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Multi-Objective Chance-Constrained Programming Problems Involving Some Continuous Random Parameters

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Abstract

In real-life applications there are some situations where the Decision Maker (DM) wishes to optimize multiple, and conflicting objective functions. This is known as Multi-Objective Programming (MOP) Problem. Stochastic programming is a branch of mathematical programming that deals with some situations in which an optimal decision is desired under some random parameters. Chance Constrained Programming (CCP) technique is a very popular approach to solve Stochastic Programming (SP) problems. This paper presents some multi-objective CCP problems by considering all the right-hand side parameters of the constraints as random variables. It is assumed that the random variables follow some continuous distributions such as Two-parameter exponential distribution, and Three-parameter gamma distribution, and Power function distribution. In this paper, we first establish equivalent deterministic models for multi-objective CCP problems, and then we apply ε-constraint

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method, weighting method, and fuzzy programming method to solve the deterministic multi-objective problems. Numerical examples are included to illustrate the solution procedures of the models.

Keywords: Stochastic programming, Chance Constrained Programming, Two parameter exponential distribution, Three parameter gamma distribution, Power function distribution

1 Introduction

In many concrete real life decision-making problems, a decision maker has to deal with the problems having multiple, conflicting and non-commensurable objectives. This has given rise to the field of Multi-Objective Programming (MOP). In case of multiple conflicting objectives, there does not exist any solution point which is optimal with respect to all objectives. So, we have to go for a compromise solution. In a typical multi-objective optimization problem, there exists a set of solutions which are superior to the rest of solutions in the search space when all objectives are considered but are inferior to other solutions in the space with one or more objectives. These solutions are known as pareto-optimal solutions or non-dominated solutions. The rest of the solutions are known as dominated solutions.

The above mentioned multi-objective optimization problem may involve some level of uncertainty about the values to be assigned to the various parameters. If the uncertainty in the problem is random in nature, then the problem is called Multi-Objective Stochastic Programming (MOSP) problem. The mathematical model of a MOSP with K stochastic objective and m stochastic constraints is given by:

$$\max \quad \mathbf{C}(\omega).\mathbf{x} \tag{1.1}$$

$$subject\ to \tag{1.2}$$

$$\mathbf{A}(\omega).\mathbf{x} \leq \mathbf{b}(\omega) \tag{1.3}$$

$$\mathbf{x} \in \mathbb{R} \tag{1.4}$$

where **C** is a random (K,n) matrix, **A** is a random (m,n) matrix and **b** is a random n-column vector defined on some probability space.

Stochastic programming (SP) is concerned with the decision making problems in which some or all parameters are treated as random variables in order to capture the uncertainty. SP is used in several real world decision making areas such as energy management, financial modelling, supply chain and scheduling, hydro thermal power production planning, transportation, agriculture, defence, environmental and pollution control, production and control management, telecommunications, etc. Several models and methodologies have been developed in the field of stochastic programming. In the literature, there exist two very popular approaches to solve SP problems, namely,
(i) Chance constrained programming, and
(ii) Two-stage programming.

Chance constrained programming was developed as a means of describing constraints in mathematical programming models in the form of probability levels of attainment. The chance constrained programming (CCP) can be used to solve problems involving chance-constraints, i.e.constraints having violation up to a pre-specified probability level. The use of chance-constraints was initially introduced by Charnes and Cooper [4].

Similarly, the two-stage programming technique was suggested by Dantzig [29] to solve the stochastic programming problem. This technique also converts the stochastic problem into an equivalent deterministic problem. Unlike the chance constrained programming, the two stage programming does not allow any constraint to be violated.

In the next Section, we have presented some literature available on multi-objective stochastic programming. In rest of the paper, we discussed about our model and methodology.

2 Literature Survey

In optimization, decision making problems under random uncertainty are modelled by using stochastic programming approach. In the literature of the stochastic linear programming [1], [2], [5], various models have been suggested by several researchers. Bibliographical review on this topic is covered by Stancu and Wets [3] and Infanger [5].

In recent past, a wide range of studies have been done in the field of multi-objective stochastic programming. There are several research articles in the literature dealing with such problems, among which we could mention the books by Goicoechea et al.[6], Stancu-Minasian [7], Slowinski and Teghem [8]. Adeyefa and Luhandjula [9] presented an up-to-date overview of how important ideas from optimization, probability theory and multi-criteria decision analysis are interwoven to address situations where several objective functions and the stochastic nature of data are considered in a linear optimization context. Hulsurkar et al.[10] presented fuzzy programming approach to solve the multi-objective stochastic linear programming (MOSLP) problem. Later, Sinha et al.[11] constructed the deterministic model of the MOSLP problem with joint probabilistic constraint having right hand side parameters as independent normal random variables. Sahoo and Biswal [12] presented a MOSLP model with joint probabilistic constraints. They established the deterministic equivalents of the model by considering the random variables as normal and log-normal random variables. Charles et al.[13] presented the equivalent deterministic form of the MOSLP with probabilistic constraints having different types of distributions like Pareto distribution, Beta distribution of first kind, Weibull distribution and Burr type XII distribution. Later, Abdelaziz [14] presented a survey of various solution approaches for multi-objective stochastic programming problems where random variables can be in both objectives and constraints parameters.

Franca et al.[15] introduced a multi-objective stochastic supply chain model to evaluate trade-offs between profit and quality. Felfel et al. [16] proposed a multi-objective two-stage stochastic programming model for a multi-site supply chain planning problem under demand uncertainty. Their proposed multi-objective model aimed simultaneously to minimize the expected total cost, to minimize the worst-case cost, and to maximize the customer demand satisfaction level. Recently, Barik et al. [17] studied stochastic programming problems involving Pareto distribution. In their paper they considered both single-objective and multi-objective stochastic programming problems. They used both chance constrained programming and two-stage stochastic programming to solve these problems.

In the literature, there is no article on the multi-objective stochastic programming problem where some parameters follow either Two-parameter exponential distribution, Three-parameter gamma distribution, and Power function distribution. So, in this paper, we propose a solution procedure of a multi-objective stochastic programming problem where the right hand side parameters follow either triangular distribution or trapezoidal distribution with known parameters.

3 Multi-Objective Chance Constrained Linear Programming Problems

In this Section, multi-objective Linear CCP problems involving Two-parameter exponential distribution, Three-parameter gamma distribution, Power function distribution are considered. Further, we establish equivalent deterministic models for multi-objective CCP problems, then we apply ε-constraint method, weighting method, and fuzzy programming method to solve the problems. The mathematical model of a multi-objective chance constrained programming problem can be stated as:

$$\max : z_k = \sum_{j=1}^{n} c_{kj}x_j, \quad k = 1,2,...,K \tag{3.5}$$

subject to

$$Pr(\sum_{j=1}^{n} a_{ij}x_j \leq b_i) \geq (1-\gamma_i), \quad i = 1,2,...,m \tag{3.6}$$

$$0 < \gamma_i < 1, \qquad i = 1,2...,m \tag{3.7}$$

$$x_j \geq 0, \qquad j = 1,2...,n \tag{3.8}$$

where a_{ij}, $(i = 1,2,...,m;\ j = 1,2,...,n)$, are the constraint coefficients, c_{kj}, $(j = 1,2,...,n);(k = 1,2,...,K)$ are the coefficients associated with the k-th objective function. In the model, only the right hand side parameters $b_i,(i = 1,2,...,m)$ are considered as random variables which follow different continuous distributions with known mean and distribution.

3.1 Equivalent Deterministic Models of Multi-Objective Chance Constrained Programming Problems

Case-I: b_i follows Two-parameter exponential distribution

In this case, we assume that b_i, $(i = 1,2,...m)$ in the model (3.5)-(3.8) are independent random variables following two-parameter exponential distribution ([19]) with parameters θ_i,σ_i where mean and variance of random variable b_i are given by:

$$E(b_i) = \theta_i + \sigma_i, i = 1,2,..,m \tag{3.9}$$

$$Var(b_i) = \sigma_i^2,\ \ i = 1,2,...,m \tag{3.10}$$

The probability density function of the i-th two-parameter exponential variable b_i is given by

$$f(b_i) = \frac{1}{\sigma_i}\exp\Big(\frac{-(b_i-\theta_i)}{\sigma_i}\Big), i = 1,2,..,m \tag{3.11}$$

where $b_i \geq \theta_i$, $\sigma_i > 0$.

To solve the problem (3.5)-(3.8), we establish the deterministic form of the problem. Then from the chance-constraint (3.6), we have

$$\begin{aligned}
& Pr(\sum_{j=1}^{n} a_{ij}x_j \leq b_i) \geq (1-\gamma_i) \\
& \Rightarrow Pr(b_i \geq \sum_{j=1}^{n} a_{ij}x_j) \geq (1-\gamma_i) \\
& \Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} f(b_i)db_i \geq (1-\gamma_i) \\
& \Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} \frac{1}{\sigma_i}\exp\Big(\frac{-(b_i-\theta_i)}{\sigma_i}\Big)db_i \geq (1-\gamma_i)
\end{aligned}$$

Integrating, we obtain

$$\sum_{j=1}^{n} a_{ij}x_j \leq \theta_i - \sigma_i ln(1-\gamma_i) \tag{3.12}$$

Using the above result in (3.6), we obtain equivalent deterministic model of (3.5)-(3.8) as follows:

$$\max : z_k = \sum_{j=1}^{n} c_{kj}x_j, \quad k = 1,2,...,K \tag{3.13}$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq \theta_i - \sigma_i ln(1-\gamma_i) \tag{3.14}$$

$$0 < \gamma_i < 1, \qquad i = 1,2...,m \tag{3.15}$$

$$x_j \geq 0, \qquad j = 1,2...,n \tag{3.16}$$

The above model is a Multi-Objective Linear Programming (MOLP) model. Using a suitable technique for MOLP technique, the model can be solved.

Case-II: b_i follows Three-parameter gamma distribution

In this case, we assume that b_i, $(i = 1,2,...m)$ in the model (3.5)-(3.8) are independent random variables following three-parameter gamma distribution ([20]) with parameters α_i, β_i and θ_i where mean and variance of random variable b_i are given by:

$$E(b_i) = \alpha_i\beta_i + \theta_i, i = 1,2,..,m \tag{3.17}$$

$$Var(b_i) = \alpha_i\beta_i^2, \quad i = 1,2,...,m \tag{3.18}$$

The probability density function (pdf) of the i-th three-parameter gamma variable b_i is given by

$$f(b_i) = \frac{1}{\Gamma(\alpha_i)\beta_i^{\alpha_i}}(b_i-\theta_i)^{\alpha_i-1}\exp\left(\frac{-(b_i-\theta_i)}{\beta_i}\right), i = 1,2,....m \tag{3.19}$$

where $\alpha_i > 0, \beta_i > 0$ and $b_i \geq \theta_i$. It is further assumed that α_i is a positive integer.

To solve the CCP problem (3.5)-(3.8), we establish the deterministic model of the problem. In this case, right hand side parameters of the chance-constraints follow three-parameter gamma distribution. Then from the chance-constraint (3.6), we have

$$Pr(\sum_{j=1}^{n} a_{ij}x_j \leq b_i) \geq (1-\gamma_i)$$

$$\Rightarrow Pr(b_i \geq \sum_{j=1}^{n} a_{ij}x_j) \geq (1-\gamma_i)$$

$$\Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} f(b_i)db_i \geq (1-\gamma_i)$$

$$\Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} \frac{1}{\Gamma(\alpha_i)\beta_i^{\alpha_i}}(b_i-\theta_i)^{\alpha_i-1}\exp\Big(\frac{-(b_i-\theta_i)}{\beta_i}\Big)db_i \geq (1-\gamma_i)$$

Integrating, we obtain

$$\exp\left(\frac{-(\sum_{j=1}^{n} a_{ij}x_j-\theta_i)}{\beta_i}\right)\left(\sum_{k=0}^{\alpha_i-1}\frac{1}{k!}\Big(\frac{(\sum_{j=1}^{n} a_{ij}x_j-\theta_i)}{\beta_i}\Big)^k\right) \geq (1-\gamma_i) \tag{3.20}$$

Using the above result in (3.6), we obtain equivalent deterministic model of (3.5)-(3.8) as follows:

$$\max: z_k = \sum_{j=1}^{n} c_{kj}x_j, \quad k=1,2,...,K \tag{3.21}$$

subject to

$$\exp\left(\frac{-(\sum_{j=1}^{n} a_{ij}x_j-\theta_i)}{\beta_i}\right)\left(\sum_{r=0}^{\alpha_i-1}\frac{1}{r!}\Big(\frac{(\sum_{j=1}^{n} a_{ij}x_j-\theta_i)}{\beta_i}\Big)^r\right) \geq (1-\gamma_i) \tag{3.22}$$

$$0<\gamma_i<1, \qquad i=1,2...,m \tag{3.23}$$

$$x_j \geq 0, \qquad j=1,2...,n \tag{3.24}$$

Case-III: b_i follows Power function distribution

Power function distribution ([18]) is a very popular random variable used to estimate the reliability and hazard rates of a electrical component. Here we consider, b_i, $(i=1,2,...m)$ in the model (3.5)-(3.8) are independent random variables following power function distribution. The probability density function (pdf) of the i-th random variable b_i, $(i=1,2,...,m)$ is given by:

$$f(b_i)=\begin{cases}\frac{\beta_i b_i^{\beta_i-1}}{\alpha_i^{\beta_i}}, & \text{if} \quad 0<b_i<\alpha_i, \beta_i>0\\ 0, & \text{otherwise}\end{cases} \tag{3.25}$$

where α_i, $(i = 1,2,...,m)$, and β_i,, $(i = 1,2,...,m)$ are known as positive scale parameter α_i and positive shape parameter β_i respectively. The mean and variance of $b_i\,(i = 1,2,...,m)$ are given by:

$$E(b_i) = \frac{\alpha_i\beta_i}{1+\beta_i}, \qquad i = 1,2,..,m \tag{3.26}$$

$$Var(b_i) = \frac{\alpha_i^2\beta_i}{(2+\beta_i)(1+\beta_i)^2}, \quad i = 1,2,...,m, \tag{3.27}$$

respectively.

To solve the problem (3.5)-(3.8), we establish the deterministic form of the problem. Then from the chance-constraint (3.6), we have

$$Pr(\sum_{j=1}^{n} a_{ij}x_j \leq b_i) \geq (1-\gamma_i)$$

$$\Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\alpha_i} f(b_i)db_i \geq (1-\gamma_i)$$

$$\Rightarrow \int_{\sum_{j=1}^{n} a_{ij}x_j}^{\alpha_i} \frac{\beta_i b_i^{\beta_i-1}}{\alpha_i^{\beta_i}} db_i \geq (1-\gamma_i)$$

Integrating, we obtain

$$\sum_{j=1}^{n} a_{ij}x_j \leq \alpha_i\gamma_i^{\frac{1}{\beta_i}}, \quad i = 1,2,...,m \tag{3.28}$$

Hence, the equivalent deterministic form of the model (3.5)-(3.8) is given by:

$$\max : z_k = \sum_{j=1}^{n} c_{kj}x_j, \quad k = 1,2,...,K \tag{3.29}$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq \alpha_i\gamma_i^{\frac{1}{\beta_i}}, \quad i = 1,2,...,m \tag{3.30}$$

$$0 < \gamma_i < 1, \quad i = 1,2...,m \tag{3.31}$$

$$x_j \geq 0, \qquad j = 1,2...,n \tag{3.32}$$

The above model is a Multi-Objective Linear Programming (MOLP) model. Using a suitable technique for MOLP technique, the model can be solved.

In the next section, we will discuss some of the popular techniques used to solve Multi-Objective Programming (MOP) problems.

4 Solution Methodology for MOP Problem

MOP problems arise in every branch of science, engineering, and social science. Due to the conflicting nature of the criteria, a unique feasible solution optimizing all the requirements does not exist. Two different efficient solutions are characterized by the fact that each of them is better in one criterion but worse in another. The primary goal of MOP is to seek suitable solutions and Pareto outcomes of multi-objective programs and, if possible, support the Decision-Maker (DM) in choosing a final preferred solution. Therefore, it is of interest to design methods for obtaining a complete or partial description of the Pareto set and efficient set referred to as the solution sets.

Several methods have been developed for solving MOP problems. Some of the widely used efficient methods are discussed in the following Subsections.

4.1 ε-Constraint Method

The ε-constraint method was developed by Haimes [22]. It is used to generate the Pareto optimal solutions for MOP problems. It It makes use of a single-objective optimizer which handles constraints, to generate one point of the Pareto front at a time. For transforming the MOP problem into several single-objective problems with constraints it uses the following procedure:

Step-1: Optimize one of the objective functions (i.e. $z_k(X), k = 1$) considering the other objective functions as constraints, incorporating them in the constraint part of the model as shown below:

$$\max : z_1(x) = \sum_{j=1}^{n} c_j^1 x_j \tag{4.33}$$

subject to

$$z_k(x) = \sum_{j=1}^{n} c_j^k x_j \geq \varepsilon_k, (k = 2, \ldots, K), k \neq 1 \tag{4.34}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, (i = 1, 2, \ldots, m) \tag{4.35}$$

$$x_j \geq 0, (j = 1, 2, \ldots, n) \quad (4.36)$$

where $\varepsilon_k \in R$ is the minimum tolerable objective level.The value of ε_k is chosen for which the formulated objective constraints in the above model are binding at the optimal solution.

Step-2: Solve the remaining $(K-1)$ number of objective functions w. r. t. the constraints separately and find the ranges of each objective functions.

Step-3: Using the ranges of each objective function, the value of the ε_k, $(k = 2, 3, \ldots, K)$ are assigned by the decision maker to find the Pareto optimal solution of the formulated single objective mathematical programming model.

Step-4: Continue the process with the remaining objective functions for finding the suitable Pareto optimal solution.

4.2 Weighting Method

The most widely used method for solving MOP problems is the weighting method. It has been proposed by Zadeh [21]. This method transforms multiple objective functions into a single objective function, by multiplying each objective function by a weighting factor, and summing up all weighted objective functions. This method generates non-dominated solutions by parametrically varying weights. It is assumed that the objective functions are measured in the same unit. If the objective functions are not in the same unit, it can be transformed into the same unit before applying the method.

Mathematically, the weighting method can be stated as follows:

$$\max : z = \sum_{k=1}^{K} w_k z_k(x) \quad (4.37)$$

subject to

$$\sum_{k=1}^{K} w_k = 1, w_k \geq 0, (k = 1, 2, 3, \ldots, K) \quad (4.38)$$

$$x \in S \quad (4.39)$$

where S is the feasible region of the multi-objective optimization problem. The coefficient w_k operating on the k-th objective function z_k, is called a weight and can be interpreted as the relative weight of that objective function when compared to the other objective functions.

4.3 Fuzzy Programming Method

Fuzzy Programming approach is an important tool for solving Multi-Objective Programming (MOP) Problems. This technique gives compromise solution of the MOP problem. Fuzzy set theory was first introduced by Zadeh [23]. Later on, Zimmermann [24] used a fuzzy set theory concept with a suitable choice of membership function and derived a fuzzy linear program, which is identical to the present-day max-min problem. Steps of the fuzzy programming technique are as follows:

Step-1: Select the first objective function (i.e. $z_k, k = 1$) and solve it as a single objective optimization problem subject to the given constraints. Let $x^{(1)}$ be the ideal solution. Then select the second objective function and find the ideal solution as $x^{(2)}$, continue the process K number of times for K different objective functions. Let $x^{(1)}, x^{(2)}, \ldots, x^{(K)}$ be the ideal solutions for the objective functions z_1, z_2, …, z_k respectively.

Step-2: Evaluate all these objective functions at all these ideal solutions and formulate a pay-off matrix (Table 1) of size $K \times K$ as follows.

Table 1: Pay-Off Matrix

	$z_1(x)$	$z_2(x)$	$\ldots$	$z_K(x)$
$x^{(1)}$	z_{11}	z_{12}	$\cdots$	z_{1K}
$x^{(2)}$	z_{21}	z_{22}	$\cdots$	z_{2K}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x^{(K)}$	z_{K1}	z_{K2}	$\cdots$	z_{KK}

Step-3: From the pay-off matrix (Table 1) determine the bounds for k-th objective function $z_k(x)$, ($k = 1, 2, \ldots, K$). If an objective function is of maximization type find the best upper bound U_k^* and worst lower bound l_k^-. If an objective function is of minimization type find the best lower bound l_k^* and worst upper bound u_k^-, $(k = 1, 2, \ldots, K)$.

Step-4: Associate a linear membership function $\mu_{z_k}(x)$ to the k-th objective function $z_k(x)$ as:

$$\mu_{z_k}(x) = \begin{cases} 1, & if\ z_k(x) \geq u_k^* \\ \frac{z_k(x) - l_k^-}{u_k^* - l_k^-}, & if\ l_k^- < z_k(x) < u_k^*, \\ 0, & if\ z_k(x) \leq l_k^- \end{cases} \quad (k = 1, 2, 3, \ldots, K) \tag{4.40}$$

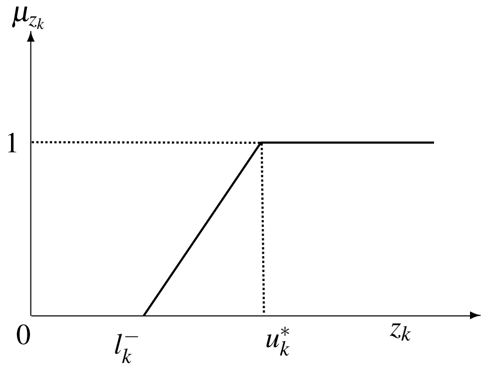

Figure 1: Membership Function of a Vector Maximization Problem

$$\mu_{z_k}(x) = \begin{cases} 1, & if\ z_k(x) \leq l_k^* \\ \frac{u_k^- - z_k(x)}{u_k^- - l_k^*}, & if\ l_k^* < z_k(x) < u_k^-, \\ 0, & if\ z_k(x) \geq u_k^- \end{cases} \quad (k = 1, 2, 3, \ldots, K) \tag{4.41}$$

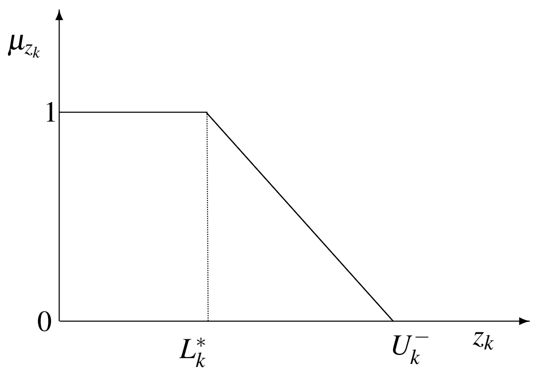

Figure 2: Membership Function of a Vector Minimization Problem

Step-5: **(a)** Use max-min operator with an augmented variable λ and formulate a single objective crisp optimization problem as:

$$\max : \lambda \tag{4.42}$$

subject to

$$\lambda \leq \mu_{z_k}(x),\ (k = 1,2,3,\ldots,K) \tag{4.43}$$

$$x \in S \tag{4.44}$$

where S is the feasible region of the optimization model.

(b) Similarly, if we use min-max operator with an augmented variable λ, a single objective crisp optimization problem can be formulated as:

$$\min : \lambda \tag{4.45}$$

subject to

$$\lambda \geq \mu_{z_k}(x),\ (k = 1,2,3,\ldots,K) \tag{4.46}$$

$$x \in S \tag{4.47}$$

where S is the feasible region of the optimization model.

Step-6: Solve the crisp model by using a suitable mathematical programming technique to find an optimal compromise solution x^*. Then evaluate all the objective functions at the optimal compromise solution x^*.

If the weights of multiple objective functions are interpreted as the relative preference of some DM, then the solution is equivalent to the best compromise solution i.e. the optimal solution relative to a particular preference structure. Also the optimal solution to the problem is a non-dominated solution provided all the weights are positive.

5 Numerical Examples

Here, we consider a Multi-objective CCP problem where the right hand side parameters of the constraints follow two-parameter exponential distribution. We first obain the deterministic equivalent of the problem, then we solve the obtained deterministic Multi-objective CCP problem using ε-constraint method, weighting method, and fuzzy programming method.

Example-1: Solution by ε-constrained Method:

$$max : z_1 = 5x_1 + 8x_2 + 7x_3 \tag{5.48}$$

$$max : z_2 = 2x_1 + 3x_2 + x_3 \tag{5.49}$$

subject to

$$Pr(2x_1 + 6x_2 + 5x_3 \leq b_1) \geq 0.99 \tag{5.50}$$

$$Pr(5x_1 + 11x_2 + 4x_3 \leq b_2) \geq 0.95 \tag{5.51}$$

$$Pr(4x_1 + 5x_2 + x_3 \leq b_3) \geq 0.90 \tag{5.52}$$

$$x_j \geq 0, \quad j = 1,2,3 \tag{5.53}$$

Here, we assume that $b_i\,(i = 1,2,3)$ are random variables following two parameter exponential distribution with following parameters:

$E(b_1) = 161, E(b_2) = 144, E(b_3) = 106$ and

$Var(b_1) = 25, Var(b_2) = 36, Var(b_3) = 64.$

Using (3.9) and (3.10), the parameters are calculated as follows:

$\theta_1 = 156, \sigma_1 = 5,\ \theta_2 = 138, \sigma_2 = 6,\ \theta_3 = 98$ and $\sigma_3 = 8.$

Now, using (3.13)-(3.16), the equivalent deterministic model of (5.48)-(5.53) can be formulated as follows:

$$max : z_1 = 5x_1 + 8x_2 + 7x_3 \tag{5.54}$$

$$max : z_2 = 2x_1 + 3x_2 + x_3 \tag{5.55}$$

subject to

$$2x_1 + 6x_2 + 5x_3 \leq 156.05 \tag{5.56}$$

$$5x_1 + 11x_2 + 4x_3 \leq 138.308 \tag{5.57}$$

$$4x_1 + 5x_2 + x_3 \leq 98.4103 \tag{5.58}$$

$$x_j \geq 0, \quad j = 1, 2, 3 \tag{5.59}$$

To obtain the Pareto optimal solution, we have applied ε-constrained method. Thus, ideal solutions are obtained as

$$\max \quad z_1 = 227.1846, \qquad x_1 = 3.961176, x_2 = 0, x_3 = 29.62553$$

$$\max \quad z_2 = 51.98608 \qquad x_1 = 23.2121, x_2 = 0, x_3 = 5.561864$$

Using the ideal solutions we can find the lower bound L_i and upper bound U_i of the function $Z_i, i = 1, 2$. $L_1 = 154.993584 < z_1 < 227.1864 = U_1$ and $L_2 = 37.547882 < z_2 < 51.98608 = U_2$, i.e, $154.993584 < \varepsilon_1 < 227.1864$ and $37.547882 < \varepsilon_2 < 51.98608$. Considering ε_1 and ε_2 as defined in the intervals, two different LP problems are formulated as follows:

$$\text{(I)} \quad \max \quad z_1 = 5x_1 + 8x_2 + 7x_3,$$

subject to

$$\begin{aligned} 2x_1+3x_2+x_3 &\geq \varepsilon_2, \\ 2x_1+6x_2+5x_3 &\leq 156.05, \\ 5x_1+11x_2+4x_3 &\leq 138.308, \\ 4x_1+5x_2+x_3 &\leq 98.4103, \\ x_1,x_2,x_3 &\geq 0. \end{aligned}$$

$$\text{(II)} \quad \max \quad z_1 = 2x_1+3x_2+x_3,$$

subject to

$$\begin{aligned} 5x_1+8x_2+7x_3 &\geq \varepsilon_1, \\ 2x_1+6x_2+5x_3 &\leq 156.05 \\ 5x_1+11x_2+4x_3 &\leq 138.308 \\ 4x_1+5x_2+x_3 &\leq 98.4103 \\ x_1,x_2,x_3 &\geq 0. \end{aligned}$$

We solved the problem by using LINGO 11.0, and the obtained solution is presented in the Table 2, and Table 3

Example-2:Solution by Weighting method:

Now, we solve the deterministic MO problem (5.54)-(5.59) using weighted sum method. Thus, the problem becomes,

$$\max \quad z^* = w_1(5x_1+8x_2+7x_3)+w_2(2x_1+3x_2+x_3),$$

Table 2: Pareto-optimal solution of problem (I)

x_1	x_2	x_3	ε_2	Z_1
3.961176	0	29.62553	37.5	227.1864
7.230667	0	25.53867	40	214.9240
9.98733	0	22.20533	42	204.9240
13.59067	**0**	**17.58867**	**44.77**	**191.0740**
17.89733	0	12.20533	48	174.9240
23.2040	0	5.5720	51.98	155.0240

Table 3: Pareto-optimal solution of problem (II)

x_1	x_2	x_3	ε_1	Z_2
23.21175	0	5.561106	154.99	51.98592
21.87707	0	7.230667	160	50.9848
16.54373	0	13.89733	180	46.9848
13.61040	**0**	**17.5640**	**191**	**44.7848**
8.543733	0	23.89733	210	40.9848
3.9624	0	29.624	227.18	37.5488

subject to

$$
\begin{aligned}
2x_1 + 6x_2 + 5x_3 &\leq 156.05 \\
5x_1 + 11x_2 + 4x_3 &\leq 138.308 \\
4x_1 + 5x_2 + x_3 &\leq 98.4103 \\
w_1 + w_2 &= 1 \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

We solved the problem by using Lingo software [30] and present the solution in the Table 4,

Example-3: Solution by Fuzzy Programming Method:

We solve the detereministic MO problem (5.54)-(5.59)using Fuzzy Programming Method.

Table 4: Pareto-optimal solution

w_1	w_2	z_1	z_2	Decision variable	z^*
0.1	0.9	154.9936	51.98608	$x_1 = 23.2121, x_2 = 0, x_3 = 5.561864$	62.28683
0.2	0.8	227.1864	37.54788	$x_1 = 3.961176, x_2 = 0, x_3 = 29.62553$	113.4026
0.5	0.5	227.1864	37.54788	$x_1 = 3.961176, x_2 = 0, x_3 = 29.62553$	132.3662
0.6	0.4	227.1864	37.54788	$x_1 = 3.961176, x_2 = 0, x_3 = 29.62553$	151.3299
0.9	0.1	227.1864	37.54788	$x_1 = 3.961176, x_2 = 0, x_3 = 29.62553$	208.2203

Here, ideal solution of the problem is obtained as:

$$\max \quad z_1 = 227.1846, \qquad x_1 = 3.961176, x_2 = 0, x_3 = 29.62553$$

$$\max \quad z_2 = 51.98608 \qquad x_1 = 23.2121, x_2 = 0, x_3 = 5.561864$$

Evaluating all these objective functions at all a pay-off matrix is formulated as: using Fuzzy linear

Table 5: Pay-off Matrix

	$z_1(x)$	$z_2(x)$
$x^{(1)}$	227.1846	37.547882
$x^{(2)}$	154.993584	51.98608

membership functions and max-min operator, we have

$$\max : \lambda \tag{5.60}$$

subject to

$$5x_1+8x_2+7x_3+(227.1864-154.993504)\lambda \geq 227.1864 \quad (5.61)$$

$$2x_1+3x_2+x_3+(51.98608-37.547882)\lambda \geq 51.98608 \quad (5.62)$$

$$2x_1+6x_2+5x_3 \leq 156.05 \quad (5.63)$$

$$5x_1+11x_2+4x_3 \leq 138.308 \quad (5.64)$$

$$4x_1+5x_2+x_3 \leq 98.4103 \quad (5.65)$$

$$x_j \geq 0, \ , \quad j=1,2,3, \ \lambda > 0 \quad (5.66)$$

The above model is solved using LINGO(11.0), and the compromise solution is obtained as follows: $\lambda = 0.50000062$, $x_1 = 13.5865, x_2 = 0, x_3 = 17.59385$, $z_1 = 191.0895, z_2 = 44.76668$

6 Conclusions

In this paper, we consider a multi-objective stochastic linear programming problem where some chance-constraints are involved. In these chance-constraints, the right hand side parameters are considered as random variables. By considering the fact that the random variables follows Two-parameter exponential distribution, Three-parameter gamma distribution, and Power function distribution with known parameters, we establish the equivalent deterministic form of the chance-constraints. Fuzzy programming technique is used to solve the multi-objective deterministic models. It will be interesting to study the chance-constraint problem with technological coefficients and cost coefficients as Two-parameter exponential distribution, Three-parameter gamma distribution, and Power function distribution. We can apply the result of this study in portfolio optimization. The study can be extended for nonlinear chance-constrained problem and in hierarchical decision making framework

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