alpha$ monotonicity for bifunctions and establish certain existence results for the following mixed invex equilibrium problem (for short, (MIEP)): Find $x \in K$ such that

$$\Phi(x, \eta(y,x)) + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in K, \tag{4.8}$$

where K is a nonempty subset of a real reflexive Banach space $\mathbb{B}$, $\eta : K \times K \to \mathbb{B}$, $\varphi : K \to \mathbb{R}$ is a mapping and $\Phi : K \times K \to \mathbb{R}$ is a real-valued bifunction such that $\Phi(x,x) = 0$ for all $x \in K$.

The generalized relaxed $\eta - \alpha$ monotonicity for bifunctions defined by Sahu et al. [32] is given in the following definition.

Definition 4.5. A bifunction $\Phi : K \times K \to \mathbb{R}$ is said to be generalized relaxed $\eta - \alpha$ monotone if there exists a function $\eta : K \times K \to \mathbb{B}$ and a function $\alpha : K \times K \to \mathbb{R}$

with $\lim_{t\to 0^+} \frac{\alpha(t\eta(y,x),x)}{t} = 0$ for all $(y,x) \in K \times K$ such that for any $x,y \in K$, we have

$$\Phi(y,\eta(y,x)) - \Phi(x,\eta(y,x)) \geq \alpha(\eta(y,x),x). \tag{4.9}$$

The following results for the mixed invex equilibrium problem (4.8) was established by Sahu et al. [32] using generalized relaxed $\eta - \alpha$ monotone mappings for bifunction in reflexive Banach spaces.

Theorem 4.9. [32] *Let K be a nonempty closed, convex and bounded subset of a real reflexive Banach space $\mathbb{B}$ and $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous mapping. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is generalized relaxed $\eta - \alpha$ monotone, hemicontinuous in the first argument and positively homogeneous in the second argument. Furthermore, assume that*

(i) for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(x,y))$ is convex;
(ii) for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(y,x))$ is upper semicontinuous;
(iii) for each $y \in \mathbb{B}$, the function $x \to \alpha(\eta(y,x),x)$ is weakly lower semicontinuous;
(iv) $\eta(x,x) = 0$, $\forall x \in K$;
(v) $\eta(tx+(1-t)y,z) = t\eta(x,z)+(1-t)\eta(y,z)$, $\forall x,y,z \in K$, $t \in [0,1]$.

Then the mixed invex equilibrium problem (4.8) has a solution.

Theorem 4.10. [32] *Let K be a nonempty closed, convex and unbounded subset of a real reflexive Banach space $\mathbb{B}$ and $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is generalized relaxed $\eta - \alpha$ monotone, hemicontinuous in the first argument and positively homogeneous in the second argument. Furthermore, assume that*

(i) for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(x,y))$ is convex;
(ii) for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(y,x))$ is upper semicontinuous;
(iii) for each $y \in \mathbb{B}$, the function $x \to \alpha(\eta(y,x),x)$ is weakly lower semicontinuous;

(iv) $\eta(x,x) = 0,\ \forall x \in K$;

(v) $\eta(tx+(1-t)y,z) = t\eta(x,z)+(1-t)\eta(y,z),\ \forall x,y,z \in K,\ t \in [0,1]$;

(vi) *(Coercivity condition) there exists $x_0 \in K$ and $R > 0$ such that $\Phi(x,\eta(x_0,x)) + \varphi(x_0) - \varphi(x) < 0$, whenever $\|x\| > R$ and $x \in K$.*

Then the mixed invex equilibrium problem (4.8) has a solution.

Theorem 4.11. [32] *Let K be a nonempty closed and convex subset of a real reflexive Banach space $\mathbb{B}$ and $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is generalized relaxed $\eta - \alpha$ monotone, hemicontinuous in the first argument and positively homogeneous in the second argument. Furthermore, assume that*

(i) *for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(x,y))$ is convex;*

(ii) *for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(y,x))$ is upper semicontinuous;*

(iii) *for each $y \in \mathbb{B}$, the function $x \to \alpha(\eta(y,x),x)$ is weakly lower semicontinuous;*

(iv) $\eta(x,x) = 0,\ \forall x \in K$;

(v) $\eta(tx+(1-t)y,z) = t\eta(x,z)+(1-t)\eta(y,z),\ \forall x,y,z \in K,\ t \in [0,1]$.

Then MIEP (4.8) and the following problem (4.10) are equivalent:
Find a vector $x \in K$ such that the set

$$B_x^0 = \{y \in K : f(y,\eta(y,x)) + \varphi(y) - \varphi(x) < \alpha(\eta(y,x),x)\}, \tag{4.10}$$

is bounded.

Sahu et al. [32] have also introduced the notion of relaxed $\rho - \theta$ invariant pseudomonotonicity for bifunctions and proved certain existence results of the invex equilibrium problem (4.2) in reflexive Banach spaces.

Definition 4.6. [32] Let $\mathbb{B}$ be a real reflexive Banach space and K be a nonempty subset of $\mathbb{B}$. Assume $\eta : K \times K \to \mathbb{B}$ and $\theta : K \times K \to \mathbb{R}$ are the functions and $\rho \in \mathbb{R}$ is a constant. A bifunction $\Phi : K \times K \to \mathbb{R}$ is said to be relaxed $\rho - \theta$ invariant pseudomonotone with respect to η if for any $x,y \in K$, we have

$$\Phi(x,\eta(y,x)) \geq 0 \implies \Phi(y,\eta(y,x)) \geq \rho|\theta(y,x)|^2. \tag{4.11}$$

Theorem 4.12. [32] *Let K be a nonempty closed, convex and bounded subset of a real reflexive Banach space $\mathbb{B}$. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is relaxed $\rho - \theta$ invariant pseudomonotone with respect to η, hemicontinuous in the first argument and positively homogeneous in the second argument. Furthermore, assume that*

- *(i) for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(x,y))$ is convex;*
- *(ii) for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(y,x))$ is upper semicontinuous;*
- *(iii) $\theta(x,y) + \theta(y,x) = 0$, $\forall x,y \in K$;*
- *(iv) $\theta(x,y)$ is convex in second argument, concave in first argument and lower semicontinuous with respect to second argument;*
- *(v) $\eta(x,x) = 0$, $\forall x \in K$.*

Then, the invex equilibrium problem (4.2) has a solution.

Theorem 4.13. [32] *Let K be a nonempty closed, convex and unbounded subset of a real reflexive Banach space $\mathbb{B}$. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is relaxed $\rho - \theta$ invariant pseudomonotone with respect to η, hemicontinuous in the first argument and positively homogeneous in the second argument. Furthermore, assume that*

- *(i) for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(x,y))$ is convex;*
- *(ii) for any fixed y, z, the mapping $x \mapsto \Phi(z,\eta(y,x))$ is upper semicontinuous;*
- *(iii) $\theta(x,y) + \theta(y,x) = 0$, $\forall x,y \in K$;*
- *(iv) $\theta(x,y)$ is convex in second argument, concave in first argument and lower semicontinuous with respect to second argument;*
- *(v) $\eta(x,x) = 0$, $\forall x \in K$;*
- *(vi) (Coercivity condition) there exists $x_0 \in K$ and $R > 0$ such that $\Phi(x,\eta(x_0,x)) < 0$, whenever $\|x\| > R$ and $x \in K$.*

Then, the invex equilibrium problem (4.2) has a solution.

Theorem 4.14. [32] *Let K be a nonempty closed convex subset of a real reflexive Banach space $\mathbb{B}$. Let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that Φ is relaxed $\rho - \theta$ invariant pseudomonotone with respect to η, hemicontinuous in the first argument and positively homogeneous in the second argument. Furthermore, assume that*

(i) for any fixed y, z, the mapping $x \mapsto \Phi(z, \eta(x,y))$ is convex;

(ii) for any fixed y, z, the mapping $x \mapsto \Phi(z, \eta(y,x))$ is upper semicontinuous;

(iii) $\theta(x,y) + \theta(y,x) = 0$, $\forall x, y \in K$;

(iv) $\theta(x,y)$ is convex in second argument, concave in first argument and lower semicontinuous with respect to second argument;

(v) $\eta(x,x) = 0$, $\forall x \in K$.

Then, the invex equilibrium problem (4.2) and the following problem (4.12) are equivalent:

Find an $x \in K$, such that the set

$$B_x^0 = \{y \in K : \Phi(y, \eta(y,x)) < \rho|\theta(y,x)|^2\}, \tag{4.12}$$

is bounded.

5. Brézis Pseudomonotone Equilibrium Problems

The pseudomonotonicity notion of operators in the topological sense was introduced by Brézis [33] in 1968, it is defined as the following.

Definition 5.1. [33] Let X and Y be topological vector spaces over $\mathbb{R}$ and $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ be a bilinear form. A single-valued mapping $F : K \to Y$ is said to be pseudomonotone in the sense of Brézis if, for any generalized sequence $\{x_\alpha\}_{\alpha \in I}$ satisfying $\{x_\alpha\}_{\alpha \in I}$ stays in a compact set and converges to $\bar{x}$ and $\limsup \langle F(x_\alpha), x_\alpha - \bar{x} \rangle \leq 0$, its limit $\bar{x}$ satisfies

$$\langle F(\bar{x}), \bar{x} - z \rangle \leq \liminf \langle F(x_\alpha), x_\alpha - z \rangle, \text{ for all } z \in K.$$

The pseudomonotonicity in the sense of Brézis was then extended to bifunctions by J. Gwinner [71] in the year of 1978 and further by himself in the couple of papers [72, 73].

Definition 5.2. [71] Let X be a topological vector space over $\mathbb{R}$ and K be a nonempty closed and convex subset of X. We shall say that a bifunction $\Psi : K \times K \to \mathbb{R}$ is pseudomonotone in the sense of Brézis, for short B-pseudomonotone, if for any generalized sequence $\{x_\alpha\}_{\alpha\in I}$ satisfying $\{x_\alpha\}_{\alpha\in I}$ stays in a compact set and converges to $\bar{x}$ and $\liminf \Psi(x_\alpha, \bar{x}) \geq 0$, its limit $\bar{x}$ satisfies

$$\Psi(\bar{x}, z) \geq \limsup \Psi(x_\alpha, z), \text{ for all } z \in K.$$

This concept of pseudomonotonicity has also been considered by Aubin [74] with an aim to relax the continuity properties for the study of various problems related to minimax formulations in game theory as well as fixed points theory. The attractive property of Brézis pseudomonotonicity is that it provides a unified approach to both monotonicity and compactness arguments, since for instance if F_1 is a monotone and hemicontinuous operator and F_2 is a strongly continuous operator, then $F = F_1 + F_2$ is a pseudomonotone operator in the sense of Brézis, see Zeidler [75, page 586]. Further, if the bifunction $\Psi : K \times K \to \mathbb{R}$ defined by $\Psi(x,y) = \langle F(x), y - x\rangle$ is upper semicontinuous with respect to the first argument, then it is B-pseudomonotone. The converse of which is not true as shown by Steck [76] in 2019, where he has provided a counter example to a claim of Sadeqi and Paydar [77] that these two properties are equivalent.

On the other hand most of the researchers in literature find the existence of solutions for equilibrium problems using the techniques of KKM principle as well as arguments from generalized monotonicity and convexity, see [5, 47]. A twist on this trend has been reached when Chadli et al. [36] studied the existence of anti-periodic solutions for nonlinear evolution equations by means of an equilibrium problem approach. They have used a new approach which is based on the notions of maximal monotonicity for bifunctions, initiated by Blum and Oettli [5], Hadjisavvas and Khatibzadeh [78] as well as the concept of pseudomonotonicity in the sense of Brézis. They proceed by a Browder-Tikhonov regularization procedure by considering the following regularized mixed equilibrium: Given $\varepsilon > 0$, find $x \in K$ such that

$$\Phi(x,y) + \Psi(x,y) + \varepsilon\langle J(x), y - x\rangle \geq 0, \text{ for all } y \in K \tag{5.1}$$

where K is a nonempty, closed and convex subset of a reflexive Banach space $\mathbb{B}$, $\Phi, \Psi : K \times K \to \mathbb{R}$ are two bifunctions and $J : \mathbb{B} \rightrightarrows \mathbb{B}^*$ is the duality mapping defined by

$$J(x) = \left\{x^* \in \mathbb{B}^* : \langle x^*, x\rangle = \|x^*\|^2 \text{ and } \|x^*\| = \|x\|\right\}.$$

The following existence result for the regularized problem (5.1) has been proved in [36].

Theorem 5.1. [36] *Let K be a nonempty, closed and convex subset of a reflexive Banach space $\mathbb{B}$ and $\Phi, \Psi : K \times K \to \mathbb{R}$ be real-valued bifunctions such that $\Phi(x,x) = \Psi(x,x) = 0$ for all $x \in K$. Suppose that*

(i) Φ is monotone and maximal monotone in the sense of Blum and Oettli;

(ii) Ψ is B-pseudomonotone;

(iii) Φ is weakly lower semicontinuous with respect to the second argument;

(iv) for each finite subset A of K and each y in K fixed, the function $x \in K \mapsto \Psi(x,y)$ is upper semicontinuous on $co(A)$;

(v) Φ and Ψ are convex with respect to the second argument;

(vi) there exists a weakly compact subset $W \in K$ such that for each $\varepsilon > 0$, there exists a weakly compact and convex subset B_ε of K satisfying the following condition:

$$\forall x \in K \setminus W,\ \exists y \in B_\varepsilon \text{ such that } \Psi(x,y) + \varepsilon\langle J(x), y - x\rangle < \Phi(y,x).$$

Then, the regularized mixed equilibrium problem (5.1) has at least one solution.

Recently, Liu, Migórski and Zeng [79] have established many existence results for the following mixed quasi-equilibrium problem:

$$\begin{cases} \text{Find } x \in D(\Phi) \cap D(\Psi) \text{ and } x^* \in F(x) \text{ such that} \\ \langle x^*, \eta(x,y) \rangle + \Phi(x,y) + \Psi(x,y) \geq 0, \\ \forall y \in K, \end{cases} \tag{5.2}$$

where K is a nonempty, closed and convex subset of Banach space $\mathbb{B}$ with dual space $\mathbb{B}^*$, $F : K \rightrightarrows \mathbb{B}^*$ is a set-valued mapping, $\eta : K \times K \to \mathbb{B}$ is a mapping and $\Phi, \Psi : K \times K \to \mathbb{R} \cup \{-\infty, +\infty\}$ are two bifunctions with $D(\Phi) \cap D(\Psi) \neq \emptyset$. Here $D(\Phi)$ and $D(\Psi)$ represent, respectively, the domain of Φ and Ψ, i.e. $D(\Phi) := \{x \in K : \Phi(x,y) \neq -\infty, \forall y \in K\}$.

They have then used the maximal monotonicity concept for bifunctions and proved the following existence result for the mixed quasi-equilibrium problem (5.2).

Theorem 5.2. [79] *Let K be a nonempty, compact and convex subset of a real Banach space $\mathbb{B}$. Assume that:*

$\eta : K \times K \to \mathbb{B}$ *is a mapping satisfying the following conditions:*

(i) $\eta(x,x) = 0$ *for all* $x \in K$,

(ii) *for all* $y \in K$, $\eta(\cdot, y)$ *is continuous,*

(iii) *for all* $x \in K$, $n \in \mathbb{N}$, $y_j \in K$, $\lambda_j \in [0,1]$, $j = 1, 2, \cdots, n$ *with* $\Sigma_{j=1}^n \lambda_j = 1$, *one has*

$$\eta\left(x, \Sigma_{j=1}^n \lambda_j y_j\right) = \Sigma_{j=1}^n \lambda_j \eta(x, y_j);$$

$\Phi : K \times K \to \mathbb{R} \cup \{-\infty, +\infty\}$ *is a mapping satisfying the following conditions:*

(iv) $\Phi(x,x) = 0$ *for all* $x \in K$,

(v) *for all* $y \in K$, $\Phi(\cdot, y)$ *is upper semicontinuous,*

(vi) *for all* $x \in K$, $\Phi(x, \cdot)$ *is convex;*

$\Psi : K \times K \to \mathbb{R} \cup \{-\infty, +\infty\}$ *is a mapping satisfying the following conditions:*

(vii) $\Psi(x,x) = 0$ *for all* $x \in K$,

(viii) *for all* $y \in K$, $\Psi(\cdot, y)$ *is concave,*

(ix) *for all* $x \in K$, *we have* $\limsup_{n \to \infty} \Psi(x, y_n) \geq \Psi(x,y)$ *as* $y_n \to y$;

$F : K \to \mathbb{B}^*$ is strongly-weakly closed and quasi*-compact.*

Then, there exists $x \in K$ and $x^* \in F(x)$ such that

$$\langle x^*, \eta(x,y)\rangle + \Phi(x,y) \geq \Psi(y,x), \ \forall y \in K. \tag{5.3}$$

Furthermore, if Ψ is maximal monotone then problem (5.2) has at least one solution.

Motivated by the wide applicability of the Brézis pseudomonotonicity and the papers by Liu, Migórski and Zeng [79] and Chadli et al. [36], very recently Sahu et al. [38] defined the following mixed equilibrium problems involving a set-valued mapping in a general setting vector spaces in duality. Let X be a topological vector space over $\mathbb{R}$ and Y be a real vector space. Let $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ be a bilinear form. Suppose that the vector spaces X and Y and the bilinear form $\langle \cdot, \cdot \rangle$ are such that the family of linear functions $\{\langle \cdot, x\rangle\}_{x\in X}$ separates the points of Y. Further, suppose that the vector space Y is endowed with the topology $\sigma(Y,X)$ generated by the family $\{V(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$ as a basis of the neighbourhood system at $\mathbf{0}$, where $V(x,\varepsilon) := \{y \in Y : |\langle y,x\rangle| < \varepsilon\}$. By Aliprantis and Border [80, page 48], the vector space Y endowed with the topology $\sigma(Y,X)$ is a Hausdorff topological vector space as the family of linear functions $\{\langle \cdot, x\rangle\}_{x\in X}$ separates the points of Y. Let K be a nonempty closed and convex subset of X, $\Phi, \Psi : K \times K \to \mathbb{R}$ be two real-valued bifunctions and $F : K \rightrightarrows Y$ be a set-valued mapping. Then the mixed equilibrium problem defined by Sahu et al. [38] is as the following:

$$\begin{cases} \text{Find } x \in K \text{ and } \varpi \in F(x) \text{ such that} \\ \langle \varpi, y - x\rangle + \Phi(x,y) + \Psi(x,y) \geq 0, \text{ for all } y \in K. \end{cases} \tag{5.4}$$

Three solutions concepts of the problem (5.4) are defined below.

Definition 5.3. An element $x \in K$ is called,

(i) *strong solution* of the problem (5.4) if and only if,

$$\langle \varpi, y - x\rangle + \Phi(x,y) + \Psi(x,y) \geq 0, \text{ for all } y \in K \text{ and all } \varpi \in F(x).$$

(ii) *solution* of the problem (5.4) if and only if, there exists $\varpi \in F(x)$ such that

$$\langle \varpi, y - x\rangle + \Phi(x,y) + \Psi(x,y) \geq 0, \text{ for all } y \in K.$$

(iii) *weak solution* of the problem (5.4) if and only if, for each $y \in K$, there exists $\varpi \in F(x)$ such that

$$\langle \varpi, y-x\rangle + \Phi(x,y) + \Psi(x,y) \geq 0.$$

Theorem 5.3. [38] *Let X be a Hausdorff topological vector space, K be a nonempty closed and convex subset of X, and Y be a vector space endowed with the topology $\sigma(Y,X)$ such that for each $y \in Y$, the function $x \mapsto \langle y,x\rangle$ is continuous. Let $\Phi, \Psi : K \times K \to \mathbb{R}$ be real-valued bifunctions such that $\Phi(x,x) = \Psi(x,x) = 0$ for all $x \in K$, $F : K \rightrightarrows Y$ be a set-valued mapping such that F is closed with respect to Y endowed with the topology $\sigma(Y,X)$ and quasi-$\sigma(Y,X)$-compact, i.e. for any relatively compact set $M \subset X$, $F(M)$ is relatively compact with respect to the topology $\sigma(Y,X)$ of Y. Furthermore, suppose that*

(i) *Φ is monotone;*

(ii) *Ψ is B-pseudomonotone;*

(iii) *$\Phi(x,\cdot)$ and $\Psi(x,\cdot)$ are convex functions;*

(iv) *$\Phi(x,\cdot)$ is lower semicontinuous;*

(v) *For each $y \in K$, $\Psi(\cdot,y)$ is upper semicontinuous on co(Z) and $\Phi(\cdot,y)$ is continuous on co(Z), for each $Z \in \mathscr{F}(K)$;*

(vi) *There exists a nonempty compact set $D \subset K$ and a nonempty compact convex subset $C \subset K$, such that for each $x \in K \setminus D$, there exists $z \in C$ satisfying*

$$\langle \varpi, z-x\rangle + \Psi(x,z) < \Phi(z,x), \text{ for each } \varpi \in F(x).$$

Then the problem (5.4) has at least a weak solution $x \in K$. Moreover, if $F(x)$ is convex, then x is a solution of (5.4).

Corollary 5.1. [38] *If the B-pseudomonotonicity assumption on the bifunction Ψ in Theorem 5.3 is replaced by the following weaker assumption*

(ii)$'$ If for any generalized sequence $\{x_\alpha\}_{\alpha\in I}$ satisfying $\{x_\alpha\}_{\alpha\in I}$ stays in a compact set and converges to x and $\liminf \Psi(x_\alpha, x) \geq 0$, its limit x satisfies

$$\Psi(x,z) \geq \liminf \Psi(x_\alpha, z), \text{ for all } z \in K.$$

Then the conclusion of Theorem 5.3 is also true.

The following notion of pseudomonotonicity defined by Wangkeeree et al. [81] and Liu et al. [79] will be needed to establish strong solutions of the problem (5.4) in Banach spaces.

Definition 5.4. Let $F : K \rightrightarrows \mathbb{B}^*$ be a set-valued mapping and $\Psi, \Phi : K \times K \to \mathbb{R}$ be two bifunctions. Then F is said to be

(i) (Ψ, Φ)-pseudomonotone, if for each $x, y \in K$

$$\langle x^*, y - x\rangle + \Psi(x,y) \geq \Phi(y,x) \Longrightarrow \langle y^*, y - x\rangle + \Psi(x,y) \geq \Phi(y,x),$$

for all $x^* \in F(x)$ and $y^* \in F(y)$.

(ii) stably (Ψ, Φ)-pseudomonotone with respect to a set $U \subset \mathbb{B}^*$, if $F(\cdot) - \varpi$ is (Ψ, Φ)-pseudomonotone for every $\varpi \in U$.

Theorem 5.4. [38] *Let $\mathbb{B}$ be a reflexive Banach space and K be a nonempty closed and convex subset of $\mathbb{B}$. Let $\Phi, \Psi : K \times K \to \mathbb{R}$ be two bifunctions satisfying $\Phi(x,x) = \Psi(x,x) = 0$ for all $x \in K$ and $F : K \rightrightarrows \mathbb{B}^*$ be a set-valued mapping such that for any $Z \in \mathscr{F}(K)$, the restriction of F to $co(Z)$ is l.s.c. with respect to the weak*-topology of $\mathbb{B}^*$. Suppose that*

(i) *Φ is monotone;*

(ii) *Ψ is B-pseudomonotone with respect to the weak topology;*

(iii) *For each $x \in K$, the functions $\Phi(x,\cdot)$ and $\Psi(x,\cdot)$ are convex;*

(iv) *For each $y \in K$, $\Psi(\cdot, y)$ is upper semicontinuous on $co(Z)$ and $\Phi(\cdot, y)$ is continuous on $co(Z)$, for each $Z \in \mathscr{F}(K)$;*

(v) *For each $x \in K$, the functions $\Phi(x,\cdot)$ is lower semicontinuous;*

(vi) *F is (Ψ, Φ)-pseudomonotone;*

(vii) *(Coercivity) There exists a nonempty weakly compact subset D and a weakly compact convex subset B of K such that for each $x \in K \setminus D$, there exists $x^* \in$*

$F(x)$ and $y \in B$ satisfying

$$\langle x^*, y-x\rangle + \Psi(x,y) < \Phi(y,x).$$

Then the problem (5.4) has at least one strong solution.

Recently, Sahu et al [38] studied the existence of solutions for quasi-hemivariational inequalities involving a set-valued mapping. The problem considered is described as follows. Let $\mathbb{B}$ be a Banach space with its dual $\mathbb{B}^*$, K be a nonempty, closed and convex subset of $\mathbb{B}$ and $F : K \rightrightarrows \mathbb{B}^*$ be a set-valued mapping. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $T : X \to L^p(\Omega;\mathbb{R}^k)$ a linear continuous mapping, where $1 < p < +\infty$, $k \in \mathbb{N}^*$ and $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ a mapping which is locally Lipschitz with respect to the second argument. Let us denote by $\hat{x} := Tx$ and by $j^0(s,r;v)$ the Clarke's generalized directional derivative of $j(s,\cdot)$ at the point $r \in \mathbb{R}^k$ in the direction $v \in \mathbb{R}^k$, where $s \in \Omega$. Let $\Theta : K \times K \to \mathbb{R}$ be a real-valued bifunction satisfying the equilibrium condition $\Theta(x,x) = 0$ for all $x \in K$, and $h : K \to \mathbb{R}$ is a given nonnegative functional. Then the quasi-hemivariational inequality considered by Sahu et al [38] is defined as the following:

$$\left\{ \begin{array}{l} \text{Find } x \in K \text{ and } x^* \in F(x) \text{ such that} \\ \langle x^*, y-x\rangle + \Theta(x,y) + h(x)\displaystyle\int_\Omega j^0(s,\hat{x}(s);\hat{y}(s)-\hat{x}(s))ds \geq 0, \text{ for all } y \in K. \end{array} \right. \tag{5.5}$$

The following assumptions are considered on the functional $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$:

[H1] The function $s \in \Omega \mapsto j(s,z)$ is measurable, for all $z \in \mathbb{R}^k$;

[H2] There exists $\tau \in L^q(\Omega,\mathbb{R})$, where q is the conjugate exponent of p, such that

$$|j(s,z_1) - j(s,z_2)| \leq \tau(s)|z_1 - z_2|, \ \forall s \in \Omega, \ \forall z_1, z_2 \in \mathbb{R}^k;$$

[H3] The mapping $z \in \mathbb{R}^k \mapsto j(s,z)$ is locally Lipschitz, for all $s \in \Omega$;

[H4] There exists a constant $c > 0$ such that

$$|\xi| \leq c(1+|z|^{p-1}), \ \forall s \in \Omega, \ \forall \xi \in \partial j(s,z).$$

Theorem 5.5. [38] *Let K be a nonempty closed and convex subset of a real Banach space $\mathbb{B}$ with topological dual space $\mathbb{B}^*$. Let $\Theta : K \times K \to \mathbb{R}$ be a bifunction such that*

$\Theta(x,x) = 0$ *for all* $x \in K$. *Let* $T : \mathbb{B} \to L^p(\Omega;\mathbb{R}^k)$ *be a linear continuous mapping, where* $1 < p < +\infty$, $k \in \mathbb{N}^*$ *and* Ω *is a bounded open subset in* $\mathbb{R}^N$, *and* $h : \mathbb{B} \to \mathbb{R}$ *is a continuous nonnegative functional. Let* $F : K \rightrightarrows \mathbb{B}^*$ *be a set-valued mapping which is closed with respect to* $\mathbb{B}^*$ *endowed with the topology* $\sigma(\mathbb{B}^*,\mathbb{B})$ *and quasi-*$\sigma(\mathbb{B}^*,\mathbb{B})$ *compact. Let* $J : L^p(\Omega;\mathbb{R}^k) \to \mathbb{R}$ *be a functional defined by* $J(x) = \int_\Omega j(s,x(s))ds$, *where* j *satisfies either* ***[H1]*** *and* ***[H2]***, *or,* ***[H1]*** *and* ***[H3]-[H4]***. *Furthermore, suppose that*

(i) Θ *is B-pseudomonotone with respect to the strong topology of* $\mathbb{B}$, *and convex with respect to the second argument;*

(ii) *For all* $Z \in \mathscr{F}(K)$ *and fixed* $y \in K$, $\Theta(\cdot,y)$ *is u.s.c. on* $co(Z)$;

(iii) *There exists a nonempty compact set* $D \subset K$ *and a nonempty compact convex subset* $C \subset K$, *such that for all* $x \in K \setminus D$, *there exists* $z \in C$ *satisfying*

$$\sup_{\xi \in F(x)} \langle \xi, z-x \rangle + \Theta(x,z) + h(x)J^0(\hat{x};\hat{z}-\hat{x}) < 0.$$

Then the problem (5.5) *has at least one solution.*

Theorem 5.6. [38] *Let* K *be a nonempty closed and convex subset of a real reflexive Banach space* $\mathbb{B}$ *with topological dual space* $\mathbb{B}^*$. *Let* $\Theta : K \times K \to \mathbb{R}$ *be a bifunction such that* $\Theta(x,x) = 0$ *for all* $x \in K$. *Let* $T : \mathbb{B} \to L^p(\Omega;\mathbb{R}^k)$ *be a linear compact operator, where* $1 < p < +\infty$, $k \in \mathbb{N}^*$ *and* Ω *is a bounded open subset in* $\mathbb{R}^N$, *and* $h : \mathbb{B} \to \mathbb{R}$ *be a nonnegative functional. Let* $F : K \rightrightarrows \mathbb{B}^*$ *be a set-valued mapping such that* F *is a set-valued mapping such that for any* $Z \in \mathscr{F}(K)$, *the restriction of* F *to* $co(Z)$ *is l.s.c. with respect to the weak***-topology of* $\mathbb{B}^*$. *Let* $J : L^p(\Omega;\mathbb{R}^k) \to \mathbb{R}$ *be a functional defined by* $J(x) = \int_\Omega j(s,x(s))ds$, *where* j *satisfies either* ***[H1]*** *and* ***[H2]***, *or,* ***[H1]*** *and* ***[H3]-[H4]***. *Furthermore, suppose that*

(i) Θ *is B-pseudomonotone with respect to the weak topology of* $\mathbb{B}$;

(ii) *For all* $x \in K$, $\Theta(x,\cdot)$ *is convex;*

(iii) *For all* $Z \in \mathscr{F}(K)$ *and fixed* $y \in K$, $\Theta(\cdot,y)$ *is u.s.c. on* $co(Z)$;

(vi) $h : K \to \mathbb{R}$ *is weakly continuous;*

(v) *F is stably* Θ*-pseudomonotone with respect to the subset* $U(h,J,T)$ *defined by*

$$U(h,J,T) = \{-h(x)\varpi_x^* : x \in K \text{ and } \varpi_x^* = T^*\varpi_x\},$$

where $T^* : L^q(\Omega;\mathbb{R}^k) \to X^*$ *be the adjoint operator of* T*, i.e.* $\langle Tx, y\rangle = \langle x, T^*y\rangle$ *for all* $x \in \mathbb{B}$ *and* $y \in L^q(\Omega;\mathbb{R}^k)$*;*

(vi) *There exists a nonempty weakly compact set* $D \subset K$ *and a nonempty weakly compact convex subset* $C \subset K$*, such that for all* $x \in K \setminus D$*, there exists* $z \in C$ *satisfying*

$$\sup_{\xi \in F(x)} \langle \xi, z - x\rangle + \Theta(x,z) + h(x)J^0(\hat{x};\hat{z} - \hat{x}) < 0.$$

Then the problem (5.5) *has at least one strong solution.*

Acknowledgements.

The first and second authors are grateful to the Mohapatra Family Foundation for it's support during this research.

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