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Impact of stress jump condition and heterogeneous reaction on velocity, temperature and concentration during bio-fluid flow through a permeable microvessel

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Abstract

Understanding how fluid flow in human bodies is crucial in Biomedical Engineering. Studying blood rheology is crucial as it may help in detecting , if not designing a treatment for some blood related diseaseses or understanding them better . The aim of this paper is to study the heterogeneous reaction of blood flow velocity, temperature and diffusion through microvessel with the stress-jump condition at the interface of the clear and peripheral region and velocity slip condition at the wall of microvessel .We have considered a two phase model where the radius of the microvessel is divided into two parts. The flow nature at the clear region is defined by non-Newtonian Casson fluid and the flow at peripheral region is defined by Newtonian fluid. The wall of the microvessel is considered as permeable and the nature defined by Brinkman model. The governing equations are solve numerically and written in the form of Bessel function. Impact of the velocity, temperature and concentration profile with respect to the different parameters such as stress-jump condition, permeability parameter, yield stress, velocity and concentration slip condition are displayed graphically.

Key words: Blood Flow, Two phase model, non-Newtonian fluid, stress-jump condition, heterogeneous reaction.

AMS Classification: 5J25, 35K05, 76Mxx, 76Zxx.

1. Introduction

The study of fluid dynamics of basic biological fluids such as blood has considered as a great tool of biomedical engineering in recognizing the cause of certain diseases and making easy to come up with ways to cure the diseases (Mazumdar [1]). Blood is a complex body fluid is a liquid tissue consisting of several types of formed elements (cells) suspended in an aqueous fluid matrix (plasma). Blood flows in different ways depending on the channel it is flowing in e.g closed circulating system (veins, arteries and capillaries) and open circulatory system (heart etc). Nanda and Basu Mallik [2] pointed out that blood behaves like a homogeneous Newtonian fluid in large blood vessels while in narrow blood vessels behaves as a non-Newtonian fluid. Study of heat transfer in a living tissue is always interesting and development mathematical models in purpose to focusing thermal regulation, comfort or other phenomena where significant heat exchanges taken place (Chen and Holmes [3]. The bio-heat equation during blood flow through vessel has been expressed by Pennes [4] based on his experimental outcomes. An analytical solution of the Pennes equation on bioheat has been studied by Huang et al. [5] and Yue et al. . Porous medium (peripheral layer for the micovessel) plays a vital role in heat transfer in blood vessel (Khaled and Vafai [7]). Sinha et al. [8]

highlighted the heat transfer for a unsteady blood flow in a permeable vessel. They have introduced non-uniform heat source. The effect of magnetic field on the heat transfer of two-phase blood flow through a stenosed artery has been discussed by Ponalagusamy and Selvi [9]. The governing parameters those influence the heat transfer and corresponding mathematical models are discussed by Fasano and Sequeira [10] in details. Solutal transport along with the heat transfer are responsible for different activities such as secretion of insulin, gastric acid etc. and it is more prompted during drug delivery (Sushma et al. [11]. Impact of both thermal diffusion and solutal reaction on blood flow plays a vital role in the concentration difference and rate of change in heat transfer (Xu et al. [12]). Heat and mass transfer for a physiological fluid has been studied by Misra and Adhikary [13]. Das and Chakraborty [14] studied the electoviscous effect on the velocity, temperature and concentration distribution of non-Newtonian biofluid. The purpose of the present paper is to study blood flow velocity , temperature and diffusion through a permeable microvessel with stress jump condition and velocity slip condition. Considering a two phase non-Newtonian fluid model where the radius of the microvessel is divided into two parts clear region and pheriheral layer of plasma. clear region defined to be non-Newtonian Casson fluid mostly containing cells such as Red blood cells, white blood cells and platelets (plug region) and cell-depleted region, while peripheral layer of plasma defined to be Newtonian fluid. The concentration profile is divided into two-phase as same as the velocity profile. However the temperature profile is considered as a single phase. The governing equations for velocity, temperature and concentration are solved analytically and established results through graphs.

2. Mathematical formulation

Since blood vessels are kind of circular, we consider a cylindrical polar coordinate system (r, θ, z) where the z-axis is along the axis of the microvessel, where r and θ are coordinates along the radial and circumferential directions respectively. The flow along the microvessel is described by a two phase non-Newtonian Casson model and the wall of the microvessel is assumed to be permeable following Brinkman Model nature, with slip condition at the wall. Clear region is taken to be non-Newtonian Casson fluid which is of radius h_2. The radius of the plug region is h_1, $h-h_2$ being the thickness of the peripheral region taken to be Newtonian fluid as shown in Fig. 2.1. It is assumed that the flow is fully developed and axi-symetric.

FIGURE 1. schematic diagram of the two-phase non-Newtonian Casoon model of two-phase blood flow in a permeable microvessel

2.1. **Velocity Profile.** Due to small radius, the flow in the microvessel is steady, incompressible and uni-directional. The axial velocity u is a function of r only. We have considered the length of the microvessel is larger that the radius of the microvessel and hence the pressure gradient is constant [9]. The governing equations for the two phase Casson fluid with Brinkman model at the peripheral region may be written as

$$\frac{\partial U_p}{\partial r} = 0,\ \ 0 \leq r \leq h_1, \tag{1}$$

$$\frac{\partial p}{\partial z} = \frac{-1}{r}\frac{\partial}{\partial r}\left[r\left(\tau_y^{1/2} + \left(-\mu_1\frac{\partial U_1}{\partial r}\right)^{1/2}\right)^2\right], \quad h_1 \leq r \leq h_2, \tag{2}$$

$$\frac{\partial p}{\partial z} = \frac{\mu_2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_2}{\partial r}\right) - kU_2, h_2 \leq r \leq h, \tag{3}$$

where p is the pressure, U_p, U_1 and U_2 are axial velocities in the plug region, core region and peripheral region respectively. τ_y is the yield stress, μ_1 and μ_2 are viscosities of the fluid in the core region and peripheral region, respectively.

Associated boundary conditions are

$$\frac{\partial U_p}{\partial r} = 0, \text{ at } r = 0, \tag{4}$$

$$U_p = U_1, \text{ at } r = h_1, \tag{5}$$

$$U_1 = U_2 \text{ and } \mu_1 \frac{\partial U_1}{dr} = \mu_2 \left(\frac{\partial U_2}{\partial r} - \beta U_2 \right) \text{ at } r = h_2, \tag{6}$$

$$\frac{\partial U_2}{\partial r} + \gamma U_2 = 0, \text{ at } r = h. \tag{7}$$

Equation (2.4) represents the velocity profile constant at the plug region. Equation (2.5) is the continuity of the velocity at the plug and core region. Stress jump condition at the interface of the clear and peripheral region is define in equation (2.6) with the continuity of the velocity. Velocity slip condition (equation 2.7) is considered at the inner wall of the microvessel as the microvessel wall is porous in nature. Solving the above governing equation analytically and using the associated boundary condition we get our velocities as follows

$$U_p = \frac{h_1^2}{4\mu_1}\frac{\partial p}{\partial z} - \frac{h_1 \tau_y}{\mu_1} + \frac{4h_1^{3/2}}{3\mu_1}\sqrt{\frac{-\tau_y}{2}\frac{\partial p}{\partial z}} + B, \tag{8}$$

$$U_1 = \frac{r^2}{4\mu_1}\frac{\partial p}{\partial z} - \frac{r \tau_y}{\mu_1} + \frac{4r^{3/2}}{3\mu_1}\sqrt{\frac{-\tau_y}{2}\frac{\partial p}{\partial z}} + B, \tag{9}$$

$$U_2 = m_1 J_0(\lambda r) + m_2 Y_0(\lambda r) + \frac{\kappa}{\lambda^2}, \tag{10}$$

where $\lambda^2 = \frac{-k}{\mu_2}$, $\kappa = \frac{1}{\mu_2}\frac{dp}{dz}$ $m_1,\ m_2$ and B is found in the appendix.

Volumetric flow rate

$$Q = 2\pi \int_0^h r u dr, \tag{11}$$

expressed as

$$Q = 2\pi \left[\int_0^{h_1} r U_p dr + \int_{h_1}^{h_2} r U_1 dr + \int_{h_2}^{h} U_2 dr \right], \tag{12}$$

Thus the average velocity of blood flow through the microvessel

$$\cup = \frac{Q}{\pi h^2}. \tag{13}$$

2.2. **Temperature.** Using the concept of the Pennes' equation (Pennes [4], Yue et al. [6]), the one dimensional bioheat equation for the steady state and absent of the external spatial heating can be written as

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{q_m}{K_b}=\frac{W_bC_b}{K_b}(T-T_A), \tag{14}$$

where T is the temperature which is a function of r, q_m is the metabolic heat generation per unit volume, W_b is the perfusion rate of blood, C_b is the specific heat of the blood, K_b is the thermal conductivity of the surrounding tissue of the blood vessel and T_A is the arterial temperature. Solving the governing equation using the boundary conditions below

$$\frac{\partial T}{\partial r}=0 \ at \ r=0, \tag{15}$$

$$T=T_w \ at \ r=h, \tag{16}$$

where T_w is the wall temperature due to the surrounding tissue of the blood vessel. We get

$$T=L_1J_0(r\sqrt{-G})+L_2Y_0(r\sqrt{-G})+T_A+\frac{q_m}{w_bC_b}, \tag{17}$$

with $G=W_bC_b/K_b$. Since Temperature is finite at $r=0$,we have L_2 is zero, Hence

$$T=L_1J_0(r\sqrt{-G})+T_A+\frac{q_m}{w_bC_b}, \tag{18}$$

where $L_1=\frac{T_w-T_A}{J_0(h\sqrt{-G})}$.

2.3. **Concentration Profile.** Following the approach proposed by Taylor [15], a cylindrical frame of reference (r,z) is considered as in fig. 2.1. The governing advection-Diffusion equation is given by

$$\frac{U(r)}{D_m}\frac{\partial C}{\partial z}+\frac{1}{D_m}\frac{\partial C}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)+\frac{\partial^2 C}{\partial z^2}, \tag{19}$$

where u(r) is non-uniform axial velocity, C(r,z,t) is solute concetration and Dm is the Diffussion coefficient. Moving on with Taylor's approximation, the governing equation can be reduced (Taylor [15]), and the reduced equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dC}{dr}\right)=\frac{\hat{U}(r)}{Dm}\frac{dC}{dz}, \tag{20}$$

where $\hat{U}(r)=U(r)-\cup$ is the velocity deviation from the mean $\cup$ (Gentile et al. [16]).

Solving the reduced equation (2.20) by using the plug, core and peripheral region velocities together with the following boundary conditions

$$C=\frac{\partial C}{\partial r}=0 \quad \text{at } r=0, \tag{21}$$

$$C_1=C_p \quad at\ r=h_1, \tag{22}$$

$$C_1=C_2, \qquad \frac{\partial C_1}{\partial r}=\frac{\partial C_2}{\partial r} \quad \text{at } r=h_2, \tag{23}$$

$$\frac{\partial C_2}{\partial r}+\gamma C_2=-\frac{1}{D_m}\frac{\partial C}{\partial z} \text{ at } r=h, \tag{24}$$

we get

$$C_p=\frac{U_p}{D_m}\frac{\partial C}{\partial z}\frac{r^2}{4}, \tag{25}$$

$$C_1=\left(\frac{r^4}{64\mu_1}\frac{\partial p}{\partial z}-\frac{r^3}{9}\frac{\tau_y}{\mu_1}+\frac{16}{147}\frac{r^{\frac{7}{2}}}{\mu_1}\sqrt{\frac{-\tau_y}{2}\frac{\partial p}{\partial z}}+\frac{Br^2}{4}\right)\frac{1}{D_m}\frac{\partial C}{\partial z}+B_1lnr+B_2, \tag{26}$$

$$C_2=\left(\frac{m_1}{\lambda^2}J_0(\lambda r)-\frac{m_2}{\lambda^2}Y_0(\lambda r)+\frac{r^2}{4}\frac{\kappa}{\lambda^2}\right)\frac{1}{D_m}\frac{\partial C}{\partial z}+B_3lnr+B_4, \tag{27}$$

where $B_1,\ B_2,\ B_3$ and B_4 are found in the appendix.

3. Results and Discussion

The governing equations for velocity, temperature and solute concentration of blood flow in permeable microvessel are solved analytically with the help of boundary conditions and written in the form of general and modified Bessel functions. Then, these profiles where plotted against radius, for some fixed parameters $\beta=0.1, \gamma=0.02, \tau_y=0.15, \frac{dp}{dz}=10$ and $k=1$. The impact of different parameters such as stress jump constant, slip constant, yield stress, pressure gradient and permeability constant are shown through Fig. 2.2 - 2.16.

The blood flow through a microvessel is very interesting and complicated in the present of RBSc which creates an additional region near to the axis. Due to rotation nature, RBCs are accumulated near to the axis of the microvessel and behaves as a semisolid cylinder of radius h_p, width of the plug region. The velocity of this region is constant or other way can say that it followed with a zero velocity gradient. From the governing equations (2.1) - (2.3), it is clear that at the plug region the velocity of the fluid is constant which appeared in all the graphs, and at the core region the fluid follow the Casson nature which gives a parabolic shape and it continue at the peripheral region to follow the boundary condition at the inner wall of the microvessel. In the present problem, we have considered the width of the plug region is 0.3 and hence the constant velocity profile will be continue till $r = 0.3$. However, a significant changes has been noticeable at $r = 0.9$, which is the interface of the clear region and peripheral region. A similar profile of the velocity is observed for all cases. Stress jump condition is taken place at the interface of the fluid

FIGURE 2. Velocity radius graph with different stress jump constant β

FIGURE 3. Velocity radius graph with different slip constant γ

region and peripheral region which is a porous medium. It represent a jump of stress between two regions. It is evident that with increase in the stress jump condition, the stress difference between two region is increasing and it introduce an additional stress which may cause for the reduction in velocity profile as shown in Fig. 2.2.

The slip condition at the interface of the microvessel taken place due to permeability nature of the ineer wall of the microvessel. It gives a non-zero velocity at the inner surface. The nature of the velocity is more significant near the wall, at the peripheral region. From Fig. 3, it is observed that for higher value of slip constant (γ), the slope of the velocity is going upward at the peripheral region and reduces the velocity difference. The stiffness is more for higher slip constant ($\gamma = 0.5$) and almost slit for $\gamma = 0.02$. The

FIGURE 4. Velocity radius graph with different permeability constant K

FIGURE 5. Velocity radius graph with different yield stress τ_y

permeability related with the porous medium which is containing the pores and fluid is passing through those pores which gives a restriction on the flow. Hence the flow is not that much faster than the clear region and as a result the fluid velocity decreases with increase of the permeability parameter as shown in Fig. 4.

Yield stress is another very important parameter which appears due to the Casson fluid nature of the blood at the clear region. Yield stress is directly proportional to the pressure gradient. With increase in yield stress, the velocity of the fluid increases but the slope is more stiffen for the higher value of yield stress as clearly observed in Fig. 5.

Fig. 6 is displayed the influence of pressure gradient on the velocity profile. It is evident that the velocity of the fluid is higher near to the axis as increase in the pressure gradient. The velocities for different pressure gradient is coincide at a point which is $r = 0.65$ in the present case. It is interesting to note that the stiffness of the velocity is higher for higher value of the pressure gradient.

Figure 6. Velocity radius graph with different pressure gradient parameters

Concentration profile followed a zero concentration at the plug region. After that there is an enhancement in the concentration with the radial direction. This profile is followed in all the cases. The concentration of the solute is decreasing with increase in the stress jump condition. It is because of the decreasing in velocity which again retrained the solute concentration (see Fig. 7).

From Fig. 8, it is evident that the slip constant at the surface of the microvessel increases the concentration of the solute. The profile of the concentration is the same as the previous. The value the slip constant is considered $0.02, 0.04, 0.06$ and 0.08 and it is evident that the concentration difference is higher between the lower values of slip constant i.e., between $\gamma = 0.02$ and 0.04.

The concentration of the solute is a decreasing function of the permeability parameter as same as the velocity (see Fig. 9). May be this is the reason of the reduction in concentration. Again the difference of the two consecutive concentration is higher for the difference between two consecutive lower permeability parameter.

Figure 7. Velocity radius graph with different stress jump constant β

Figure 8. Concentration radius graph with different slip constant γ

The yield stress, which is related with the nature of Casson fluid act mainly at the clear region, is enhanced the concentration profile through the radius of the microvessel. However, an opposite phenomena is observed when compare the difference of the two consecutive concentration, which is higher for this case for the difference between two consecutive higher yield stress as visible in Fig. 10. Initially, the pressure gradient enhanced the concentration of the solute but for higher value it shows a stable concentration profile. From the figure 11, it evident that the concentration profile not significantly change for $dp/dz = 30$ and $dp/dz = 40$.

FIGURE 9. Concentration radius graph with different permeability constant K

FIGURE 10. Concentration radius graph with different yield stress τ_y

FIGURE 11. Concentration radius graph with different pressure gradient

Temperature profile looks different and interesting. The temperature of the blood (T_A) and temperature at the inner surface of the microvessel which is basically due to the temperature of the surrounding tissue, are considered different in values. However, the temperature at the surface of the inner wall is considered higher than the temperature of the blood. The motion of the RBCs are constant near the axis and they are unable to distribute the temperature through out. As a result, the temperature decreasing significantly at the plug region and it continue at the outer region which is basically a cell depleted region (absence of RBCs). Due to this cell depleted nature the temperature reduction is continuing till $r = 0.65$. After that there is an increment in the temperature profile which continue till the surface of the wall to follow the boundary condition. This enhancement is related with the temperature of the inner surface which is higher than the temperature of the blood and hence it influence the heat transfer towards the axis and increase the temperature near to the surface. This general phenomena is observed for all the cases. The nature of the temperature profile for different value of stress jump constant is same as velocity and concentration profile i.e., the temperature profile is increasing by enhancement in the stress jump constant (see Fig. 12). It is noted that the temperature at near the axis is higher than the temperature at the surface except for $\beta = 0.1$.

Figure 12. Temperature radius graph with different stress jump constant β

Figure 13. Temperature radius graph with different slip constant γ

Temperature profile curve is more interesting and significant with respect to the slip velocity condition at the inner surface of the wall. Initially the temperature gradient is negative, while after $r = 0.65$, the gradient becomes positive. It observed that the stiffness of the positive gradient is higher that the stiffness of the negative gradient (see Fig. 13). The temperature profile is increasing with increase in the slip constant, however the difference is not much significant.

The permeability of the microvessel and peripheral region reduced the temperature at the axis of the vessel. This permeability parameter gives a significant increment through out the region and it decreasing the temperature with increase of the permeability. It is very clear that the temperature distribution is much faster at the clear region than the peripheral region as reflected in Fig. 14. In all the cases, the temperature at the axis is higher than the temperature at the surface of the microvessel. It is interesting to note that the temperature profile is constant with respect to radial direction for $K = 1$ and it carried a constant value 0.5, the surface temperature.

FIGURE 14. Temperature radius graph with different permeability constant K

FIGURE 15. Temperature radius graph with different yield stress τ_y

FIGURE 16. Temperature radius graph with different pressure gradient

It is very interesting to see that the temperature at the axis is always lower that the temperature at the surface of the microvessel under consideration of the yield stress values as $0.15, 0.30, 0.45$ and 0.60 (see Fig. 15). The temperature at the axis is decreasing with increase in the yield stress. Initially $r = 0.5$, the temperature profile not significantly behaves with respect to the radial direction but after a critical point there is a sharp peak in the temperature profile. This peak is more sharper for higher yield stress and the critical point moves away from the surface with increase on the yield stress. With increase in pressure gradient, it enhances the velocity of the solute which equally responsible to distribute the temperature through out. Hence, it is very natural that the pressure gradient enhanced the temperature at the axis and through out the radial direction. The temperature at the axis is higher than the temperature at the surface for most of the cases ($dp/dz = 20$, 30, 40) while it is lower for the lower pressure gradient. Hence we have observed two different pattern of temperature profile for larger and smaller value of the pressure gradient as displayed in Fig. 16. For lower pressure gradient (here $dp/dz = 10$), the temperature profile not significant but slightly increases with radial direction.

Conclusion

The present problem represent the velocity, temperature and concentration of blood flow through a microvessel contain peripheral layer. The velocity and concentration profile are divided into three region e.g., plug region, outer region and peripheral region. The influence of the stress jump condition, slip condition of velocity and concentration , yield stress, pressure gradient and permeability of the peripheral region plays important roles which reflected through graphs. In general it observed that the velocity and concentration at the plug region is constant while after that velocity decreases through out the radius but concentration increases continuously in radial direction. Velocity is non-zero at the walls of the microvessel because of the slip constant γ and stress jump constant cause rapid decrease in velocity between the core region and the peripheral region. Temperature profile is challenging for all the parameters. This works may give an overall idea of blood flow in sense of velocity, temperature and concentration under certain condition.

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4. Appendix

$\lambda^2 = \frac{-k}{\mu_2}$

$\kappa = \frac{1}{\mu_2}\frac{dp}{dz}$

$T_1 = \frac{h_2^2}{4\mu_1}\frac{dp}{dz} - \frac{h_2\tau_y}{\mu_1} + \frac{4h_2^{3/2}}{3\mu_1}\sqrt{\frac{-\tau_y}{2}\frac{dp}{dz}} - \frac{\kappa}{\lambda^2}$

$T_2 = \frac{h_2^2}{2\mu_1}\frac{dp}{dz} - \frac{\tau_y}{\mu_1} + \frac{2h_2^{1/2}}{\mu_1}\sqrt{\frac{-\tau_y}{2}\frac{dp}{dz}}$

$T_3 = \frac{h_1^2}{4\mu_1}\frac{dp}{dz} - \frac{h_1\tau_y}{\mu_1} + \frac{4h_1^{3/2}}{3\mu_1}\sqrt{\frac{-\tau_y}{2}\frac{dp}{dz}}$

$S_1 = -\mu_2\lambda J_0(\lambda h_2) - \mu_2\beta J_0(\lambda h_2)$

$S_2 = -\mu_2\lambda Y_1(\lambda h_2) - \mu_2\beta Y_0(\lambda h_2)$

$S_3 = \lambda J_1(\lambda h) - \gamma J_0(\lambda h)$

$S_4 = \lambda Y_1(\lambda h) - \gamma Y_0(\lambda h)$

$m_1 = (T_2 + \mu_2\beta\frac{\kappa}{\lambda^2} - \gamma\frac{\kappa}{\lambda^2}\frac{\S_1}{\S_3})(\frac{S_3}{S_1\S_4+S_2\S_3})$

$m_2 = \gamma\kappa(\frac{S_3}{S_3^2\lambda^2-m_2\S_4})$

$B = m_1 J_0(\lambda h_2) + m_2 Y_0(\lambda h_2) - T_1$

$G_1 = \frac{U_p}{D_m}\frac{dC}{dz}\frac{h_1^2}{4} - (\frac{h_1^4}{64\mu_1}\frac{dp}{dz} - \frac{h_1^3}{9}\frac{\tau_y}{\mu_1} + \frac{16}{147}\frac{h_1^{\frac{7}{2}}}{\mu_1}\sqrt{\frac{-\tau_y}{2}\frac{dp}{dz}} + \frac{Bh_1^2}{4})\frac{1}{D_m}\frac{dC}{dz}$

$G_2 = (\frac{m_1}{\lambda^2}J_0(\lambda h_2) - \frac{m_2}{\lambda^2}Y_0(\lambda h_2) + \frac{h_2^2}{4}\frac{\kappa}{\lambda^2})\frac{1}{D_m}\frac{dC}{dz} - (\frac{h_2^4}{64\mu_1}\frac{dp}{dz} - \frac{h_2^3}{9}\frac{\tau_y}{\mu_1} + \frac{16}{147}\frac{h_2^{\frac{7}{2}}}{\mu_1}\sqrt{\frac{-\tau_y}{2}\frac{dp}{dz}} + \frac{Bh_2^2}{4})\frac{1}{D_m}\frac{dC}{dz}$

$G_3 = (\frac{h_2^3}{16\mu_1}\frac{dp}{dz} - \frac{h_2^2}{3}\frac{\tau_y}{\mu_1} + \frac{8}{21}\frac{h_2^{\frac{5}{2}}}{\mu_1}\sqrt{\frac{-\tau_y}{2}\frac{dp}{dz}} + \frac{Bh_2}{2})\frac{1}{D_m}\frac{dC}{dz} + (\frac{m_1}{\lambda}J_1(\lambda h_2) + \frac{m_2}{\lambda}Y_1(\lambda h_2) + \frac{h_2}{2}\frac{\kappa}{\lambda^2})\frac{1}{D_m}\frac{dC}{dz}$

$G_4 = (\frac{m_1}{\lambda}J_1(\lambda h) + \frac{m_2}{\lambda}Y_1(\lambda h) + \frac{h}{2}\frac{\kappa}{\lambda^2})\frac{1}{D_m}\frac{dC}{dz} + \gamma(\frac{m_1}{\lambda^2}J_0(\lambda h) - \frac{m_2}{\lambda^2}Y_0(\lambda h) + \frac{h^2}{4}\frac{\kappa}{\lambda^2})\frac{1}{D_m}\frac{dC}{dz})$

$B_1 = (G_2 - h_2 G_3 lnh_2 + \frac{G_4}{\gamma} + \frac{1}{\gamma D_m}\frac{dC}{dz} + G_3 h_2 lnh + G_1)\frac{1}{lnh_1 - lnh}$

$B2 = G_1 - B_1 lnh_1$

$B_3 = G_3 h_2 + B_1$

$B_4 = -(\frac{G_4}{\gamma} + \frac{1}{\gamma D_m}\frac{dC}{dz} + B_3 lnh)$

$\tau_{yp} = \sqrt{\frac{-\tau_y}{2}\frac{dp}{dz}}$

$A_1 = B - U_{av}$