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On the Absolute Hausdorff Summability of Fourier Series and Series Conjugate to Fourier Series

by

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Abstract

We study the absolute Hausdorff summability problem of Fourier's series and its conjugate series generalizing some known results in the literature.

Key words: Hausdorff means, Fourier series, conjugate series, mass function

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1. Definition and Notations

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series and let $\{S_n\}$ be the sequence of its partial sums. Corresponding to a given sequence $\{\mu_n\}$ of real or complex numbers , the sequence to sequence Hausdorff transformation is defined by [4]

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \left(\Delta^{n-k} \mu_k\right) S_k$$

and the series to series Hausdorff Transformation is defined by [8]

$$C_n = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\left(\Delta^{n-k}\mu_k\right)ka_k, (n \geq 1)$$
$$C_0 = a_0, \text{where for } n \geq 0$$
$$\Delta^m \mu_n = \sum_{k=0}^{m}\binom{m}{k}(-1)^k \mu_{n+k}$$

The sequence $\{S_n\}$ (or the series $\sum_{n=0}^{\infty} a_n$) is said to be summable (H, μ_n) to s if

$$\lim_{n\to\infty} t_n = s$$

Further if $\sum_{n=1}^{\infty} |C_n| < \infty$ (or if $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$)

We say the series $\sum_{n=0}^{\infty} a_n$ (or the sequence $\{s_n\}$) is absolutely summable $|H, \mu_n|$ or summable $|H, \mu_n|$.

It is well known [4] that the necessary and sufficient condition for the method (H, μ_n) to be conservative is the existence of a mass function $\chi(x)$ defined over closed interval $[0, 1]$ such that

$$(i)\ \chi(x) \in BV(0, 1)$$
$$(ii)\ \mu_n = \int_0^1 x^n d\chi(x) \quad (n = 0, 1, 2...)$$

If further $\chi(x)$ satisfies the conditions

$$(iii)\ \chi(0^+) = \chi(o) = o$$
$$(iv)\ \chi(1) = 1$$

then $\mu(u)$ is called a regular moment constant, (H, μ_n) is called a regular Hausdorff Transformation and $\chi(x)$ is called a regular mass function.

If the mass function $\chi(x) = 1 - (1 - x)^{\alpha}, \alpha > 0, 0 \leq x \leq 1$ then $\mu_n = \binom{n+\alpha}{n}^{-1}$

and (H, μ_n) method reduces to familiar (C, α) method [4]. On the other hand if

$$\chi(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{1+q} \\ 1, & \frac{1}{1+q} \leq x \leq 1 \end{cases}$$

where $q > 0$, then (H, μ_n) method reduces to familiar Euler or (E, q) method. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. We assume without loss of generality that the constant term in the Fourier series of $f(t)$ is zero.

We write

$$\varphi(t) = \frac{1}{2}\{f(u+t) + f(u-t)\}$$
$$\psi(t) = \frac{1}{2}\{f(u+t) - f(u-t)\}$$

Let the Fourier series of $f(t)$ at $t = u$ be given by

$$\sum_{n=1}^{\infty} A_n(u)$$

Where $A_n(u) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \varphi(t) \cos ntdt, (n \geq 1)$ and the series conjugate to the Fourier series of $f(t)$ at $t = u$ be given by

$$\sum_{n=1}^{\infty} B_n(u)$$

Where $B_n(u) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \psi(t) \sin ntdt, (n \geq 1)$

2. Introduction

Hille and Tamarkin[7] have studied the (H, μ_n) summability of Fourier series and associated series by imposing direct or indirect conditions on the mass function $\chi(x)$. Concerning the absolute Hausdorff summability of Fourier series Tripathy [12] obtained a result which turned out to be equivalent to Cesaro case. [see 10].

Bosanquet proved the following Theorem regarding the absolute Cesaro summability of Fourier series.

Theorem A[1] $\varphi(t) \in BV(0,\pi) \Rightarrow \sum_{n=1}^{\infty} A_n(u) \in |C,\alpha|, \alpha > 0$. Further, Bosanquet and Hyslop proved the following Theorem regrading the absolute Cesaro summability of Conjugate Fourier series.

Theorem B[2] $\psi(t) \in BV(0,\pi)$ and $\dfrac{\psi(t)}{t} \in L(0,\pi) \Rightarrow \sum_{n=1}^{\infty} B_n(u) \in |C,\alpha|, \alpha > 0$.

3. Main Results

With a view to generalise Theorem A in Hausdorff summability set up, we prove the following

Theorem 1

Let

(i) (H,μ_n) is conservative

(ii) $\chi(x)$ is absolutely continuous over $(0,1)$

(iii) $\chi^1(x)$ is monotonic increasing in $(0,1)$

(iv) $\int_0^1 |\chi^1(x)| \log \dfrac{1}{1-x} dx < \infty$

Then $\varphi(t) \in BV(0,\pi) \Rightarrow \sum_{n=1}^{\infty} A_n(u) \in |H,\mu_n|$.

Generalising result of Bosanquet and Hyslop (Theorem B) in Hausdorff summability set up we prove the following.

Theorem 2 Let

(i) (H,μ_n) is conservative

(ii) $\chi(x)$ is absolutely continuous over $(0,1)$

(iii) $\chi^1(x)$ is monotonic increasing in $(0,1)$

(iv) $\int_0^1 |\chi^1(x)| \log \dfrac{1}{1-x} dx < \infty$

Then

(a) $\psi(t) \in BV(0,\pi)$

(b) $\dfrac{\psi(t)}{t} \in L(0,\pi)$

$\Rightarrow \sum_{n=1}^{\infty} B_n(u) \in |H, \mu_n|$

Remarks (i) The conditions imposed on the mass function $\chi(x)$ appears to be stringent but Theorem 1 fails to hold if we merely assume the absolute continuity of $\chi(x)$. We prove

Theorem 3 There exists Conservative matrix (H, μ_n) with absolutely continuous mass function $\chi(x)$ and a function $f(t)$ of the class L such that

$$\varphi(t) \in AC(0,\pi)$$

but the Fourier series of $f(t)$ at $t = u$ is not summable $|H, \mu_n|$.

Remarks (ii) By taking $\chi(x) = 1 - (1-x)^{\alpha}, 0 < \alpha < 1$ in Theorem 1 and Theorem 2 we obtain respectively Theorem A and Theorem B.

4. Lemmas

We need the following additional notations.

$$\rho^2 = 1 - 4x(1-x)\sin^2 t/2$$
$$\Theta = \tan^{-1}\frac{x\sin t}{1-x+x\cos t}$$
$$P_n(x,t) = \sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\cos\nu t$$
$$Q_n(x,t) = \sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\sin\nu t$$
$$R_n(x,t) = \sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\nu.\sin\nu t$$
$$S_n(x,t) = \int_t^\pi \frac{R_n(x,\nu)}{\nu}d\nu$$
$$L_n(x,t) = \int P_n(x,t)dx$$
$$M_n(x,t) = \int Q_n(x,t)dx$$

Lemma 1

$$P_n(x,t) = O(1)$$
$$Q_n(x,t) = O(1)$$
$$L_n(x,t) = O\left(\frac{1}{nt}\right)$$
$$M_n(x,t) = O\left(\frac{1}{nt}\right)$$

Proof By formal computation

$$\begin{aligned}
P_n(x,t) &= \rho^n \cos n\Theta \\
Q_n(x,t) &= \rho^n \sin n\Theta \\
L_n(x,t) &= \frac{\rho^{n+1}}{n+1}\left[\frac{\sin(n+1)\Theta}{2\tan t/2} - \frac{\cos(n+1)\Theta}{2}\right] \\
M_n(x,t) &= -\frac{\rho^{n+1}}{n+1}\left[\frac{\cos(n+1)\Theta}{2\tan t/2} + \frac{\sin(n+1)\Theta}{2}\right]
\end{aligned}$$

As $0 < \rho^n \leq 1$, for $0 \leq x \leq 1$ and $0 < t \leq \pi$, the proof of the lemma follows.

Lemma 2[3] Suppose that $f_n(x)$ is measurable in (a,b) where $b-a \leq \infty$ for $n = 1,2,3....$ Then a necessary and sufficient condition that for every $\lambda(x)$ integrable (L) over (a,b) the functions $f_n(x)\lambda(x)$ should be integrable (L) over (a,b) and

$$\sum_{n=1}^{\infty}\left|\int_a^b \lambda(x).f_n(x)dx\right| < \infty$$

is that $\sum_{n=1}^{\infty}|f_n(x)|$ should be essentially bounded for x in (a,b).

Lemma 3 [14] $\sum_{\nu=1}^{n}\nu\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu} = nx.$

Proof It is easy to verify.

Lemma 4[11] If $\eta > 0$ and $\lambda > 0, t^{\lambda}h(t) = H(t)$ then necessary and sufficient conditions that

(i) $h(t)$ should be of bounded variation in $(0,\eta)$

(ii) $\frac{|h(t)|}{t}$ should be integrable in $(0,\eta)$ are that

$$\int_0^{\pi} t^{-\lambda}|dH(t)| < \infty \qquad \text{and} \qquad H(0+) = 0.$$

Lemma 5[12] For $N = \left[\frac{1}{t}\right] + 1$ and $M = \left[\frac{1}{t^2}\right]$

$$\sum_{n=N}^{m}\frac{|Q_n(x,t)|}{n} \longrightarrow \infty \text{ as } t \longrightarrow 0+.$$

Lemma 6 The Integral $\int_0^1 \chi^1(x).\log \frac{1}{1-x} dx$ exists if and only if the integral

$$\int_0^1 \frac{\chi(1)-\chi(x)}{1-x} dx \quad \text{exists.}$$

Proof

$$\begin{aligned} &\int_0^1 \frac{\chi(1)-\chi(x)}{1-x} dx \\ &= \int_0^1 \frac{d}{dx}\left(\log \frac{1}{1-x}\right) \int_x^1 \chi^1(u) du \\ &= \int_0^1 \chi^1(u) \left\{ \int_0^u \frac{d}{dx}\left(\log \frac{1}{1-x}\right) dx \right\} du \\ &= \int_0^1 \chi^1(u) \log \frac{1}{1-u} du \end{aligned}$$

and hence the lemma follows.

Lemma 7 For $n > \frac{\pi}{t}$ and $0 < t < \pi$

(i) $\int_0^{1-\pi/nt} P_n(x,t)\chi^1(x)dx = O\left(\frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{nt}\right)$

(ii) $\int_{1-\frac{\pi}{nt}}^1 P_n(x,t)\chi^1(x)dx = O\left\{\chi(1)-\chi(1-\pi/nt)\right\}$

(iii) $\int_0^{1-\pi/nt} Q_n(x,t)\chi^1(x)dx = O\left\{\frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{nt}\right\}$

(iv) $\int_{1-\pi/nt}^1 Q_n(x,t).\chi^1(x)dx = O\left\{\chi(1)-\chi\left(1-\pi/nt\right)\right\}$

Proof of(i) We have by mean value theorem for some ζ with $0 < \zeta < 1 - \frac{\pi}{nt}$

$$\begin{aligned} &\int_0^{1-\pi/nt} P_n(x,t)\chi^1(x)dx = \chi^1\left(1-\frac{\pi}{nt}\right) \int_\xi^{1-\frac{\pi}{nt}} P_n(x,t)dx \\ &= \chi^1\left(1-\frac{\pi}{nt}\right)\left[L_n\left(1-\frac{\pi}{nt}, t\right) - L_n(\zeta, t)\right] \\ &= O\left(\frac{\chi^1(1-\pi/nt)}{nt}\right), \text{using Lemma 1.} \end{aligned}$$

Proof of (ii) Integrating by parts and then using mean value theorem we get

$$\int_{1-\pi/nt}^{1} P_n(x,t).\chi^1(x)dx = [-(\chi(1)-\chi(x))P_n(x,t)]_{x=1-\pi/nt}^{1}$$
$$+\int_{1-\pi/nt}^{1} -(\chi(1)-\chi(x))\frac{d}{dx}P_n(x,t)dx$$
$$= [-(\chi(1)-\chi(x))P_n(x,t)]_{x=1-\pi/nt}^{1} + (\chi(1)-\chi(1-\pi/nt))$$
$$\int_{1-\pi/nt}^{\xi} \frac{d}{dx}P_n(x,t)dx(1-\pi/nt<\xi<1)$$
$$= O(\chi(1)-\chi(1-\pi/nt)) + (\chi(1)-\chi(1-\pi/nt))$$
$$\left\{-P_n\left(1-\frac{\pi}{nt},t\right)+P_n(\xi,t)\right\}$$
$$= O\left(\chi(1)-\chi\left(1-\frac{\pi}{nt}\right)\right) \quad \text{using Lemma 1}$$

Proofs of (iii) and (iv) are respectively same as that of (i) and (ii).

Lemma 8 For $0<t<\pi$

$$\int_0^1 S_n(x,t).\chi^1(x)dx = O(n)$$

Proof We have

$$\int_0^1 S_n(x,t)\chi^1(x)dx = \int_0^1\left(\int_t^\pi \frac{R_n(x,v)}{v}dv\right)\chi^1(x)dx$$
$$= \int_0^1\left(\sum_{\nu=1}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}.\nu.\int_t^\pi\frac{\sin\nu v}{v}dv\right)\chi^1(x)dx$$
$$= O(1)\int_0^1\left(\sum_{\nu=1}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}.\nu.\int_t^\pi\frac{\sin\nu v}{v}dv\right)\chi^1(x)dx$$
$$= O(1)\int_0^1\left(\sum_{\nu=1}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}.\nu\right)\chi^1(x)dx$$
$$= O(\nu)\int_0^1 x|\chi^1(x)|dx, \quad \text{using Lemma 3}$$
$$= O(n).$$

Lemma 9 If $\chi(x)$ satisfies the hypothesis of Theorem 1 (on Theorem 2) then
(i) $\sum_{n>\frac{\pi}{t}} \frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{n^2} = O(t), 0<t<\pi$
(ii) $\sum_{n>\frac{\pi}{t}} \frac{1-\chi\left(1-\frac{\pi}{nt}\right)}{n} = O(1), 0<t<\pi$
Proof of (i) Since

$$\sum_{n>\frac{\pi}{t}} \frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{n^2} = \sum_{n>\pi/t} \frac{n+1}{n} \frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{n(n+1)}$$

and $\max\left(\frac{n+1}{n}\right) = 2$ for $n \geq 1$, we have

$$\begin{aligned}
&\sum_{n>\frac{\pi}{t}} \frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{n^2} \leq 2 \sum_{n>\pi/t} \frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{n(n+1)} \\
&= \frac{2t}{\pi} \sum_{n=\left[\frac{\pi}{t}\right]+1}^{\infty} \left(1-\frac{\pi}{nt}\right) \left\{\left(1-\frac{\pi}{(n+1)t}\right) - \left(1-\frac{\pi}{nt}\right)\right\} \\
&< \frac{2t}{\pi} \int_{\delta}^{1} \chi^1(t)dt, \text{where } \delta = \frac{\pi}{t\left(\left[\frac{\pi}{t}\right]+1\right)} \\
&= O(t).
\end{aligned}$$

Proof of (ii) We have

$$\begin{aligned}
\sum_{n>\frac{\pi}{t}} \frac{\chi(1)-\chi\left(1-\frac{\pi}{nt}\right)}{n} &\leq \int_{\frac{\pi}{t}}^{\infty} \frac{\chi(1)-\chi\left(1-\frac{\pi}{tx}\right)}{x} dx \\
&= \int_0^1 \frac{\chi(1)-\chi(y)}{1-y} dy.
\end{aligned}$$

which exists by hypothesis and Lemma 6.

5. Proof of Theorem 1

Integrating by parts we have

$$A_n(u) = -\frac{2}{\pi} \int_0^{\pi} \frac{\sin nt}{n} d\varphi(t), n \geq 1.$$

and using it with the definition of Absolute Hausdorff summability of Fourier series, now we obtain , by simplification

$$C_n = -\frac{2}{\pi}\int_0^{\pi} d\varphi(t)\int_0^1 Q_n(x,t).\chi^1(x)dx$$

and by Lemma 2 the series $\sum_{n=1}^{\infty}|C_n| < \infty$ if and only if

$$\sum = \sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^1 Q_n(x,t)\chi^1(x)dx\right| = O(1)$$

uniformly for $0 < t \leq \pi$.

Write

$$\begin{aligned}\sum &= \left(\sum_{n\leq\pi/t} + \sum_{n>\pi/t}\right)\frac{1}{n}\left|\int_0^1 Q_n(x,t)\chi^1(x)dx\right| \\ &= \sum_1 + \sum_2 \quad \text{(say)}\end{aligned}$$

Since $|\sin\nu t| \leq \nu t$ for all $\nu \geq 1$ and $0 < t < \pi$,

We have

$$|Q_n(x,t)| \leq nxt \quad \text{(by Lemma 3)}$$

Now by Lemma 3 and hypothesis

$$\begin{aligned}\sum_1{}' &\leq \sum_{n\leq\pi/t}\frac{1}{n}\int_0^1 Q_n(x,t)|\chi^1(x)|dx \\ &\leq t\sum_{n\leq\pi/t}\int_0^1 x|\chi^1(x)|dx \\ &= O(1)\end{aligned}$$

By Lemma 7 and Lemma 9

$$\begin{aligned}\sum_2 &= O(t^{-1})\sum_{n>\pi/t}^{1}\frac{\chi^1(1-\pi/nt)}{n^2} + O(1)\sum_{n>\pi/t}\frac{\chi(1)-\chi(1-\pi/nt)}{n} \\ &= O(1)\end{aligned}$$

Collecting the above results, we have for $0 < t < \pi$

$$\sum = \sum_1 + \sum_2 = O(1).$$

and this completes the proof of Theorem 1.

6. Proof of Theorem 2

By Lemma 4 (taking $\lambda = 1$) Theorem 2 is equivalent to Theorem 2a. Let conditions (i), (ii), (iii) and (iv) of Theorem 2 hold. Then

$$H(+0) = 0, \int_0^\pi t^{-1}|dH(t)| < \infty \Rightarrow \sum_{n=1}^\infty B_n(x) \in |H, \mu_n|$$

Proof of Theorem 2a Writing $H(t) = t\psi(t)$ and integrating by parts we get

$$\begin{aligned} B_n(u) &= \frac{2}{\pi}\int_0^\pi H(t)\frac{\sin\ nt}{t}dt \\ &= \int_0^\pi \left(\int_t^\pi \frac{\sin nv}{v}dv\right) dH(t), (n \geq 1) \end{aligned} \tag{6.1}$$

the integrated part vanishes as $H(+0) = 0$. From the definition the series $\sum_{n=1}^\infty B_n(u)$ is summable $|H, \mu_n|$ if

$$\sum{}' |D_n| < \infty$$

where

$$D_n = \frac{1}{n}\sum_{\nu=1}^n \binom{n}{\nu}(\Delta^{n-\nu}\mu_\nu)(\nu B_\nu(u)) \tag{6.2}$$

using (6.1) and (6.2) we get

$$\begin{aligned} D_n &= \frac{2}{\pi n}\int_0^\pi dH(t)\int_t^\pi \frac{dv}{v}\left\{\sum_{\nu=1}^n \binom{n}{\nu}\nu \sin \nu v \int_0^1 x^\nu(1-x)^{n-\nu}\chi^1(x)dx\right\} \\ &= \frac{2}{\pi n}\int_0^\pi dH(t)\int_0^1 S_n(x,t).\chi^1(x)dx. \end{aligned}$$

By Lemma 8, the integral $\frac{1}{n}\int_0^1 S_n(x,t).\chi^1(x)dx$ is finite for all $n \geq 1$ and hence by Lemma 2 for the convergence of the series $\sum_{n=1}^{\infty}|D_n|$, it is necessary and sufficient to show that

$$\sum^{*} = \sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^1 S_n(x,t)\chi^1(x)dx\right| = O(t^{-1}) \tag{6.3}$$

uniformly for $0 < t \leq \pi$.

We write

$$\sum^{*} = \left(\sum_{n<\frac{\pi}{t}} + \sum_{n>\pi/t}\right)\frac{1}{n}\left|\int_0^1 S_n(x,t)\chi^1(x)dx\right| = \sum_1^{*} + \sum_2^{*} \quad \text{(say)}$$

By Lemma 8 $\sum_1^{*} = \sum_{n\leq\pi/t}\frac{1}{n}\left|\int_0^1 S_n(x,t)\chi^1(x)dx\right| = O(t^{-1})$.

By mean value theorem for some ξ with $t < \xi < \pi$

$$\begin{aligned}
&\int_0^1 S_n(x,t)\chi^1(x)dx \\
=&\int_0^1\left(\sum_{\nu=1}^{n}\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}\int_t^{\pi}\frac{\sin\nu v}{v}dv\right)\chi^1(x)dx \\
=&t^{-1}\left\{\int_0^1 P_n(x,\xi)\chi^1(x)dx - \int_0^1 P_n(x,t)\chi^1(x)dx\right\} \\
=&t^{-1}\left\{I_n^{(1)} - I_n^{(2)}\right\} \quad \text{say}
\end{aligned} \tag{6.4}$$

using (i) and (ii) of Lemma 7, we get

$$I_n^{(2)} = O\left(\frac{\chi^1(1-\pi/nt)}{nt}\right) + O(\chi(1) - \chi(1-\pi/nt)) \tag{6.5}$$

Similarly

$$I_n^{(1)} = O\left(\frac{\chi^1(1-\pi/nt)}{n\xi}\right) + O\left(\chi(1) - \chi\left(1-\frac{\pi}{n\xi}\right)\right) \tag{6.6}$$

Now

$$\sum_2^* = \sum_{n>\pi/t} \frac{1}{n} \left| \int_0^1 S_n(x,t)\chi^1(x)dx \right|$$
$$= O(t^{-1}) \left\{ \sum_{n>\pi/t} \frac{|I_n(1)|}{n} + \sum_{n>\pi/t} \frac{|I_n(2)|}{n} \right\} \tag{6.7}$$

From (6.5)

$$\sum_{n>\pi/t} \frac{|I_n^{(2)}|}{n} = O(t^{-1}) \sum_{n>\pi/t} \frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{n^2} + O(1) \sum_{n>\pi/t} \frac{\chi(1)-\chi(1-\pi/nt)}{n}$$
$$= O(1)(\text{by Lemma 9})$$

Further

$$\sum_{n>\pi/t} \frac{|I_n^{(1)}|}{n} = O(1) \sum_{n>\pi/t} \frac{\chi^1\left(1-\frac{\pi}{n\xi}\right)}{n^2\xi} + O(1) \sum_{n>\pi/t} \frac{\chi(1)-\chi\left(1-\frac{\pi}{n\xi}\right)}{n}$$
$$= O(1)(\text{using Lemma 9}) \tag{6.9}$$

Collecting the results of (6.7), (6.8) and (6.9) we get

$$\sum_2^* = O(t^{-1})$$

$$\text{Hence } \sum^* = O(t^{-1}), 0 < t \leq \pi$$

and this completes the proof of Theorem 2a.

7. Proof of Theorem 3

As $\varphi(t) \in AC(0,\pi)$, it is clear from the proof of Theorem 1 that $\sum_{n=1}^{\infty} A_n(u)$ is summable $|H, \mu_n|$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^1 Q_n(x,t)\chi^1(x)dx \right| \tag{7.1}$$

is essentially bounded for $0 < t \leq \pi$.

The functions $\chi^1(x)$ and $Q_n(x,t)$ are Lebesgue integrable functions of x over

$(0,1)$ for each $n \geq 1$.

Hence by an appeal to Lemma 2 for the validity of (7.1) it is necessary that $\sum_{n=1}^{\infty} \frac{|Q_n(x,t)|}{n}$ should be essentially bounded for $0 \leq x \leq 1$ and $0 < t \leq \pi$.

Now for $N = \left[\frac{1}{T}\right] + 1$ and $M = [t^{-2}]$

$$\sum_{n=1}^{\infty} \frac{|Q_n(x,t)|}{n} \geq \sum_{n=N}^{M} \frac{\rho^n(t)|\sin n\Theta|}{n} \longrightarrow \infty \text{ as } t \to 0^+.$$

by an appeal to Lemma 5, for every x in $(0,1)$. This completes the proof of Theorem 3.

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