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Upper Bounds for the Spectral Radius of Block Hadamard Product of Block H-matrices

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Abstract

In this paper, we discuss some upper bounds for the spectral radius of block Hadamard product of block H-matrices. By using norm structure of block matrices, we establish estimations for the spectral radius.

Keywords:Block H-Matrix, Block Hadamard Product, Spectral Radius, Norm, Diagonally Dominant

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1 Introduction

Spectral radius of matrices and H-matrices are used in many fields such as numerical analysis, control theory, mathematical physics, image and signal processing. In recent years, there are many studies on bounds for the eigenvalues and spectral radius of matrices [2, 3, 4, 5, 8, 9, 10, 12]. Particularly, these papers are interested in the upper and lower bounds for the spectral radius of Hadamard product of nonnegative or positive semidefinite matrices. In this paper, we investigate bounds for the spectral radius of block Hadamard product of two block H-matrices.

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For a positive integer n, the set of all $n \times n$ complex (or real) matrices is denoted by $\mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$) throughout the paper. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real $n \times n$ matrices. Then $A \geq B(> B)$ if $a_{ij} \geq b_{ij}(> b_{ij})$ for all $i, j \in \{1, \dots, n\}$. If $A \geq 0(> 0)$, we say A is nonnegative (positive) matrix. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, then $\rho(A) \in \sigma(A)$, by the Perron-Frobenius theorem, where $\sigma(A)$ is the spectrum of A.

A matrix $A \in \mathbb{C}^{n\times n}$ is said to be reducible if there exists a permutation matrix P such that

$$P^T AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B and D are square matrices of order at least one. If A is not reducible, then it is called irreducible. We note that any 1×1 complex matrix is irreducible. For an irreducible nonnegative matrix A, there exists a positive vector u such that $Au = \rho u$ where u is called right Perron eigenvector of A.

A matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if $A^* = A$ where A^* is the conjugate transpose of A. A Hermitian matrix A is said to be positive definite (positive semidefinite) if $x^*Ax > 0$ ($x^*Ax \geq 0$) for all nonzero $x \in \mathbb{C}$.

Let $\mathbf{A} = (A_{ij}) \in \mathbb{C}^{n\times n}$ be a block matrix partitioned into $p \times p$ blocks in the following form,

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp}, \end{pmatrix}$$

in which $A_{ij} \in \mathbb{C}^{n_i, n_j}$ and $\sum_{i=1}^{p} n_i = n$. If each diagonal block A_{ii} is nonsingular and

$$\|A_{ii}^{-1}\|^{-1} > \sum_{j\neq i} \|A_{ij}\| \quad \text{for all } i = 1, 2, \dots, p,$$

then $\mathbf{A}$ is said to be block strictly diagonally dominant with respect to $\|\cdot\|$. If there exits $x_1, x_2, \dots, x_k > 0$ such that

$$x_i\|A_{ii}^{-1}\|^{-1} > \sum_{j\neq i} x_j\|A_{ij}\| \quad \text{for all } i = 1, 2, \dots, p,$$

then $\mathbf{A}$ is said to be a block H-matrix.

Let $\mathbf{A} = (A_{ij}) \in \mathbb{C}^{n\times n}$ and $\mathbf{B} = (B_{ij}) \in \mathbb{C}^{n\times n}$ for $A_{ij}, B_{ij} \in \mathbb{C}^{n_i,n_j}$ and $\sum_{i=1}^{p} n_i = n$. Then the block Hadamard product of $\mathbf{A}$ and $\mathbf{B}$ is defined by $\mathbf{A}\Box\mathbf{B} = (A_{ij}B_{ij})$. If $\mathbf{A}$ and $\mathbf{B}$ are both positive semidefinite (or both positive definite) then $\mathbf{A}\Box\mathbf{B}$ is positive semidefinite (or positive definite, respectively) [6].

The block Kronecker product of A and $\mathbf{B}$ is defined by $A \boxtimes \mathbf{B} = (AB_{ij})_{i=1,\ldots,s}^{j=1,\ldots,t}$ where AB_{ij} is the usual matrix product of A and B_{ij}. For $\mathbf{A} = (A_{ij}) \in \mathbb{C}^{n\times n}$, the block Kronecker product is given by $\mathbf{A} \boxtimes \mathbf{B} = (A_{ij} \boxtimes \mathbf{B})_{i=1,\ldots,p}^{j=1,\ldots,q}$.

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are called block commuting if every block of $\mathbf{A}$ commutes with every block of $\mathbf{B}$.

Throughout the paper, a block matrix we consider is $p \times p$ and each block is a matrix in $\mathbb{C}^{n_i}$ for $\sum_{i=1}^{p} n_i = n$. We call each block of a matrix $\mathbf{A}$ as an entry of $\mathbf{A}$; i.e., ij'th entry of $\mathbf{A}$ is A_{ij}. Taking matrix norm of each entry of $\mathbf{A}$, we define $\tilde{A} = (\|A_{ij}\|)$ where $\|\cdot\|$ is a consistent matrix norm such as Frobenius norm $\|\cdot\|_F$, 1-norm $\|\cdot\|_1$ and ∞-norm $\|\cdot\|_\infty$.

2 Lemmas

In this section, we shall give some lemmas we use throughout.

Lemma 2.1. [7] If A is an $n \times n$ irreducible nonnegative matrix and $Az \le kz$ for a nonnegative nonzero vector z, then $\rho(A) \le k$.

Lemma 2.2. [1] Let A be an $n \times n$ nonnegative matrix. Then either A is irreducible or there exists a permutation P such that

$$P^T AP = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1p} \\ & A_2 & \cdots & A_{2p} \\ & & \ddots & \vdots \\ & & & A_k \end{pmatrix}$$

and each A_i is irreducible, $i = 1, \ldots, p$.

Lemma 2.3. [1] Let A be a nonnegative matrix and A_α be a principal submatrix of A. Then $\rho(A_\alpha) \le \rho(A)$. If A is irreducible and $A_\alpha \ne A$, $\rho(A_\alpha) < \rho(A)$.

Lemma 2.4. [12] Let $A = (A_{ij})$ be a block $p \times p$ matrix where A_{ij} are nonnegative $n_i \times n_j$ matrices. If $\tilde{A} = (\|A_{ij}\|)$ then $\rho(A) \leq \rho(\tilde{A})$ where $\|\cdot\|$ is a consistent matrix norm.

Lemma 2.5. [11] Let $A = (a_{ij})$ be a nonnegative matrix. Then

$$\rho(A) \leq \max_{i \neq j} \frac{1}{2}\Big\{a_{ii} + a_{jj} + \big[(a_{ii} - a_{jj})^2 + 4\sum_{k \neq i} a_{ik} \sum_{k \neq j} a_{jk}\big]^{1/2}\Big\}.$$

Lemma 2.6. [7] Let A and B be $n \times n$ matrices and D and E be diagonal $n \times n$ matrices. Then

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) = (AE) \circ (DB) = A \circ (DBE).$$

By [4] and [11], we have following results on the upper bound of spectral radius of Hadamard product of two matrices.

Lemma 2.7. If A and B are two nonnegative matrices, then

(i) $\rho(A \circ B) \leq \max_{i} \big\{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\big\}$,

(ii) $$\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} + \Big[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})(\rho(A) - a_{jj})(\rho(B) - b_{jj})\Big]^{1/2}\Big\}.$$

3 Main Results

It is known that for the Hadamard product of two nonnegative $n \times n$ matrices A and B, we have $\rho(A \circ B) \leq \rho(A)\rho(B)$. It is natural to ask whether the same is valid for the block Hadamard product. The answer to the question is negative in view of the following example.

Example 3.1. Consider the following two nonnegative 2×2 block matrices A and B.

$$A=\begin{pmatrix} 3 & 1 & \vdots & 0 & 1\\ 1 & 2 & \vdots & 1 & 1\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & 1 & \vdots & 2 & 2\\ 1 & 2 & \vdots & 1 & 3 \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} 1 & 0 & \vdots & 1 & 0\\ 1 & 1 & \vdots & 0 & 1\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \vdots & 1 & 0\\ 1 & 0 & \vdots & 1 & 1 \end{pmatrix}$$

Then $\rho(A)=5.7669$, $\rho(B)=1$ and $\rho(A\square B)=7.1875$, but

$$\rho(A\square B)>\rho(A)\rho(B).$$

Below we show that the upper bound holds for positive (semi)definite matrices.

Theorem 3.1. Let $\mathbf{A}$ and $\mathbf{B}$ be positive semidefinite and block commuting matrices. Then

$$\rho(\mathbf{A}\square\mathbf{B})\leq\rho(\mathbf{A})\rho(\mathbf{B}). \tag{3.1}$$

Proof. First note that $\mathbf{A}\boxtimes\mathbf{B}$ is a positive semidefinite matrix and $\mathbf{A}\square\mathbf{B}$ is a principal submatrix of $\mathbf{A}\boxtimes\mathbf{B}$. This implies that $\rho(\mathbf{A}\square\mathbf{B})\leq\rho(\mathbf{A}\boxtimes\mathbf{B})$ by Lemma 2.3. Then since $\lambda_{\max}(\mathbf{A}\boxtimes\mathbf{B})\leq\lambda_{\max}(\mathbf{AB})$ by Proposition 2.11 in [6], we get $\rho(\mathbf{A}\boxtimes\mathbf{B})\leq\rho(\mathbf{AB})$. Using the fact that $\rho(\mathbf{AB})\leq\rho(\mathbf{A})\rho(\mathbf{B})$ for positive semidefinite matrices, we obtain the result. □

As it is obvious, this bound is restrictive. In general, we get similar upper bounds for the spectral radius of block Hadamard product of any two matrices to the results related to Hadamard product in [4] and [11].

Let $\tilde{A}=(\|A_{ij}\|)$ and $\tilde{B}=(\|B_{ij}\|)$. We note that by Lemma 2.4, we have

$$\rho(\mathbf{A}\square\mathbf{B})=\rho\left((A_{ij}B_{ij})\right)\leq\rho\Big(\left(\|A_{ij}B_{ij}\|\right)\Big)\leq\rho\Big(\left(\|A_{ij}\|\|B_{ij}\|\right)\Big)=\rho(\tilde{A}\circ\tilde{B}). \tag{3.2}$$

Then the followings are straightforward by Lemma 2.7.

Theorem 3.2. Let $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (B_{ij})$ are $p \times p$ block matrices. Then

(1) $\rho(\mathbf{A}\Box\mathbf{B}) \leq \max_i \Big\{2\|A_{ii}\|\|B_{ii}\| + \rho(\tilde{A})\rho(\tilde{B}) - \|A_{ii}\|\rho(\tilde{B}) - \|B_{ii}\|\rho(\tilde{A})\Big\}$.

(2)$\rho(\mathbf{A}\Box\mathbf{B}) \leq \max_{i\neq j} \frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\| + \|A_{jj}\|\|B_{jj}\| + \Big[(\|A_{ii}\|\|B_{ii}\| - \|A_{jj}\|\|B_{jj}\|)^2$

$+4(\rho(\tilde{A}) - \|A_{ii}\|)(\rho(\tilde{B}) - \|B_{ii}\|)(\rho(\tilde{A}) - \|A_{jj}\|)(\rho(\tilde{B}) - \|B_{jj}\|)\Big]^{1/2}\Big\}$.

Now, we give some upper bounds for the spectral radius of block Hadamard product of block H-matrices.

Theorem 3.3. Let $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (B_{ij})$ be block H-matrices. Then,

$$\rho(\mathbf{A}\Box\mathbf{B}) < \max_i\{\|A_{ii}\|\|B_{ii}\| + \|A_{ii}^{-1}\|^{-1}\|B_{ii}^{-1}\|^{-1}\}. \tag{3.3}$$

Proof. Since $\mathbf{A}$ and $\mathbf{B}$ are block H-matrices, there exists positive numbers $u_1,\dots,u_p$ and $v_1,\dots,v_p$ such that

$$u_i\|A_{ii}^{-1}\|^{-1} > \sum_{j\neq i} u_j\|A_{ij}\| \quad \text{for all } i, \tag{3.4}$$

$$v_i\|B_{ii}^{-1}\|^{-1} > \sum_{j\neq i} v_j\|B_{ij}\| \quad \text{for all } i. \tag{3.5}$$

Let $\tilde{A} = (\|A_{ij}\|)$ and $\tilde{B} = (\|B_{ij}\|)$, and $w = u \circ v$. Then for any $i = 1,\dots,p$,

$$\begin{aligned}
\big((\tilde{A}\circ\tilde{B})w\big)_i &= \|A_{ii}\|\|B_{ii}\|w_i + \sum_{j\neq i}\|A_{ij}\|\|B_{ij}\|w_j \\
&\leq \|A_{ii}\|\|B_{ii}\|w_i + \sum_{j\neq i}\|A_{ij}\|u_j\sum_{j\neq i}\|B_{ij}\|v_j \\
&< \|A_{ii}\|\|B_{ii}\|w_i + \|A_{ii}^{-1}\|^{-1}u_i\|B_{ij}^{-1}\|^{-1}v_i \\
&= (\|A_{ii}\|\|B_{ii}\| + \|A_{ii}^{-1}\|^{-1}\|B_{ij}^{-1}\|^{-1})w_i.
\end{aligned}$$

If $\tilde{A}\circ\tilde{B}$ is irreducible, by (3.2) and Lemma 2.1, we have

$$\rho(\mathbf{A}\Box\mathbf{B}) \leq \rho(\tilde{A}\circ\tilde{B}) < \max_i\big\{\|A_{ii}\|\|B_{ii}\| + \|A_{ii}^{-1}\|^{-1}\|B_{ij}^{-1}\|^{-1}\big\}.$$

If $\tilde{A} \circ \tilde{B}$ is reducible, let $T = (t_{ij})$ be an $p \times p$ permutation matrix with $t_{12} = t_{23} = \cdots = t_{p-1p} = t_{p1} = 1$ and remaining $t_{ij} = 0$. Then both $\tilde{A} + \varepsilon T$ and $\tilde{B} + \varepsilon T$ are irreducible nonnegative matrices for any positive real number ε. Substituting $\tilde{A} + \varepsilon T$ and $\tilde{B} + \varepsilon T$ for $\tilde{A}$ and $\tilde{B}$, respectively, in the first case and letting $\varepsilon \to 0$, the result follows by continuity.

□

The upper bound in Theorem 3.3 can be improved in the following theorem.

Theorem 3.4. Let **A** and **B** be block H-matrices, then

$$\rho(\mathbf{A}\Box\mathbf{B}) < \max_{i \neq j} \frac{1}{2}\Big\{ \|A_{ii}\|\|B_{ii}\| + \|A_{jj}\|\|B_{jj}\| + \big[(\|A_{ii}\|\|B_{ii}\| - \|A_{jj}\|\|B_{jj}\|)^2 + 4\|A_{ii}^{-1}\|^{-1}\|B_{ii}^{-1}\|^{-1}\|A_{jj}^{-1}\|^{-1}\|B_{jj}^{-1}\|^{-1}\big]^{1/2}\Big\}. \tag{3.6}$$

Proof. Since **A** and **B** be block H-matrices, we have positive numbers $u_1, \ldots, u_p$ and $v_1, \ldots, v_p$ that satisfies (3.4) and (3.5). Consider $\tilde{A}$ and $\tilde{B}$ and let U and V be positive diagonal matrices such that $U = diag(u_1, u_2, \ldots, u_p)$ and $V = diag(v_1, v_2, \ldots, v_p)$. Then, define $\hat{A} = U^{-1}\tilde{A}U$ and $\hat{B} = V^{-1}\tilde{B}V$ such that

$$\hat{A} = \begin{pmatrix} [1.5]\|A_{11}\| & \frac{u_2}{u_1}\|A_{12}\| & \cdots & \frac{u_p}{u_1}\|A_{1p}\| \\ \frac{u_1}{u_2}\|A_{21}\| & \|A_{22}\| & \cdots & \frac{u_p}{u_2}\|A_{2p}\| \\ \vdots & \vdots & \ddots & \vdots \\ \frac{u_1}{u_p}\|A_{p1}\| & \frac{u_2}{u_p}\|A_{p2}\| & \cdots & \|A_{pp}\| \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} [1.5]\|B_{11}\| & \frac{v_2}{v_1}\|B_{12}\| & \cdots & \frac{v_p}{v_1}\|B_{1p}\| \\ \frac{v_1}{v_2}\|B_{21}\| & \|B_{22}\| & \cdots & \frac{v_p}{v_2}\|B_{2p}\| \\ \vdots & \vdots & \ddots & \vdots \\ \frac{v_1}{v_p}\|B_{p1}\| & \frac{v_2}{v_p}\|B_{p2}\| & \cdots & \|B_{pp}\|. \end{pmatrix}.$$

Note that by Lemma 2.6, we have $\hat{A} \circ \hat{B} = (U^{-1}\tilde{A}U) \circ (V^{-1}\tilde{B}V) = (VU)^{-1}(\tilde{A} \circ \tilde{B})(VU)$. Then by (3.2) and Lemma 2.5,

$$\begin{aligned}\rho(\mathbf{A}\square\mathbf{B}) &\le \rho(\tilde{A}\circ\tilde{B}) = \rho(\hat{A}\circ\hat{B})\\ &\le \max_{i\neq j}\frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\Big[\big(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|\big)^2\\ &+4\sum_{k\neq i}\frac{\|A_{ik}\|u_k}{u_i}\frac{\|B_{ik}\|v_k}{v_i}\sum_{k\neq j}\frac{\|A_{jk}\|u_k}{u_j}\frac{\|B_{jk}\|v_k}{v_j}\Big]^{1/2}\Big\}\\ &\le \max_{i\neq j}\frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\Big[\big(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|\big)^2\\ &+4\sum_{k\neq i}\frac{\|A_{ik}\|u_k}{u_i}\sum_{k\neq i}\frac{\|B_{ik}\|v_k}{v_i}\sum_{k\neq j}\frac{\|A_{jk}\|u_k}{u_j}\sum_{k\neq j}\frac{\|B_{jk}\|v_k}{v_j}\Big]^{1/2}\Big\}\\ &< \max_{i\neq j}\frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\Big[\big(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|\big)^2\\ &+4\|A_{ii}^{-1}\|^{-1}\|B_{ii}^{-1}\|^{-1}\|A_{jj}^{-1}\|^{-1}\|B_{jj}^{-1}\|^{-1}\Big]^{1/2}\Big\}.\end{aligned}$$

□

Remark 3.1. The upper bound in (3.6) is sharper than the bound in (3.3) . Without loss of generality, for $i\neq j$, assume that

$$\|A_{ii}\|\|B_{ii}\|+\|A_{ii}^{-1}\|^{-1}\|B_{ii}^{-1}\|^{-1}\ge\|A_{jj}\|\|B_{jj}\|+\|A_{jj}^{-1}\|^{-1}\|B_{jj}^{-1}\|^{-1}.$$

Then

$$\begin{aligned}&\|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\big[(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|)^2\\ &\qquad\qquad+4\|A_{ii}^{-1}\|^{-1}\|B_{ii}^{-1}\|^{-1}\|A_{jj}^{-1}\|^{-1}\|B_{jj}^{-1}\|^{-1}\big]^{1/2}\\ &\le \|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\big[(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|)^2\\ &\qquad\qquad+4\|A_{ii}^{-1}\|^{-1}\|B_{ii}^{-1}\|^{-1}\big(\|A_{ii}\|\|B_{ii}\|+\|A_{ii}^{-1}\|^{-1}\|B_{ii}^{-1}\|^{-1}-\|A_{jj}\|\|B_{jj}\|\big)\big]^{1/2}\\ &= \|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\big[\big(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|+2\|A_{ii}^{-1}\|^{-1}\|B_{ii}^{-1}\|^{-1}\big)^2\big]^{1/2}\\ &= 2\big(\|A_{ii}\|\|B_{ii}\|+\|A_{ii}^{-1}\|^{-1}\|B_{ii}^{-1}\|^{-1}\big).\end{aligned}$$

Next upper bound for the spectral radius of block Hadamard product improves the bound in (3.6) for block strictly diagonally dominant matrices.

Theorem 3.5. Let **A** and **B** be block H-matrices. Then

$$\begin{aligned}\rho(\mathbf{A}\Box\mathbf{B}) < \min\Big\{ &\max_{i\neq j}\frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\big[(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|)^2 \\ &+4\alpha_i\alpha_j\|B_{ii}^{-1}\|^{-1}\|B_{jj}^{-1}\|^{-1}\big]^{1/2}\Big\}, \\ &\max_{i\neq j}\frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\big[(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|)^2 \\ &+4\beta_i\beta_j\|A_{ii}^{-1}\|^{-1}\|A_{jj}^{-1}\|^{-1}\big]^{1/2}\Big\}\Big\},\end{aligned} \tag{3.7}$$

where $\alpha_i = \max_{k\neq i}\|A_{ik}\|$ and $\beta_i = \max_{k\neq i}\|B_{ik}\|$.

Proof. We first show the first inequality, the second follows in a similar way. Since **B** is block H-matrix, there exists positive numbers $v_1,\ldots,v_p$ such that $V = diag(v_1,\ldots,v_p)$ is positive and $\hat{B} = V^{-1}\tilde{B}V$ is block strictly diagonally dominant where $\tilde{B} = (\|B_{ij}\|)$. Let $\tilde{A} = (\|A_{ij}\|)$. Here we note that $\tilde{A}\circ(V^{-1}\tilde{B}V) = V^{-1}(\tilde{A}\circ\tilde{B})V$ by Lemma 2.6. Then, by (3.2) and Lemma 2.5 we get,

$$\begin{aligned}\rho(\mathbf{A}\Box\mathbf{B}) &\le \rho(\tilde{A}\circ\tilde{B}) = \rho(\tilde{A}\circ\hat{B}) \\ &\le \max_{i\neq j}\frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\Big[\big(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|\big)^2 \\ &+4\sum_{k\neq i}\|A_{ik}\|\frac{\|B_{ik}\|v_k}{v_i}\sum_{k\neq j}\|A_{jk}\|\frac{\|B_{jk}\|v_k}{v_j}\Big]^{1/2}\Big\} \\ &\le \max_{i\neq j}\frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\Big[\big(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|\big)^2 \\ &+4\max_{k\neq i}\|A_{ik}\|\sum_{k\neq i}\frac{\|B_{ik}\|v_k}{v_i}\max_{k\neq j}\|A_{jk}\|\sum_{k\neq j}\frac{\|B_{jk}\|v_k}{v_j}\Big]^{1/2}\Big\} \\ &< \max_{i\neq j}\frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\|+\|A_{jj}\|\|B_{jj}\|+\big[\big(\|A_{ii}\|\|B_{ii}\|-\|A_{jj}\|\|B_{jj}\|\big)^2 \\ &+4\alpha_i\alpha_j\|B_{ii}^{-1}\|^{-1}\|B_{jj}^{-1}\|^{-1}\big]^{1/2}\Big\}\end{aligned}$$

where $\alpha_i = \max_{k\neq i}\|A_{ik}\|$. □

Remark 3.2. In (3.7), if additionally **A** and **B** are block strictly diagonally dominant matrices, the bound in (3.7) is sharper than the one in (3.6) since $\alpha_i = \max_{k\neq i}\|A_{ik}\| \leq \sum_{k\neq i}\|A_{ik}\| < \|A_{ii}^{-1}\|^{-1}$ and $\beta_i = \max_{k\neq i}\|B_{ik}\| \leq \sum_{k\neq i}\|B_{ik}\| < \|B_{ii}^{-1}\|^{-1}$.

Remark 3.3. If only one of the matrices **A** and **B** is block H-matrix, for instance **A** is block H-matrix, then

$$\rho(\mathbf{A}\Box\mathbf{B}) < \max_{i\neq j}\frac{1}{2}\Big\{\|A_{ii}\|\|B_{ii}\| + \|A_{jj}\|\|B_{jj}\| + \big[(\|A_{ii}\|\|B_{ii}\| - \|A_{jj}\|\|B_{jj}\|)^2 + 4\beta_i\beta_j\|A_{ii}^{-1}\|^{-1}\|A_{jj}^{-1}\|^{-1}\big]^{1/2}\Big\}.$$

Lastly, we shall give an upper bound for the spectral radius of block Hadamard product of a class of positive definite matrices. Here, we shall use the notation $|A| \geq |B|$ when $|a_{ij}| \geq |b_{ij}|$ for all i, j.

Definition 3.1. A block matrix **A** is said to be block diagonally dominant of its block column entries (or block row entries) if

$$|A_{ii}| \geq |A_{ji}| \quad (\text{or } |A_{ii}| \geq |A_{ij}|)$$

for each $i = 1,\ldots,p$ and all $j \neq i$. Similarly, **A** is said to be block diagonally subdominant of its block column (or row) entries if the inequalities are reversed.

Lemma 3.6. Let **A** and **B** be block matrices and **D** and **E** are block diagonal matrices. Then, if

i) **A** block commutes with **D**, $\mathbf{D}(\mathbf{A}\Box\mathbf{B})\mathbf{E} = \mathbf{A}\Box(\mathbf{DBE})$,

ii) **B** block commutes with **E**, $\mathbf{D}(\mathbf{A}\Box\mathbf{B})\mathbf{E} = (\mathbf{DAE})\Box\mathbf{B}$.

Theorem 3.7. Let **A**, **B** be nonnegative positive semidefinite matrices. If there exists positive block diagonal matrix **D** such that

(1) $\mathbf{DBD}^{-1}$ is block diagonally dominant of its block column entries and **D** block commutes with either **A** or **B**, then

$$\rho(\mathbf{A}\Box\mathbf{B}) \leq \rho(\mathbf{A})\max_i \rho(B_{ii}).$$

(2) If $\mathbf{DBD}^{-1}$ is block diagonally subdominant of its block column entries and $\mathbf{D}$ block commutes with either $\mathbf{A}$ or $\mathbf{B}$, then

$$\rho(\mathbf{A}) \min_i \rho(B_{ii}) \leq \rho(\mathbf{A}\Box\mathbf{B}).$$

Proof. First note that $\rho(\mathbf{A}\Box\mathbf{B}) = \rho(\mathbf{A}\Box(\mathbf{DBD}^{-1}))$ by Lemma 3.6. Moreover, the diagonal blocks of $\mathbf{B}$ and $\mathbf{DBD}^{-1}$ are the same. We may assume that $\mathbf{B}$ is block diagonally dominant of its column entries since $\mathbf{DBD}^{-1}$ is so. Then,

$$\begin{pmatrix} [1.2]A_{11}B_{11} & A_{12}B_{12} & \cdots & A_{1p}B_{1p} \\ A_{21}B_{21} & A_{22}B_{22} & \cdots & A_{2p}B_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1}B_{p1} & A_{p2}B_{p2} & \cdots & A_{pp}B_{pp} \end{pmatrix} \leq \begin{pmatrix} [1.2]A_{11}B_{11} & A_{12}B_{22} & \cdots & A_{1p}B_{pp} \\ A_{21}B_{11} & A_{22}B_{22} & \cdots & A_{2p}B_{pp} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1}B_{11} & A_{p2}B_{22} & \cdots & A_{pp}B_{pp} \end{pmatrix}$$

$$\leq \begin{pmatrix} [1.2]A_{11}B_{kk} & A_{12}B_{kk} & \cdots & A_{1p}B_{kk} \\ A_{21}B_{kk} & A_{22}B_{kk} & \cdots & A_{2p}B_{kk} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1}B_{kk} & A_{p2}B_{kk} & \cdots & A_{pp}B_{kk} \end{pmatrix}$$

where $B_{kk} \geq B_{ii}$ for all i. In other words,

$$\mathbf{A}\Box\mathbf{B} \leq \mathbf{A}\, diag(B_{11}, \ldots, B_{pp}) \leq \mathbf{A}\, diag(B_{kk}, \ldots, B_{kk}).$$

This implies that

$$\rho(\mathbf{A}\Box\mathbf{B}) \leq \rho(\mathbf{A}\, diag(B_{11}, \ldots, B_{pp})) \leq \rho(\mathbf{A})\rho(B_{kk}),$$

where the last inequality follows from the submultiplicative property of spectral radius of positive semidefinite matrices.

The proof related to block diagonal dominance of block row entries and the second part of the theorem follows similarly. □

4 Examples

In this section, we shall give three examples that compares upper bounds obtained throughout the paper. First two examples demonstrates that upper bound in Theorem 3.4 might be better than the one in Theorem 3.2 or vice versa. On the other hand, the upper bound in Theorem 3.7 is more accurate than the others in some cases for matrices that are block diagonally dominant of its column entries.

Example 4.1. Let A and B be block H-matrices such that

$$A = \begin{pmatrix} 1 & 3 & \vdots & 0 & 1 \\ 2 & 1 & \vdots & 1 & 1 \\ \cdots & \cdots & & \cdots & \cdots \\ 1 & 1 & \vdots & 2 & 2 \\ 2 & 1 & \vdots & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & \vdots & 0 & 0 \\ 1 & 2 & \vdots & 1 & 1 \\ \cdots & \cdots & & \cdots & \cdots \\ 1 & 1 & \vdots & 1 & 2 \\ 0 & 1 & \vdots & 1 & 1 \end{pmatrix}.$$

With respect to ∞-norm, we have

$$\tilde{A} = \begin{pmatrix} 4 & 2 \\ 3 & 4 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},$$

and their spectral radii are $\rho(\tilde{A}) = 6.4495$ and $\rho(\tilde{B}) = 5$. Inverses of block diagonal entries of block matrices are

$$A_{11}^{-1} = \begin{pmatrix} -.2 & .6 \\ .4 & -.2 \end{pmatrix}, \quad A_{22}^{-1} = \begin{pmatrix} -.25 & .5 \\ .75 & -.5 \end{pmatrix}, \quad B_{11}^{-1} = \begin{pmatrix} 1 & 0 \\ -.5 & .5 \end{pmatrix}, \quad B_{22}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Then we have $\rho(A \Box B) = 12.2661$, and $\rho(A)\rho(B) = 20.778$ for which $\rho(A) = 5.6909$ and $\rho(B) = 3.6511$. According to inequalities given in Theorems 3.2-3.5, respectively, we have $\rho(A \Box B) \leq 16.899$, $\rho(A \Box B) < 13.25$, $\rho(A \Box B) < 12.5774$, and $\rho(A \Box B) < 13.4142$.

Example 4.2. Let A and B be H-matrices such that

$$A = \begin{pmatrix} 2 & -1 & \vdots & .5 & 0 \\ -1 & 2 & \vdots & .25 & .25 \\ \cdots & \cdots & & \cdots & \cdots \\ .5 & 0 & \vdots & 2 & -1 \\ .25 & .25 & \vdots & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & \vdots & .125 & .125 \\ 1 & 1 & \vdots & 0 & .25 \\ \cdots & \cdots & & \cdots & \cdots \\ .125 & .125 & \vdots & 2 & 0 \\ 0 & .25 & \vdots & 1 & 1 \end{pmatrix}.$$

Then $\rho(A) = 3.25$, $\rho(B) = 2.25$, $\rho(A \Box B) = 3.0312$ and $\rho(A)\rho(B) = 7.3125$. With respect to ∞-norm, we have

$$\tilde{A} = \begin{pmatrix} 3 & .5 \\ .5 & 3 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 2 & .25 \\ .25 & 2 \end{pmatrix},$$

and their spectral radii are $\rho(\tilde{A}) = 3.5$ and $\rho(\tilde{B}) = 2.25$. Inverses of block diagonal entries of block matrices are

$$A_{11}^{-1} = A_{22}^{-1} = \begin{pmatrix} .6667 & .3333 \\ .3333 & .6667 \end{pmatrix}, \quad B_{11}^{-1} = B_{22}^{-1} = \begin{pmatrix} .5 & 0 \\ -.5 & 1 \end{pmatrix}.$$

With respect to the inequalities given in Theorems 3.2-3.5, we have, respectively, $\rho(A \Box B) \leq 6.125$, $\rho(A \Box B) < 6.6667$, $\rho(A \Box B) < 6.6667$, and $\rho(A \Box B) < 6.5$.

Example 4.3. Let matrices A and B be block diagonally dominant of its column entries such that

$$A = \begin{pmatrix} 3 & 1 & \vdots & 0 & 0 \\ 1 & 2 & \vdots & 0 & 1 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & \vdots & 2 & 2 \\ 0 & 1 & \vdots & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & \vdots & 1 & 0 \\ 1 & 1 & \vdots & 0 & 1 \\ \cdots & \cdots & & \cdots & \cdots \\ 1 & 0 & \vdots & 1 & 1 \\ 0 & 1 & \vdots & 1 & 2 \end{pmatrix}.$$

Then $\rho(A) = 4.8422$, $\rho(B) = 3.5616$, $\rho(A\Box B) = 11.949$ and $\rho(A)\rho(B) = 17.246$. With respect to ∞-norm, we have

$$\tilde{A} = \begin{pmatrix} 4 & 1 \\ 1 & 5 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

and their spectral radii are $\rho(\tilde{A}) = 5.618$ and $\rho(\tilde{B}) = 4$. Inverses of block diagonal entries of block matrices are

$$A_{11}^{-1} = \begin{pmatrix} .4 & -.2 \\ -.2 & .6 \end{pmatrix}, \quad A_{22}^{-1} = \begin{pmatrix} 1.5 & -1 \\ -1 & 1 \end{pmatrix}, \quad B_{11}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_{22}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then we have $\rho(A\Box B) \leq 15.3028$, $\rho(A\Box B) < 15.1333$, $\rho(A\Box B) < 15.0184$, and $\rho(A\Box B) < 15.0366$ according to Theorems 3.2-3.5, respectively.. With respect to Theorem 3.7, $\rho(A\Box B) \leq \rho(A)\rho(B_{11}) = 12.6769$ where $\rho(B_{11}) = \rho(B_{22}) = 2.618$.

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