Journal of Orissa Mathematical Society ISSN: 0975-2323
Vol. 38, No. 01-02, 2019, 85-120

Variational Inequality and Complementarity Problem

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Abstract

Variational inequality and Complementarity have much in common, but there has been little direct contact between the researchers of these two related fields of mathematical sciences. Several problems arising from Fluid Mechanics, Solid Mechanics, Structural Engineering, Mathematical Physics, Geometry, Mathematical Programming etc. have the formulation of a Variational Inequality or Complementarity Problem. People working in applied mathematics mostly deal with infinite dimensional case and they deal with Variational inequality whereas people working in operations research mostly deal with finite dimensional problem and they use complementarity problem. Variational inequality is a formulation for solving the problem where we have to optimize a functional. The theory is derived by using the techniques of nonlinear functional analysis such as fixed point theory and theory of monotone operators etc.

In this paper we give a brief review of the subject. This paper is divided into four sections. Section 1 deals with nonlinear operators which are required to describe the results. Sections 2, 3, 4 and 5 deal with Variational inequality, Equilibrium Problem and Complementarity

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Problem. Section 6 describes semi-inner-product spaces and Variational Inequality in semi-inner-product spaces.

Keywords:Generalized invexity, Minimax fractional programming, optimal solutions, Parametric sufficient optimality conditions.

2010 AMS Subject Classification: 90C30, 90C32, 90C34

1 Nonlinear Operators

In this section we discuss certain nonlinear operators, which are useful in the study of variational inequalities and complementarity problem.

Let X be a real normed linear space and let X^* be the dual space of X. Let the pairing between $x \in X$ and $x^* \in X^*$ be denoted by (x^*, x). Let T be a of a map from a subset $D(T)$ of X into X^*. T is said to be **monotone** if

$$(Tx - Ty, x - y) \geq 0 \text{ for all } x, y \in D(T),$$

and **strictly monotone** if T is monotone and strict inequality holds whenever $x \neq y$. T is α**-monotone** if there is a continuous strictly increasing function $\alpha : [0,1) \to [0,1]$ with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$ such that

$$(Tx - Ty, x - y) \geq \|x - y\| \, \alpha\left(\|x - y\|\right)$$

for all $x, y \in D(T)$. T is **strongly monotone** if $\alpha(r) = cr$ for some $c > 0$. T is **coercive** on subset K of $D(T)$ if there exists a function $c : (0, \infty) \to [-\infty, \infty]$ with $c(r) \to \infty$ as $r \to \infty$ such that

$$(Tx, x) \geq \|x\| \, c\left(\|x\|\right) \text{ for all } x \in K.$$

Thus T is coercive on K if K is bounded, while T is coercive on an unbounded K if and only if

$$\frac{(Tx, x)}{\|x\|} \to \infty \text{ as } \|x\| \to \infty, x \in K.$$

T is hemicontinuous if $D(T)$ is convex for any $x, y \in D(T)$, the map $t \to T(tx + (1-t)y)$ of $[0,1]$ to X^* is continuous for the natural topology of $[0,1]$ and the weak topology of X^*.

Examples:

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Then f is a monotone operator.

(b) Let H be a Hilbert space and $T : H \to H$ be a compact self-adjoint linear operator. Then T is monotone operator if all the eigen-values of T are non-negative.

(c) Let H be a Hilbert space. An operator $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in H.$$

If T is nonexpansive, then $I - T$ is a monotone operator.

(d) Let H be a Hilbert space and C a closed convex subset of H. Let P_x denote the point of minimum distance of C from x, that is,

$$P_x = \left\{ z \in C : \|z - x\| = \inf_{y \in C} \|y - x\| \right\}.$$

Then P is a monotone operator on H.

(e) Let H be a Hilbert space. Then an operator $T : H \to H$ is said to be *accretive* if

$$\|x - y\| \leq \|Tx - Ty\| \quad \text{for all } x, y \in H.$$

Then $T : H \to H$ is monotone if $I + \lambda T$ is accretive for every $\lambda > 0$.

Theorem 1.1. *If $T : D(T) \subset X \to X^*$ is α-monotone, then it is strictly monotone (hence monotone) and coercive in particular every strongly monotone operator is strictly monotone and coercive.*

Let X be nls and let X^* be its dual. A map $T : X \to X^*$ is said to a duality map if for any $x \in X$,

(i) $(Tx, x) = \|Tx\| \; \|x\|$, and

(ii) $\|Tx\| = \|x\|$.

A duality map can be constructed in any nls in the following way : By Hahn-Banach theorem, for any $x \in X$, there exists at least one bounded linear functional $y_x \in X^*$ such that $y_x = 1$ and $(y_x, x) = \|x\|$. Taking one such functional y_x and setting $T_x = \|x\| \, y_x$ and $T(-x) = -\|x\| \, y_x$, we get $\|T_x\| = \|x\|$ and $(T_x, x) = \|T_x\| \; \|x\|$.

Theorem 1.2. *In general a duality map $T : X \to X^*$ is multivalued. It is single-valued if X^* is strictly convex.*

Theorem 1.3. *If $T : X \to X^*$ is a duality map, then it is monotone and coercive. If further X is strictly convex, then T is strictly monotone.*

Theorem 1.4. *Let X be a real Banach space and $F : X \to X^*$ be a nonlinear operator. If the Gateaux derivative $F'(x)$ exists for every $x \in X$ and is positive semidefinite, then F is monotone.*

Theorem 1.5. *Let f be a proper convex function defined on X. If f is differentiable, then ∇f is monotone.*

Theorem 1.6. *Let f be a proper differentiable function defined on X. If ∇f is monotone, then f is convex.*

2 Variational Inequalities

In this section we shall discuss some basic properties of variational inequalities. Before we state the definition we shall first discuss some examples where variational inequalities arise.

Example 2.1. Let $I = [a, b] \subset \mathbb{R}$. Let f be a real-valued differentiable function defined on I. Suppose, we seek for the points $x \in I$ for each

$$f(x) = \min_{y \in I} f(y).$$

Then three cases will arise in this case :

(i) $a < x < y \Rightarrow f'(x) = 0$

(ii) $a = x \Rightarrow f'(x) \geq 0$

(iii) $x = b \Rightarrow f'(x) \leq 0.$

All these cases can be put together as a single inequality as follows:

$$f'(x)(y-x) \geq 0 \text{ for all } y \in I.$$

This is an example of variational inequality.

Example 2.2. Let K be a closed convex set in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ be differentiable. We characterize the points $x \in K$ for which

$$f(x) = \min_{y \in K} f(y).$$

If there exists $x \in K$ which satisfies the above equation and if $F(x) = \text{grad } f(x)$, then x is a solution of the following inequality

$$x \in K : (F(x), y-x) \geq 0 \text{ for all } y \in K.$$

Conversely, if f is differentiable and convex and if the above inequality is satisfied by x, then

$$f(x) = \min_{y \in K} f(y).$$

Example 2.3. Let Ω be a bounded open domain in $\mathbb{R}^n$ with the boundary T. In some problems of mechanics we seek a real-valued function $x \to u(x)$ which, in Ω, satisfies the classical equation

$$-\nabla u - u = f,\ f \in \Omega,\ u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \tag{2.1}$$

with the boundary conditions

$$u \geq 0,\ \frac{\partial u}{\partial v} \geq 0,\ u\frac{\partial u}{\partial v} = 0 \text{ on } \Gamma. \tag{2.2}$$

where $\frac{\partial}{\partial v}$ denotes differentiation along the outward normal to Γ. If we write

$$J(v) = \frac{1}{2}a(v,v) - (f,v)$$

where

$$\begin{aligned} a(u,v) &= \sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_{\Omega} uv\, dx \\ (f,v) &= \int_{\Omega} fv\, dx \end{aligned}$$

and if we introduce the closed convex set K defined by

$$K = \{v : v \geq 0 \text{ on } \Gamma\},$$

then the problem given by (2.1) and (2.2) is equivalent to finding $u \in K$ such that

$$J(u) = \inf_{v \in K} J(v).$$

This admits a unique solution u characterized by

$$u \in K : \ a(u, v-u) \geq (f, v-u) \ \text{ for all } v \in K.$$

This is called a variational inequality problem.

We shall now state the problem in the most general setting.

Let X be a reflexive real Banach space and let X^* be its dual. Let T be a monotone hemicontinuous mapping from X to X^* and let K be a nonempty closed convex subset of the domain $D(T)$ of T. Then a variational inequality is stated as follows:

$$x \in K : (Tx, y-x) \geq 0 \ \text{ for all } y \in K. \tag{2.3}$$

Any $x \in X$ which satisfies (2.3) is called a solution of the variational inequality. Let us write $S(T,K)$ to denote the set of all solutions of variational inequality (2.3). We shall, in fact, consider a more general inequality which is stated as follows:

For each given element $w_0 \in X^*$,

$$x \in K : (Tx - w_0, y-x) \geq 0 \ \text{ for all } y \in K. \tag{2.4}$$

Inequality (2.3) can also be written by replacing the subset K of X by an extended real-valued function defined on X. For any subset K of X, let δ_k, called the indicator function of K, be the function defined on X by

$$\delta_K(y) = \begin{cases} 0, & \text{if } y \in K; \\ \infty, & \text{if } y \notin K. \end{cases}$$

Then it is easy to verify that $x \in K$ is a solution of (2.3) if and only if

$$(Tx, y-x) \geq \delta_K(x) - \delta_K(y) \ \text{ for all } y \in K.$$

Therefore we consider, as a generalization of inequality (2.3), the inequalities of the form :

$$(Tx, y-x) \geq f(x) - f(y) \ \text{ for all } y \in K, \tag{2.5}$$

where f is an arbitrary extended real-valued function defined on X.

Observe that if $f = 0$, then (2.5) reduces to the VI(2.3) and if $T = 0$, then we are in the framework of the calculus of variations where we minimize the extended real-valued functional f, i.e., we have

$$f(x) \leq f(y) \ \text{ for all } y \in X$$

or

$$f(x) = \min_{y \in X} f(y).$$

Theorem 2.4. *Let T be a monotone, hemi-continuous mapping of a subset $D(T)$ of X into X^* and K a convex subset of $D(T)$. Then for a given element $w_0 \in X^*$, any solution of inequality* (2.4) *is also a solution of the equality*

$$(Ty - w_0, y-x) \geq 0 \ \ \forall\, y \in K. \tag{2.6}$$

Theorem 2.5. *Let T be a hemi-continuous mapping of X into X^*. Suppose that for any pair of vectors $x_0 \in K$ and $w_0 \in X^*$,*

$$(Ty - w_0, y-x) \geq 0 \ \ \forall\, y \in K. \tag{2.7}$$

Then $Tx_0 = w_0$.

The following result gives uniqueness of solution when it exists.

Theorem 2.6. *If the mapping T from X into X^* is strictly monotone, then the inequality* (2.4) *can have atmost one solution.*

Theorem 2.7. *If either the mapping T is strictly monotone or the function f is strictly convex, then the inequality* (2.5) *can have atmost one solution.*

We shall now state that the following fundamental theorem for variational inequality.

Theorem 2.8. *Let T be a monotone hemicontinuous map of a closed convex subset K of a reflexive real Banach space X, with $0 \in K$, into X^* and if K is not bounded, let T be coercive on K. Then for each given element $w_0 \in X^*$ there is an $x \in K$ such that inequality* (2.4) *holds, i.e.,*

$$x \in K : \ (Tx - w_0, y - x) \geq 0 \ \ \forall\, y \in K.$$

3 Types of Variational Inequalities

(1) The Variational like Inequality Problem (VLIP)

Find $x \in K$ such that

$$(Tx, \theta(u,x)) \geq 0 \ \text{ for all } u \in K.$$

(2) Variational Type Inequality Problem (VTIP)

For $z \in K$ find $u \in K$ such that

$$\left(T\left(\frac{z+u}{2}\right), x-u\right) \geq 0 \ \text{ for all } x \in K.$$

(3) Strongly Nonlinear Variational like Inequality Problem (SNVIP)

Find $u \in K$ such that

$$(Tu, \theta(x,u)) \geq (Au, \theta(x,u)) \ \text{ for all } x \in K.$$

(4) Strongly Nonlinear Implicit Variational like Inequality Problem (SNIVIP)

Find $u \in K$ such that

$$(Tu, g(x) - g(u)) \geq (Au, g(x) - g(u)) \text{ for all } x \in K.$$

(5) Nonlinear Quasi-Variational Inequality Problem (NQVIP)

Find $u \in K$ such that $u \in S(u)$ and

$$(T(u), x - u) \geq 0 \text{ for all } x \in S(u).$$

(6) Quasi-Variational like Inequality Problem (QVLIP)

Find $u \in K$ such that $u \in S(u)$ and

$$(T(u), \theta(x, u)) \geq 0 \text{ for all } x \in S(u).$$

(7) Quasi-Variational Type Inequality Problem (QVTIP)

For $z \in K$, find $u \in K$ such that $u \in S\left(\frac{z+u}{2}\right)$, and

$$\left(T\left(\frac{z+u}{2}\right), x - u\right) \geq 0 \text{ for all } x \in S\left(\frac{z+u}{2}\right).$$

(8) Generalized Quasi-Variational Type Inequality Problem (GQVTIP)

For $z \in K$, find $u \in K$, $w \in X^*$ such that $u \in S\left(\frac{z+u}{2}\right), T\left(\frac{z+u}{2}\right)$ and

$$(w, x - u) \geq 0 \text{ for all } x \in S\left(\frac{z+u}{2}\right).$$

Theorem 3.1. *Let K be a nonempty closed convex subset of a reflexive real Banach space X with dual X^*, 0 is an interior point of K. Let $S = T - \mho$, $T, \mho : K \to K^*$ be hemicontinuous. T uniformly monotone with a gauge function $c_1(r)$, $\mho$ uniformly relaxed Lipschitz with another gauge function $c_2(r)$. Let $G \subset X \times X^*$ be such that*

$$G = \{(v, Su + z) : (x, u - z) \geq 0 \; \forall \; v \in K\}.$$

Then NVI: $u \in K :\ (Su, v - u) \geq 0$ has a unique solution.

Theorem 3.2. *$S : K \to X^*$ is p-monotone, $T : K \to X^*$ is p-Lipschitz, S,T are hemicontinuous, K is nonempty closed convex set in X. Then $u \in K$ is a solution of the VI*

$$(Su - Tu - w, v - u) \geq 0 \; \forall \; v \in K$$

iff u is a solution of

$$(Su - Tu - w, v - u) \geq (r - s) \, \|u - v\|^p$$

when $s < r$, $p > 1, r$ p-monotone constant and s p-Lip constant.

Let K be a nonempty closed convex subset of a reflexive Banach space X, $S,T : K \to X^*$, $f : X \to [-\infty, \infty]$ is convex, lsc and $f \neq \infty$. Let S be hemicontinuous, strong monotone with constant $r > 0$, T hemicontinuous and relaxed Lip with constant k.

Theorem 3.3. *Under the above conditions*

$$(Su - Tu, v - u) + f(v) - f(u) \geq 0 \; \forall \; v \in K$$

$$\Rightarrow (Sv - Tv, v - u) + f(v) - f(u) \geq c \, \|v - u\|^2$$

where $c = r - k > 0$.

Theorem 3.4. *Under the above conditions*

$$(Su - Tu, v - u) + f(v) - f(u) \geq 0$$

has a unique solution.

Let X be lc H-tvs, $G : X \to X$ is continuous, T is said to be G-monotone if $\exists$ a constant $r > 0$ such that

$$(Tu - Tv, G(u - v)) \geq r \, (p(u - v))^2$$

where p is a seminorm on X.

Theorem 3.5. *Let K be a nonempty, compact, convex subset of X, $T : K \to X^*$ is strongly G-monotone, $G(u+v) = G(u) + G(v)$, $G(tx) = tG(x)$. Then*

$$(Tu, G(v-u)) \geq 0 \; \forall \, v \in K$$

has a solution.

A set $K \subset \mathbb{R}^n$ is said to invex if

$$u, v \in K, 0 \leq t \leq 1 \Rightarrow u + t\eta(v,u) \in K$$

for some vector $\eta : K \times K \to \mathbb{R}$. $\eta : K \times K \to \mathbb{R}$ is said to be strongly monotone if $\exists \, \sigma > 0$ such that

$$(\eta(v,u), v-u) \geq \sigma \|v-u\|^2$$

Lip continuous if $\exists$ a constant $\delta > 0$ such that

$$\|\eta(x,u)\| \leq \delta \|v-u\|.$$

Assumption 1. $\eta(v,u) = -\eta(u,v)$.

Assumption 2. $\gamma = \xi\delta + \beta\sqrt{1 - 2\sigma + \delta^2} < \alpha$, $\alpha > 0$, $\sigma > 0$, $\beta > 0$, $\delta > 0$ are strong monotone an Lip constants of T and η respectively, $\xi > 0$ is the Lip constant of A.

Theorem 3.6. $T : K \to \mathbb{R}$, $\eta : K \times K \to \mathbb{R}$ *are both strongly monotone and Lip continuous,* $A : K \to \mathbb{R}$ *Lip continuous. Assumption 1 and 2 hold. Then $\exists$ a unique solution of the variational inequality*

$$u \in K : (Tu - Au, \eta(v,u)) \geq 0 \; \forall \, v \in K.$$

Definition 3.7. Let $T : K \to X^*$. T is said to be

Uniformly Monotone if $\exists$ a gauge function $c_1(r)$ such that

$$(Tx - Ty, x - y) \geq c_1(\|x-y\|) \|x-y\|.$$

Strongly Monotone if $\exists$ a constant $c_1 > 0$ such that

$$(Tx - Ty, x - y) \geq c_1 \|x - y\|^2 .$$

Uniformly Relaxed Lipschitz if $\exists$ a gauge function $c_2(r)$ such that

$$(Tx - Ty, x - y) \leq c_1 (\|x - y\|) \|x - y\| .$$

Relaxed Lipschitz if $\exists$ a constant $c_2 > 0$ such that

$$(Tx - Ty, x - y) \leq c_2 \|x - y\|^2 .$$

p-Monotone if $\exists$ a constant $r > 0$ such that

$$(Tx - Ty, x - y) \geq r \|x - y\|^p .$$

p-Lipschitz type if $\exists$ a $s > 0$ and $p > 1$ such that

$$(Tx - Ty, x - y) \leq s \|x - y\|^p .$$

Relaxed Monotone if $\exists$ a constant $k > 0$ such that

$$(Tx - Ty, x - y) \geq -k \|x - y\|^2 .$$

Strongly Lip is $(Tx - Ty, x - y) \leq -c \|x - y\|^2$.

Strongly Pseudo-contractive or Strictly Pseudo-contractive :

$$\exists\, t > 1 : \|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| .$$

Strictly accretive for $r > 0$ such that

$$(Tx - Ty, x - y) \geq r \|x - y\|^2 .$$

Remarks:

(1) T Relaxed Lip $\Leftrightarrow -T$ Relaxed Monotone

(2) T Relaxed Lip $\Leftrightarrow$

$$\begin{array}{ll} I-T \text{ is strongly monotone,} & \text{if } k<1; \\ I-T \text{ is relaxed monotone,} & \text{if } k>1. \end{array}$$

(3) T strongly monotone $\Leftrightarrow I-T$ is strongly (strictly) contractive.

(4) T strongly Lip $\Leftrightarrow I-T$ is strongly monotone.

4 Equilibrium Problem

There is another problem, called 'equilibrium problem', which is even more general than variational inequality. For a brief discussion of equilibrium problem one may refer, for example, Blum and Oettli. We quote below the problem and then mention some problems which arise as special cases.

Let X be a real Banach space and K a closed convex subset of X. Let $f : K \times K \to \mathbb{R}$ be such that $f(x,x) = 0$ for $\forall\, x \in K$.

The equilibrium problem (P) is to find

$$x_0 \in K,\ f(x,y) \geq 0\ \forall\, y \in K.$$

Examples **1. Optimization**: Let $g : K \to \mathbb{R}$. Find $x_0 \in K$ such that

$$g(x_0) = \min_{x \in K} g(x).$$

This is a special case of (P), the case for which $f(x,y) = g(y) - g(x)$.

2. Convex Optimization for differentiable map and variational inequality:

Let $g : X \to \mathbb{R}$ be convex and Gateaux differentiable with Gateaux differential $Dg(x) \in X^*$ at x. Consider the problem $\min_{x \in K} g(x)$. If x_0 solves the above problem, it is known that x_0 is a solution of the following VI

$$x_0 \in K,\ (Dg(x_0), y - x_0) \geq 0\ \forall\, y \in K.$$

Observe that the above is a special case of the problem (P) with $f(x,y) = (Dg(x), y-x)$.

3. Fixed Points:

Let X be a Hilbert space, $T : K \to K$. Find $x_0 \in K$ such that $Tx_0 = x_0$. Put $f(x,y) = (T(x)-x, y-x)$. Then the above problem is the problem (P).

5 Complimentary Problem

Several problems arising in various fields such as: mathematical problem, game theory, economics, mechanics and geometry have mathematical formulation of a complementarity problem.

Definition 5.1. *Let X be a reflective real Banach space and let X^* be its dual. Let K be a closed convex cone in X with $0 \in K$. The polar of K is the cone K^* defined by*

$$K^* = \{y \in X^* : (y,x) \geq 0 \; \forall \, x \in K\}.$$

Obviously $K^* \neq \varnothing$ since $0 \in K^*$. It is also easy to see that K^* is a closed convex cone in X^*. Let T be a map from K into X^*. Then the complementarity problem (CP in short) is to find an $x \in X$ such that

$$x \in K, Tx \in K^*, (Tx,x) = 0.$$

The following theorem proves the equivalence between the complementarity problem and variational inequality over closed convex cone. We write

$$S(T,K) = \{x : x \in K, (Tx, y-x) \geq 0 \text{ for all } y \in K\}$$

and

$$C(T,K) = \{x : x \in K, Tx \in K^*, (Tx,x) = 0\}.$$

We have

Theorem 5.2. *Karamardian [57]*

$C(T,K) = S(T,K)$.

Remarks:

(a) It should be noted that the solution of a complementarity problem, if exists, is unique if the operator is strictly monotone. Since $C(T,K) = S(T,K)$ for a closed convex cone K, the proof is same as that of Theorem 2.6.

(b) Regarding the existence it must be noted that the solution may not exist only under the assumption of hemicontinuity and monotonicity (even strict monotonicity) of the operator T. For example, let $X = \mathbb{R}$, $K = \{x \in \mathbb{R} : x \geq 0\}$, so that $K = K^*$ and K is a closed convex cone. Let $T : K \to \mathbb{R}$ be defined by

$$Tx = -\frac{1}{1+x}.$$

Then T is hemicontinuous and strictly monotone $(Tx,x) = 0$ implies $x = 0$ but $T0 = -1 \notin K^*$.

We shall now discuss the existence of solutions of the complementarity problem. We have

Theorem 5.3. *Let $T : K \to X^*$ be hemicontinuous, monotone and coercive. Then the complementarity problem has a solution. In particular if T is hemicontinuous and α-monotone, then the solution exists and is unique.*

Theorem 5.4. *Let $T : K \to X^*$ be hemicontinuous, monotone and let $T0 \in K^*$. The the complementarity problem has a solution.*

Theorem 5.5. *Let $T : K \to X^*$ be hemicontinuous and monotone such that there is an $x \in K$ with $Tx \in$ int K^*. Then there is an x_0 such that*

$$x_0 \in K,\ Tx_0 \in K^* \text{ and } (Tx_0, x_0) = 0. \tag{5.1}$$

If further T is strictly monotone, then there is a unique x_0 satisfying (5.1).

In order to prove the theorem we need the following result, which is due Browder, See Browder [12] and Mosco [74].

Theorem 5.6. *Let T be a monotone, hemicontinuous map of a closed, convex bounded subset K of X, with $0 \in K$, into X^*. Then there is an $x_0 \in K$ such that*

$$(Tx_0, y - x_0) \geq 0 \text{ for all } y \in K.$$

Now observe that if $e \in K^*$ but $e \notin$ int K^*, the sets $D_r(e)$ need not be bounded. In this case we cannot conclude that $y = 0$ from the fact that $(e, y) = 0$. Consider the case when $X = \mathbb{R}^2$, $K = \mathbb{R}^2$ and $e = (1, 0)$. Then for each $r > 0$, $D_r(e)$ contains the positive y-axis and hence is unbounded.

We note that this theorem fails to hold if the requirement that there exists $x \in K$ with $Tx \in$ int K^* is dropped.

Take $X = \mathbb{R}^3$, $K = \left\{(x,y,z) \in \mathbb{R}^3 : x, z \geq 0,\ 2xz \geq y^2\right\}$. Define T by $T(x,y,z) = (x+1, y+1, 0)$. Then T is monotone, hemicontinuous (even bounded). $(1,-1,1) \in K$ and $T(1,-1,1) = (2,0,0) \in K^*$. If $u = (x,y,z) \in K$ with $Tu \in K^*$, then $y = -1$ and hence $x > 0$. Hence for any such u, $(Tu, u) = x(x+1) > 0$.

Theorem 5.6 appears, in, some form and other Browder [9], Mosco [74] and Hartman and Stampacchia [44]. The papers were written almost at the same time and there are some overlapping results contained in those papers.

Finite Dimensional Case

Let K be a closed convex cone in $\mathbb{R}^n$ and f a map from K into $\mathbb{R}^n$ such that

$$x \in K,\ f(x) \in K^*,\ (f(x,x)) = 0.$$

In particular, if $K = \mathbb{R}^n_+$,

$$\{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, 2, \cdots, n\},$$

then the particular problem can be stated as follows:

$$x \geq 0,\ f(x) \geq 0, (f(x), x) = \sum_{i=1}^{n} x_i f(x_i) = 0.$$

If further $f(x) = Mx + b$ where M is a given real square matrix of order n and b is a given column vector in $\mathbb{R}^n$, then the above problem is called linear complementarity problem (LCP in short) and it can be stated as follows:

Find $w_1, w_2, \cdots, w_n$ and $(x_1, x_2, \cdots, x_n)$ such that

$$w = Mx + b,\ w \geq 0, x \geq 0,\ w_i x_i = 0,\ i = 1, 2, \cdots, n.$$

Otherwise, in general, the problem is known as a nonlinear complementarity problem (NCP in short).

We shall now illustrate the LCP by a numerical example.

An Example

As a specific example of an LCP in $\mathbb{R}^n$, let

$$n = 2,\ M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ q = \begin{pmatrix} -5 \\ -6 \end{pmatrix}.$$

In this case the problem is to solve

$$w_1 - 2x_1 - x_2 = -5$$

$$w_2 - x_1 - 2x_2 = -6$$

$$w_1, w_2, x_1, x_2 \geq 0,\ w_1 x_1 = w_2 x_2 = 0.$$

This can be expressed in the form of vector equation as:

$$w_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + w_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_1 \begin{pmatrix} -2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \end{pmatrix}$$
$$w_1, w_2, x_1, x_2 \geq 0,\ w_1 x_1 = w_2 x_2 = 0.$$

As special cases we have the following results for $\mathbb{R}^n$.

Theorem 5.7. *Let $f : K \to \mathbb{R}^n$ be continuous, monotone, then NCP has a solution. In particular if f is continuous and strongly monotone , then the solution exists uniquely.*

Theorem 5.8. *Let $f : K \to \mathbb{R}^n$ be continuous, monotone and such that $f(0) \in K^*$ (or $f(0) = 0$). Then there exists a solution to the NCP.*

Theorem 5.9. *Let $f : K \to \mathbb{R}^n$ be continuous, monotone and such that there exists an $x \in K$ with $f(x) \in$ int K^*. Then there exists a solution to the NCP.*

Lemke [63] and Eaves [27] discussed the existence of stationary points and nature of the set of all stationary points of the pair (f, K) in $\mathbb{R}^n$ where $K = \mathbb{R}^n_+$. Lemke [63] discussed the linear case by considering affine function. A basic theorem of Lemke [64] states as follows:

Theorem 5.10 (Lemke). *Given an affine map $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ and a $d \in \mathbb{R}^n_+$ there is a piecewise affine map $x : \mathbb{R}_+ \to \mathbb{R}^n_+$ such that $x(t)$ is a stationary point of (f, D^n_t) with*

$$d \cdot x(t) = t \text{ where}$$

$$D^n_t = \left\{x \in \mathbb{R}^n_+ : d \cdot x \leq 1\right\}.$$

There are several matrices used in LCP.

Definition 5.11. *Let M be square matrix of order n. M is said to positive definite if*

$$y^T My = \sum_{i=1}^{n} \sum_{j=1}^{n} y_i M_{ij} y_j > 0 \text{ for all } 0 \neq y \in \mathbb{R}^n.$$

Positive semi-definite if

$$y^T My \geq 0 \text{ for all } y \in \mathbb{R}^n.$$

Copositive Matrix if

$y^T My \geq 0$ *for all* $y \geq 0$ *and strictly copositive if strictly inequality holds for all* $y \geq 0$.

Copositive plus Matrix if it is copositive matrix and if

$y^T(M+M^T)=0$, *whenever* $y \geq 0$ *satisfies* $y^T My = 0$.

P-matrix if all the principal subdeterminants of M are positive.

Q-matrix if the LCP has a solution for every $q \in \mathbb{R}^n$.

nondegenerate matrix if all the principal subdeterminants are nonzero.

degenerate if it is not nondegenerate.

z-matrix if $m_{ij} \leq 0$ *for all* $i \neq j$

J-matrix if

$$Mz \geq 0,\ z^T Mz \geq 0,\ z \geq 0, \Rightarrow z = 0.$$

6 Semi Inner-Product Space and Variational Inequality

In this section we discuss the concept of semi-inner product (sip in short), which was introduced by Lumer [68] in the year 1961 and subsequently studied by Giles [38] and several other mathematicians. We then study variational inequality in sip space.

Let V be a complex vector space. A sip on V is a function [,] on $V \times V$ with the following properties: for $x,y,z \in V$ and $\lambda \in \mathbb{C}$,

(i) $[x+y,z]=[x,z]+[y,z]$,

$[\lambda x,y]=\lambda[x,y]$,

(ii) $[x,x]>0$ for $x \neq 0$,

(iii) $|[x,y]|^2 \leq [x,x][y,y]$.

V along with a sip defined on it is called a sip space. A sip space has the homogeneity property when the sip satisfies

(iv) $[x,\lambda y] = \overline{\lambda}[x,y]$.

With the aim of carrying over Hilbert space type arguments to the theory of Banach spaces Lumer [68] introduced the concept of sip. But the generality of the axiom system defining the sip is a serious limitation of any extensive development of a theory of sip spaces parallel to the theory of inner-product spaces. Let X be a normed linear space and let X^* be its dual.

The unit ball of X is

$U = \{x \in X : \|x\| \leq 1\}$ and its boundary

$S = \{x \in X : \|x\| = 1\}$ is the unit sphere of X.

$U^* = \{f \in X^* : \|f\| \leq 1\}$ and its boundary

$S^* = \{f \in X : \|f\| = 1\}$ is the unit sphere of X^*.

The conjugate norm will also be denoted by $\| \ , \ \|$.

Theorem 6.1. *A sip space V is a normed linear space with the norm $\|x\| = [x,x]^{1/2}$. Every normed linear space can be made into a sip space (in general, in infinitely many different ways) with the homogeneity property.*

Theorem 6.2. *A Hilbert space H can be made into a sip space in a unique way. A sip space is an ip space if and only if the norm it induces satisfies parallelogram law.*

Continuous and Uniform sip spaces

A continuous sip space is sip space V where the sip has the additional property:

(v) For all $(x,y) \in S \times S$,

$\mathrm{Re}[y, x+\lambda y] \to \mathrm{Re}[y,x]$ for all $\lambda \to 0$.

The space is a uniformly continuous sip space when the above limit is approached uniformly for all $(x,y) \in S \times S$.

A uniform sip space is a uniformly continuous sip space where the induced normed linear space is uniformly continuous and complete.

Examples (L_p space for $1 < p < \infty$)

The real Banach space $L_p(X,S,\mu)$, where $1 < p < \infty$, can readily be expressed as a uniform sip space with sip defined by

$$[y,x] = \frac{1}{\|x\|^{p-2}} \int_X y\,|x|^{p-1}\,\mathrm{sgn}x\; d\mu.$$

For x, y in any sip space V, x is said to be normal to y and y is transversal to x if $[y,x] = 0$. A vector $x \in V$ is normal to a subspace N and N is transversal to x, if x is normal to each $y \in N$.

A Banach space X is said to be smooth at a point $x \in S$ if and only if there exists a unique hyperplane of support of x, that is, there exists only one continuous linear functional $I_x \in E^*$ with $\|I_x\| = 1$ and $I_x(x) = 1$. x is said to be a smooth Banach space if it is smooth at every $x \in S$.

The norm of X is said to be Gateaux differentiable if for all $x, y \in S$ and real λ,

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda} \text{ exists.}$$

The norm is said to be uniformly Frechet differentiable if this limit is approached uniformly for $(x,y) \in S \times S$. Note that X is smooth at $x \in S$ if and only if the norm is Gateaux differentiable at x. We have

Theorem 6.3. *In a continuous sip space, x is normal to y if and only if $\|x+\lambda y\| > \|x\|$ for all complex numbers λ.*

Theorem 6.4. *A sip space is continuous (uniformly continuous) sip space iff the norm is Gateaux (uniformly Frechet) differentiable.*

Lemma 6.5. *In a continuous sip space which is uniformly convex and complete in its norm, there exists a nonzero vector normal to every proper closed vector subspace.*

Lemma 6.6. *A sip space is strictly convex if whenever $[x,y] = \|x\| \, \|y\|$, $x,y \neq 0$, then $y = \lambda x$ for some real $\lambda > 0$.*

Theorem 6.7 (Generalized Riesz-Fischer Theorem). *In a continuous sip space V which is uniformly convex and complete in its norm, to every continuous linear functional $f \in V^*$, there exists a unique vector $y \in V$ such that*

$$f(x) = [x,y], \; x \in V.$$

Theorem 6.8. *For a uniform sip space M, the dual space M^* is a uniform sip space w.r.t. the sip defined by*

$$\lfloor f_x . f_y \rfloor = [y,x].$$

Theorem 6.9. *Every finite dimensional strictly convex, continuous sip space is a uniform sip space.*

Theorem 6.10. *Let X be a continuous sips which is uniformly convex and complete in its norm. If A is a bounded linear operator from X into itself, then there is a unique bounded linear operator A^+ such that*

$$[Ax,y] = \lfloor x, A^+ y \rfloor .$$

A^+ is called the generalized adjoint of A: The proof uses Theorem 6.7 and is similar to that of the corresponding for Hilbert space operators. Note that if X is a Hilbert space, then the generalized adjoint is the usual Hilbert space adjoint.

We now discuss variational inequality and complementarity problem in semi-inner-product space under certain contractive type conditions on the operators.

Let K be a closed convex subset of sips. If $T : K \to K$, then a variational inequality (VI in short) is stated as follows:

$$x \in K : \; [Tx, y - x] \geq 0 \text{ for all } y \in K.$$

If K is closed convex cone, then the polar or dual of K, denoted by K^+, is defined by

$$K^+ = \{z \in X^* : [z.x] \geq 0 \text{ for all } x \in K\}.$$

If K is a closed convex cone, then the complementarity problem (CP for short) is defined as follows:

$$x \in K, Tx \in K^+ \text{ and } [Tx, x] = 0.$$

Observe that if K is a closed convex cone, then (VI) and (CP) are equivalent.

We have

Theorem 6.11. *Let X be uc and ss and K a nonempty closed convex subset of X. Let $T : K \to K$ satisfy any one of the following conditions:*

(i) $\|Tx - Ty\| \leq a\|x - y\| + b\|Tx - y\| + c\|Ty - y\|$

where

$$-1 < a < 0, b \geq 0, c \geq 0, a + b + c = 0,$$

(ii) $\|Tx - Ty\| \leq a_1\|x - y\| + a_2\|x - Tx\| + a_3\|y - Ty\| + a_4\|x - Tx\| + a_5\|y - Tx\|$

where

$$-1 < a < 0, a_2, a_3, a_4, a_5 \geq 0, \sum_{i=1}^{5} a_i = 0.$$

Then there is a unique $y_0 \in K$ such that $[Ty_0, x - y_0] \geq 0$ for all $x \in K$.

Theorem 6.12. *Let X be Hilbert space and K a closed convex cone and let the conditions of the previous theorem be satisfied. Then the CP has a unique solution, i.e., there is a unique $y_0 \in X$ such that*

$$y_0 \in K, Ty_0 \in K^* \text{ and } (Ty_0, y_0) = 0.$$

Theorem 6.13. *Let X be uc and ss and K a nonempty closed convex subset of X. Let $T : K \to K$ be nonexpansive. Then there exists some $y_0 \in K$ such that*

$$[Ty_0 + y_0, y_0] = 0.$$

References

1. George Allen, Variational Inequalities Complementarity Problems and Duality Theorems, *JMAA*, **58**, (1977), 1-10.

2. Alfred Auslender, Marc. Teboulle, Entropy Proximal Decomposition Methods for Convex Programms and Variational Inequalities, *Math. Prog. Ser A* **91**, (2001), 33 - 47.

3. A. EI-Arni, Generalized Quasi-variational Inequalities with Non-compact Sets, *JMAA*, **241**, (2000), 189 - 197.

4. A EI-Arni, Generalized Quasi-variational inequalities on Non-compact Sets with Pseudo-monotone Operators, *JMAA*, **249**, (2000), 515 - 526.

5. M. S. Bazaras, J. J. Goode and M.Z. Nashed, A nonlinear complementarity problem in mathematical programming in Banach space, *Proc. Amer. Math. Soc.* **35** (1972), 165 - 170.

6. A. Behera and G.K. Panda, Nonlinear Random Variational Inequality Problem of Mathematical Programming in Banach Spaces, *Opsearch*, **36** (1), (1999), 35 - 41.

7. A. Behera and Lopamudra Nayak, On Nonlinear-type Inequality Problem, *Ind. J. Pure Appl. Math.*, **30** (9) (1999), 911 - 923.

8. A. Behera and Lopamudra Nayak, A Variational-type Inequality,*Opsearch*, **36** (2) (1999), 107 - 112.

9. F.E. Browder, Nonlinear variational inequalities and maximal monotone mappings in Banach spaces, *Math. Annalen*, **176** (1968), 89 - 113.

10. F.E. Browder, Existence and approximation of solutions of nonlinear variational inequalities, *Proc. Acad. Sci. U. S.* **56** (1966a), 1080 - 1086.

11. F.E. Browder, On the unification of calculus of variation and the theory of monotone nonlinear operators in Banach spaces, *ibid* **(1966b)**, 419 - 425.

12. F.E. Browder, Nonlinear monotone operators and convex sets in Banach spaces, *Bull. Amer. Math. Soc.* **71** (1965a), 780 - 58.

13. F.E. Browder, On a theorem of Beurling and Livingstone, *Canad. J. Math.* **17** (1965b), 367 - 372.

14. F.E. Browder, Variational boundary value problems for quasilinear elliptic equations of arbitrary order, *Proc. Natl. Acad. Sci., U.S.* **50** (1963), 31 - 37.

15. B.D. Craven and B. Mond, Complementarity over arbitrary cone, *Z.O.R.* (1977).

16. Jean-Pierre Crouzeix, Patrice Marcotte and Daolizhu, Conditions Ensuring the Applicability of Cutting Plane Methods for Solving Variational Inequalities, *Math. Prog.* Ser A 88, (2000), 521 - 539.

17. R. Chandrasekharan, A special case of the complementarity pivot problem, *Opsearch*, **7** (1970) 263 268.

18. R.W. Cottle and G.B. Dantzig, Complementarity pivot theory of mathematical programming, *Linear Algebra and its Appl.* **1** (1968), 108 - 125.

19. R.W. Cottle and G.B. Dantzing, A Generalization of the Linear Complementarity Problem, *Journal of Cominatorial Theory*, **8** (1970), 79 - 90.

20. Aris Daniilidis and Nicolas Hadjisavvas, Coercivity Conditions and Variational inequalities, *Math. Prog.* **86** (1999), 433 - 438.

21. A.T. Dash and S. Nanda, A complementarity problem in mathematical programming in Banach space, *J. Math. Analysis Appl.* **98** (1984).

22. Androas Domokos, Solution Sensitivity of Variational Inequalities, *JMAA*, 230, (1999), 382 - 389.

23. Helmut Dietrich, A Smooth Dual Gap Function Solution to a Class of Quasi-variational Inequalities, *JMAA,* **235** (1999), 380 - 393.

24. O.D. Donato and G. Maier, The linear Complementarity Problem, *Management Sci.*, **17** (1971), 612 - 634.

25. O.D.Donato and G. Maier, Mathematical programming methods for the inelastic analysis of reinforced concrete frames allowing for limited rotation capacity, *Int. J. Numer. Methods Eng.* **4** (1972), 307 - 329.

26. Stelle Dafermous, Exchange Price Equilibria and Variational Inequality, *Math. Prog.* **46** (1990), 391 - 402.

27. B.C. Eaves, On the basis theorem of complementarity, *Math. Programming* , **1**(1971), 68 - 75.

28. M. Edelstein, On nearest points of sets in uniformly convex Banach spaces, *J. London Math. Soc.*, **43**(1968) 375 - 77.

29. C. M. Elliott, On a Variational Inequality Formulation of an Electromechanical Machine moving boundary Problem and its Approximation by the Finite Element Method, *J. Inst. Math. Appl.* **25** (1980), 121 - 131.

30. C.M. Elliott and V. Janavosky, A Variational Inequlity Approach to Hele Shaw Flow with a Moving Boundary, *Proc. Royal Soc. Edinberg* **88A** (1981), 93 - 107.

31. S.C. Fang and E.L. Peterson, Generalized Variational Inequalities, *J. Optm. Theory Appl.* **38**(3), (1982), 363 - 383.

32. Ashok Ganguly and Kamal Wadhwa, A variational Inequality Problem in Non-compact Sets, *Indian J. Math.* **39** (3), 1997, 213 - 220.

33. B. Garcia, Some classes of matrices in Linear complementarity problem Math. Programming 5 (1973), 299 - 310.

34. Bjorn Gustafssion, Application of Variational Inequalities to a Moving Boundary Problem for Hele Shaw Flows, *SIAM J. of Math. Anal*. **16**(2) (1985), 279 - 300.

35. F. Gianessi, On Some Connection among Variational Inequalities Combinatorial and Continuous Optimization, *Anals of Operation research*, **58** (1985), 181 - 200.

36. F. Gianessi, A remark on Infinite-dimentional Variational Inequalities, *Estrattoda Le Mathematiche*, **XLIX** (1994), Fascicolo II.

37. F.Gianessi and A.A. Khan, Regulation of Non-coercive Quasi Variational Inequalities, *Control and Cybernatics*, **29** (1), (2000).

38. J. R. Giles, Classes of semi-inner-product spaces, *Trans. Amer. Math. Soc.* **129** (1967), 436 - 46.

39. M.S. Gowda and Thomas I. Seidman, Generalized Complementarity Problems, *JMAA,* **46** (1990), 329 - 340.

40. M.S. Gowda and Yoon Song, On Semidefinite Linear Complementarity Problems, *Math. Prog. Ser* **A 88** (2000), 575 - 587.

41. K.M. Ghosh, Fixed-point theorems, *Pure Math Manuscript* **3** (1984), 51 - 53.

42. M. Gregus, A fixed-point theorem in Banach spaces, *Boll. Un. Mat.Ital.* **5** (17) A (1980), 193 - 198.

43. G.E. Hardy and T.D. Rogers, A generalization of a fixed-point theorem of Reich, *Canad. Math. Bull.*, **16** (1973) 201 - 206.

44. Hartman and G. Stampacchia, On some nonlinear elliptic differential functional equations, *Acta. Math.* **115** (1966), 271 - 310.

45. G.J. Habetler and A.L. Price, Existence theory of generalized nonlinear complementary problem, *J. Optim. Theory Appl.* **7** (1971), 223 - 239.

46. G.J. Habetler and M.M. Kostreva, On a direct algorithm for nonlinear complementarity problems SIAM Journ. *Control and optim.* **16**(3) (1978), 504 - 511

47. P.T. Harker and J.S. Pang, Finite Dimensional Variational Inequality and Complementarity Problem: A Survey of Theory Algorithms and Applications, *Math. Prog.* **48**, (1990), 161 - 217.

48. Keisuke Hotta and Akiko Yoshise, Global Convergence of a class of Non-interior Point Algorithms Using Chen-Harker-Kanzow-Smale Functions for Non-linear Complementarity Problems, *Math. Prog. Ser A,* **86**, (1999), 105 - 133.

49. Bingsheng He, Inexact Implicit Methods for Monotone General Variational Inequalities, *Math. Prog.* Ser A,86, (1999), 199 - 217.

50. Chena-Fing Hu, Solving Variational Inequalities in a Fuzzy Environment, *JMAA,* **249**, 527 - 538.

51. Nan-jing Huang and Chuan-xian Deng, Auxiliary Principle and Iterative Algorithms for Generalized Set-valued Strongly Non-linear Mixed Variational Inequalities, *JMAA*, **256**, (2001), 345 - 359.

52. G.Isac, Exceptional Family Elements and the Solvability of Variational Inequalities for Unbounded Sets in Infinite Dimensional Hilbert Spaces,*JMAA,* **246**, (2000), 544 - 556.

53. M.C. Joshi and R.K. Bose, Some Topics in Nonlinear Functional Analysis, *Wiley Eastern Ltd* (1985).

54. Jin-Mun Jeong, Chul-Yen Park and Jong-Yeoul Park, Parabolic Variational Inequalities and Their Lipschitz Perturbations in Hilbert Spaces, *Ind. J. Pure Appl. Math.* **31** (6), (2000), 603 - 620.

55. S. Karamaradian, The nonlinear complementarity problem with applications I, *J. Optm. Theory Appl.* **4** (1969) 87 - 98.

56. S. Karamadian, The Nonlinear Complementarity Problem with Applications, II,*J. of Optm. Theoy Appl*. **4** (1969), 167 - 181.

57. S. Karamardian, *Generalized complementarity problem*, J. Optimization Theory Appl., 8(3) (1971), 161 - 168.

58. S. Karamardian, The complementarity problem, *Math. Programming* **2** (1972), 107 - 129.

59. Akhtar A. Khan, Auxiliary Problem Principle Extended to Monotone Variational Inequalities, T*amakang Journal of Mathematics* , **27** (1996), 117 - 124.

60. D. Kinderlehrer and G. Sampacchia, An Introduction to Variational Inequalities and their Applications, *Academic Press, New York* 1980.

61. Arunima Kumari, On Some Generalized Multi-Valued Variational Inequalities, *Ind. J. Pure Appl. Math.* **31** (6) (2000), 667 - 674.

62. J.L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* **20** (1967), 493 - 519.

63. C.E. Lemke, Bimatrix equilibrium points and mathematical programming, *Management Sci.* Ser. A, **11** (1965) 168 - 169.

64. C. E. Lemke, Bimatrix equilibrium points and mathematical programming (J. B. Rosen, O. L. Mangasarian and K. Ritter, eds.) *Academic Press, New York* (1970)

65. C.E. Lemke, Recent results on complementarity problems, in Nonlinear programming (j.B. Rosen, O.L. Managasarian and K. Ritter, eds.), *Academic press, New York* (1970).

66. C.E. Lemke, A Survey of Complementarity Theory, Variational Inequalities and Complementarity Problems, (R.W. Cottle, F. Gianessi and J. Lions eds) *John wiley and Sons* 1980, Chapter 5, 213 - 239.

67. C.E. Lemke and J.T. Howson Jr., Equilibrium points of bimatrix games, *SIAM J. Appl. Math.* **12** (1964), 413 - 423.

68. G.Lumer, Semi-inner-product spaces, *Trans. Amer. Math. Soc*. **100** (1961), 29 -43.

69. Kunquan Lan and Jeffrey Webb, Variational Inequalities and Fixed Point Theorems for PM-Maps, *JMAA,* **224**, (1998), 102 - 116.

70. Li-Wei-Liu and Yu-Qiang Li, On Generalized Set-valued Variational Inclusion, *JMAA*, **261** (2001), 231 - 240.

71. Dinh The Luc, Existence Results for Densely Pseudomonotone Variational Inequalities, **JMAA, 254**, (2001), 291 - 308.

72. B. Mond, On the complex complementarity problem, *Bull. Austral. Math. Soc.* 9 (1973), 249 - 257.

73. O.G. Mancino and G.G. Stampacchia, Convex programming and variational inequalities, *J. Optm. Theory Appl.* **9**(1) (1972), 3 - 23.

74. U. Mosco, Convergence of convex sets and solutions of variational inequalities, *Adv. Math.* **3** (1966), 510 - 585.

75. U. Mosco, A remark on a theorem of F.E. Browder, *J. Math. Analysis Appl.* **20** (1967a), 90 - 93.

76. U. Mosco, Approximation of the solution of some variational inequalities, *Ann Scuolo Normals sup. Pisa* **21** (1967b), 373 - 934.

77. K.G. Murty, On a characterization of P- matrices, *SIAM J. Appl. Math.* **20**(3) (1971), 378 - 384.

78. K.G. Murty, On the number of solutions of the complementarity problem and spanning properties of complementary cones, *Linear Algebra and its Appl*. **5** (1972), 65 - 108.

79. K.G.Murty, Note on a Bard-type scheme for solving the complementarity problem, *Opsearch* **11**, 2-3 (1974), 123 - 130.

80. J. More, Coercivity Conditions in Nonlinear Complementarity Problem, *SIAM Review* **16** (1974) 1 - 16.

81. S.R. Mohan, Existence of Solutions range for Linear Complementarity Problems with Z-matrix, *Math. Programming Study* **7** (1970), 108 - 119.

82. R.N. Mukherjee and H.L. Verma, Sensitivity Analysis of Generalized Variational Inequalities, *JMAA*, **167** (2) (1992), 299 - 304.

83. Sribatsa Nanda and Sudarsan Nanda, A complex nonlinear complementarity problem, *Bull. Austral.Math. Soc.* **19** (1978), 437 - 444.

84. Sribatsa Nanda and Sudarsan Nanda, A nonlinear complemtantarity problem in Banach space ibid **21** (1980), 351 - 356.

85. Sribatsa Nanda and Sudarsan Nanda, On stationary points and the complementarity problem, *ibid* **20** (1979a), 77 - 86.

86. Sribatsa Nanda and Sudarsan Nanda, A nonlinear complementarity problem in Mathematical Programming in Hilbert space, *ibid* **20** (1979b), 233 - 236.

87. Sudarsan Nanda, A note on a theorem on a nonlinear complementarity problem *ibid* **27** (1978), 161 - 163.

88. S. Nanda, The nonlinear complementarity problem, J.Orissa Math. Soc. 1(1) (1982), 61 - 68.

89. S. Nanda, A nonlinear Complementarity Problem in Semi-inner-product-Space, *Rendiconti di math* **2** (1982), 167 - 171.

90. S. Nanda, A remark on a Nonlinear Complementarity Problem in Semi-inner-product Space, *Rendiconti di Mat.* **4** (1983), 807 - 809.

91. Sudarsan Nanda, Generalized quasi complementarity problems, *J. Math. Anal. Appl.* **120** (1986), 321 - 327.

92. Sudarsan Nanda, Nonlinear complementarity problem in mathematical programming in Banach Space, *Indian J. Pure Appl. Math.* **18** (3) (1987), 215 218.

93. S. Nanda and M. Maity, Nonlinear complementarity problem in elasticity, *Opsearch* **27** (2) (1990), 122 - 124.

94. Sudarsan Nanda, On a complementarity problem in Banach space, *Proc. Amer. Math. Soc.*,**21** (1994), 1203 - 1205.

95. S.Nanda, Nonlinear Analysis, *Norosa Pub. House,* new Delhi (2013).

96. S.Nanda, A note on complementarity problem in mathematical programming in Banach space, (Unpublished).

97. M.A Noor, Generalized quasi complementarity problems, *J.Math. Appl.*, **120** (1986), 321 - 327.

98. M.A. Noor, Convergence analysis of the iterative methods for quasi complementarity problem, Internat. *J. Math. and Math.Sci.* ,**11** (2) (1988), 319 - 334.

99. M.A. Noor, Iterative algorithms for semi-linear quasi complementarity problems, *J. Math. Anal. Appl.* ,**145**(1990), 402 - 412.

100. M.N. Noor, Nonconvex functions and variational inequality, *JOTA* , **87**(3) (1995), 615 - 630.

101. M.A. Noor, Monotone Nonlinear Variational Inequality, *Mathematica* **47** (1982), 99 - 118.

102. M.A. Noor, Strongly Nonlinear Variational Inequality, *CR Math Rep Acad Sci Canda,* **4**(1982), 213 - 218.

103. M.A. Noor, Generalized Nonlinear Variational Inequlaity, *Journ. Math. Anal. Appl.* , **126**(1987), 78 - 84.

104. M.A. Noor, Quasi-Variational Inequality, *Appl Math Letters,* (1988) 119 - 122.

105. M.A. Noor, On Nonlinear Variational Inequality, *Int. J. Math & Math Sci* **14** (2) 1991, 399 - 402.

106. M.A. Noor, On Nonlinear Complementarity Problem, *JMAA,* **128** (1987), 211 - 216.

107. Muhammad Aslam Noor, Algorithm for General Monotone Mixed Variational Inequality, *JMAA,* **229** (1999), 330 - 343.

108. Muhammad Aslam Noor, On a Class of Variational Inequalities, *JMAA,* **128** (1987).

109. Muhammad Aslam Noor, An Interative Scheme for a Class of Quasi-variational Inequalities, *JMAA*, **110** (1985), 463 - 468.

110. Muhammad Aslam Noor, Varaitional Inequalities for Fuzzy Mappings, *FSS* **55** (1993), 309 - 312.

111. Muhammad Aslam Noor, Projection Method for Monotone Variational Inequalities, *JMAA* **237** (1999), 405 - 412.

112. Muhammad Aslam Noor, Three Step Iterative Algorithms for Multivalued Quasi-variational Inequalities, **JMAA 255** (2001), 589 - 604.

113. Muhammad Aslam Noor, Solvability of Multivalued General Mixed Variational Inequalities, *JMAA* **261** (261) (2001), 390 - 402.

114. M.A. Noor and E.H. Al-shemas, Fixed point methods for nonlinear complementarity problems, Indian **J. Math. 35** (1993), 71 - 97.

115. W. Oettli and E. Blum, From optimization and variational inequalities to equilibrium problems, Math. Student **63** (1994), 123 - 145.

116. J. Parida and A. Sen, A variational-like inequality for multifunctions with applications, *J. Math. Anal. Appl.* **124** (1987), 73 - 81.

117. J. Parida and B. Sahoo, On the complex nonlinear complementarity problem , *ibid* **14** (1976a), 129 - 136.

118. J. Parida and B. Sahoo, Existence theory for the complex nonlinear complementarity problem, *ibid* **14** (1976b), 417 - 435.

119. J. Parida, A. Sen and A. Kumar, A linear Complementarity Problem Involving Subgradient, *Bull. Austral. Math. Soc.* **37** (1988), 345 - 351.

120. J. Park and Sangho Kum, An Application of a Browder type fixed point Theorem to Generalized Variational Inequalities, *JMAA,* **218** (1987), 519 - 526.

121. G.K. Panda and N. Dash, On a Variational Type Inequality and its omplementarity Problem in Hausdorff Topological Spaces, *Ind. J. Pure Appl. Math.* , **31**(2) (2000), 153 - 160.

122. Hou-Duo Qi, Li-Zhi Liao and Zheng-Hua Lin, Regularized Smoothing Aproximation to Vertical Complementarity Problems, *JMAA*, **230** (1999), 261 - 276.

123. A. Ravindran, A computer routine for quadratic and linear programming problems [H], *Commun. ACM* **15** (9) (1972), 818 - 820.

124. A. Ravindran, A comparison of the primal-simplex and complementary pivot methods for linear programming, *Naval Res. Logistics Q.* **20**(1)(1973), 96 - 100.

125. S. Reich, Some remarks concerning contraction mappings, *Canad. Math. Bull.* **14**(1971), **MR 45, 1145**, 121 - 124

126. B.E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257 - 289.

127. A.H. Siddiqi, Recent developments in applicable mathematics, *Macmillan Indian Ltd*. (1994).

128. A.H.Siddiqi, Some remarks on variational and quasivariational inequalities of monotone operators, *preprint* (IC / **90** / 215) ICTP, Italy (1990).

129. A.H. Siddiqi and Q.H. Ansari, General strongly nonlinear variational inequalities, *J. Math. Anal. Appl.* **166** (1992), 386 - 392.

130. A.H. Siddiqi and Q.H. Ansari, An algorithm for a class of quasivariational inequalities, *Journ. Math. Anal.Appl.* **145** (1990), 413 - 418.

131. H. Scarf, The approximation of fixed points of a continuous map. *SIAM J. Appl. Math*. **15** (1967), 328 - 343.

132. R. Saigal, On the class of complementary cones and Lemkes algorithm, *SIAM Journ. Appl. Math.* **23** (1972), 46 - 60.

133. R. Saigal, A note on a special linear complementarity problem, *Opsearch* **7** (3) (1970), 175 - 183.

134. S. Simons, Variational inequalities via the Hann- Banach theorem, *Arch. Math.* **31** (1978), 482 - 490.

135. M. Sakai, Applications of Variational Inequalities to the Existence Theorem on Quadrature Domains, *Trans. Amer. Math. Soc.* **276** (1983), 267 - 279.

136. Michel Thera, Existence results for the nonlinear complementarity problem and applications to nonlinear analysis, *J. Math. Anal.Appl.* **154** (1991), 572 - 584.

137. Ram U Verma, Nonlinear Uniformly monotone operators and associated nonlinear variational inequality, *Applicable Analysis*, **66** (1997), 101 - 106.

138. Ram U Verma, Variational inequality in locally convex Hausdroff topological vector space, *Arch Math.* **71** (1998), 246 - 248.

139. Ram U Verma, Nonlinear variational inequality on convex subsets of Banach Space, *Appl. Math. Letters*, **10**(4)(1997), 25 - 27.

140. Ram U Verma, On monotone nonlinear variational inequality problems, *Comment Math Univ. Corolinal.* **39**(1) (1998), 9 - 98.

141. Ram U Verma, Nonlinear monotone variatiomnal inequality, *Comm. Appl. Anal* **2** (1998), 151 - 156.

142. L.T. Watson, Solving nonlinear complementarity problem by homotopy methods, *SIAM J. Control and Optm*. **17**(1) (1979) 36.

143. J.C. Yao, Variational inequality, *Appl Math Letters* **5** (1992), 39 - 42.

144. J.C. Yao, Application of variational inequality to nonlinear analysis, *Appl Math Letters* **4** (1991), 89 - 92.