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Some Results for Error Function with Complex Argument

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Abstract

In this basic research, the well-known error functions with the complex variable and some of their properties are first introduced, then some interesting results concerning error functions are presented. Finally, several conclusions and recommendations for the applications of error functions with complex argument are pointed out..

Keywords: Complex plane, Domains in the complex plane, Functions with the complex variable, Analytic functions, Equations and inequalities in certain domains of the complex plane, The error function with the complex variable

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1 Introduction and Motivation

In the literature, from time to time, we come across the well-known error functions, which have different independent variables (parameters *or* arguments). As is known, the first is the error function with the

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real variable and the second is the error function with the *complex* variable. These functions, which are considered in special functions in mathematics, have important roles both in theory and in practice. From the theory of approximation to probability theory, these functions, which have a wide range of use and are special functions, are also encountered in many areas of technology. As certain references relating to those functions, one may refer to the works in [7,8,12,24,31,34,40,42], for probability and statistics, [35,13,20,34,35,41], for data analysis, [3,4,6,14,18,20,27,31,34,35,38], for heat conduction, [5,13,14,18,20,29, 31,35,37-39], for astronomy, [1,2,5,6,9,10,14,16,19,27,29,34,35,41], for fundamental role in asymptotic expansions, exponential asymptotics and approximation theory, and see also the others given by the references.

In particular, since the error functions with the complex variable and their possible consequences will be closely related to the theory of complex functions, some information and theories related to the theory of the complex function will be very important for this scientific research. In this respect, the research papers [11,17,21-25,33,34] given in the references will be the main source for our main results which will mentioned in this article. Because of this relationship between the theory of complex function and the error functions with the complex argument, we think that this research will be both a theoretical work and an unusual research for the researchers who have been working on related scientific fields. More particularly, as a novel work, the real and imaginary parts of the stated results will provide to the researchers certain implications and/or suggestions for applications in some branches of science. For such results, an interested reader is referred to [1,5,7,13-16,23,31,33,37-44]. In this novel investigation, the well-known assertion, obtained by [32], will be used for the proofs of the main results. See also the result in [21] for that assertion and one may check the special results in [21-26] for certain examples.

2 Definitions and Notations

We now introduce certain definitions and notations that may be necessary for our novel results which consist of certain inequalities and error functions in the complex plane. For those, firstly, we denote by

$$\mathbb{N} \quad , \quad \mathbb{C} \quad \text{and} \quad \mathbb{U}$$

the set of natural numbers, the set of complex numbers and the unit open disk, *i.e.,* the *open* set:

$$\mathbb{U} = \big\{ z : z \in \mathbb{C} \text{ and } |z| < 1 \big\},$$

respectively.

Also let $\mathscr{A}(m)$ be the family of the functions $f(z)$ being analytic and m-valent in the domain $\mathbb{U}$ and also consisting of the form in following Taylor-Maclaurin series:

$$f(z) = a_m z^m + a_{1+m} z^{1+m} + a_{2+m} z^{2+m} + \cdots + a_{k+m} z^{k+m} + \cdots \, , \tag{2.1}$$

where $a_i \in \mathbb{C}$ and $z \in \mathbb{U}$, and, *of course,* $i \in \mathbb{N}$ and $a_m \in \mathbb{C} - \{0\}$.

Next, we recall the special functions, which are known as the error functions with complex variable *or* the complex error functions in the literature.

The *error* function with complex variable (parameter *or* argument) z is denoted by $\mathrm{Erf}(z)$ and also defined by

$$\mathrm{Erf}\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\big(- \xi^2 \big) \, d\xi \tag{2.2}$$

for an arbitrary integration path (from the point 0 to the point z $(z = x + iy)$ in the complex plane.

By means of the well-known Taylor-Maclaurin series expansion of the real-valued function with the (real) variable κ:

$$f(\kappa) = \exp(\kappa) \, : \, \mathbb{R} = (-\infty, \infty) \to \mathbb{R}^+ = (0, \infty)$$

in the following form:

$$\exp(\kappa) = 1 + \frac{\kappa}{1!} + \frac{\kappa^2}{2!} + \cdots + \frac{\kappa^n}{n!} + \cdots, \tag{2.3}$$

the second definition of the *error* function with complex variable $\text{Erf}(z)$, defined by (2), is also defined as the series expansion in the following form:

$$\text{Erf}(z) = \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots + \frac{(-1)^n z^{2n+1}}{n!(2n+1)} + \cdots \right), \tag{2.4}$$

where $z \in \mathbb{C}$.

In view of the definition of the *error* function with complex variable z, defined by (2.2), has the *complementary error* function with complex variable (parameter *or* argument) z (*or* the *complex complementary error* function with the variable z). It is denoted by $\text{Erfc}(z)$ and defined by

$$\text{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp\big(-\xi^2\big)\, d\xi \quad \big(z \in \mathbb{C}\big). \tag{2.5}$$

In the light of the well-known result:

$$\int_0^\infty \exp\big(-\xi^2\big)\, d\xi = \frac{\sqrt{\pi}}{2}$$

and the familiar property:

$$\int_0^\infty \exp\big(-\xi^2\big)\, d\xi = \int_0^z \exp\big(-\xi^2\big)\, d\xi + \int_z^\infty \exp\big(-\xi^2\big)\, d\xi\,,$$

the well-known relation between the related error functions $\text{Erf}(z)$ and $\text{Erfc}(z)$, which is given by

$$\begin{aligned} \text{Erf}(z) &= \frac{2}{\sqrt{\pi}} \int_0^z \exp\big(-\xi^2\big)\, d\xi \\ &= \frac{2}{\sqrt{\pi}} \left(\int_0^\infty \exp\big(-\xi^2\big)\, d\xi - \int_z^\infty \exp\big(-\xi^2\big)\, d\xi \right) \\ &= 1 - \text{Erfc}(z)\,, \end{aligned} \tag{2.6}$$

can be easily obtained, *where* $z \in \mathbb{C}$.

Moreover, with the help of the series expansion of the *complex error* function $\mathrm{Erf}(z)$, given by (2), the second definition of the *complex complementary error* function with the variable z, *i.e.,* $\mathrm{Erfc}(z)$, given by the following-series expansion:

$$\mathrm{Erfc}(z) = 1 - \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots + \frac{(-1)^n z^{2n+1}}{n!(2n+1)} + \cdots\right) \tag{2.7}$$

can be also obtained, *where* $z \in \mathbb{C}$.

With a simple derivation, it is clearly seen that both the complex series in (2) and (5) are uniformly convergent on any region of the set $\mathbb{C}$. For both this and the fundamental-theoretical details of the related complex functions, as we indicated before, one can check the books given in [11], [17], [23], [31] and also [24] and [33].

In addition, we know that there are a number of important connections in relation with both the related error functions and the other special functions. For the details of them, it can be looked over one should look at the works in [5,14,15,22,23,28,30,31,34,37,38,42,43]. For the scope of this research, we think that it will be useful to emphasize only some of the relevant special relations *or* results. The following relations are easily established:

$$\mathrm{Erf}\,(-z) = -\mathrm{Erf}\,(z)\,, \tag{2.8}$$

$$\mathrm{Erf}\,(\bar{z}) = \overline{\mathrm{Erf}\,(z)}\,, \tag{2.9}$$

$$\mathrm{Erf}\,(z) = 1 - \mathrm{Erfc}\,(z)\,, \tag{2.10}$$

$$\frac{d}{dz}\Big(\mathrm{Erf}\,(z)\Big) = \frac{2}{\sqrt{\pi}}\,z\exp\left(-z^2\right), \tag{2.11}$$

$$\int \mathrm{Erf}\,(z)\,dz = z\,\mathrm{Erf}\,(z) + \frac{1}{\sqrt{\pi}}\exp\left(-z^2\right), \tag{2.12}$$

$$\sqrt{\pi}\,\mathrm{Erf}\,(z) = 2z\,{}_1F_1\left(1/2;3/2;-z^2\right), \tag{2.13}$$

and

$$\sqrt{\pi}\,\mathrm{Erf}\,(z) = \Gamma\left(1/2,z^2\right) \quad \left(\Re e(z) > 0\right) \tag{2.14}$$

and

$$\sqrt{\pi}\,\mathrm{Erf}\,(z) = 2\,z^2\,e^{-z^2}\,{}_1F_1\big(1;3/2;z^2\big)\,, \tag{2.15}$$

where the well-known-special functions:

$${}_1F_1\big(a;b;z\big) \quad \text{and} \quad \Gamma\big(a,z\big)$$

are the *confluent hypergeometric function of the first kind* and *incomplete gamma function*, respectively. As we said before, for more special results and also properties between the complex error functions and also other special functions, see the references in [12,15,24,28,30,31,34,36,39, 43].

3 A Lemma and Main Results

In this section, we will present our main results pertaining to the complex error functions, as well as the proofs of those and two lemmas, which will be used in the related proofs. We first begin by introducing the lemmas.

Lemma 1 ([32]). Let a function $w(z)$ be in the family $\mathscr{A}(m)$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at the point z_0, then

$$z\,w'(z)\big|_{z=z_0} = \kappa\,w(z)\big|_{z=z_0}\,, \tag{3.1}$$

where $\kappa \in \mathbb{R}^+$ with $\kappa \geq m$.

Noting that, by taking $m := 1$ in the Lemma mentioned above, one obtains the well-known Jack's lemma (see [26]).

We now state and prove our main results dealing with the complex error function $\mathrm{Erf}\,(z)$, defined by (2).

Theorem 1. If the complex error function $\mathrm{Erf}(z)$, given as in (4) (*or* in (2)), satisfies the inequality:

$$\sqrt{\pi}\left|z\mathrm{Erf}'\,(z) + \mathrm{Erf}\,(z)\right| < 4\rho\,, \tag{3.2}$$

then

$$\sqrt{\pi}\left|\mathrm{Erf}\,(z)\right| < 2\rho \tag{3.3}$$

for any z in the disk $\mathbb{U}$ and for some ρ $(\rho > 0)$.

Proof. By means of the complex error function $\mathrm{Erf}(z)$, given as in (4), we define a function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{\sqrt{\pi}}{2}\,\mathrm{Erf}\,(z) \qquad (3.4) \\ &= z - \frac{1}{3}z^3 + \frac{1}{10}z^5 - \frac{1}{42}z^7 + \cdots \quad \big(z \in \mathbb{U}\big). \end{aligned}$$

Then, it is clear that the function $w(z)$ above is in $\mathscr{A}(1)$, analytic in $\mathbb{U}$ and $w(0) = 0$. It follows from (19) that

$$w'(z) = \frac{\sqrt{\pi}}{2}\,\mathrm{Erf}'\,(z) \quad (z \in \mathbb{U}). \tag{3.5}$$

By combining (19) and (20), the result:

$$w(z) + zw'(z) = \frac{\sqrt{\pi}}{2}\Big(\mathrm{Erf}\,(z) + z\mathrm{Erf}'\,(z)\Big) \quad (z \in \mathbb{U}). \tag{3.6}$$

is also obtained.

Suppose next that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max\Big\{\big|w(z)\big| : |z| \leq |z_0| \;\big(z, z_0 \in \mathbb{U}\big)\Big\} = |w(z_0)| = \rho \;\; (\rho > 0).$$

By applying the hypothesis above and also the assertion (16) of Lemma 1, we then find from (21) that

$$\frac{\sqrt{\pi}}{2}\left|\mathrm{Erf}\,(z) + z\mathrm{Erf}'\,(z)\right| = \big|w(z_0)\big|\big(1+\kappa\big) = \rho\big(1+\kappa\big) \geq 2\rho\,,$$

which is in contradiction with the condition (17) of Theorem 1. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all z in $\mathbb{U}$. So that the function, defined by (19), follows that

$$\big|w(z)\big| = \left|\frac{\sqrt{\pi}}{2}\,\mathrm{Erf}\,(z)\right| = \frac{\sqrt{\pi}}{2}\left|\mathrm{Erf}\,(z)\right| < \rho$$

for all $z \in \mathbb{U}$ and for some $\rho > 0$. This completes the desired proof.

Theorem 2. If the complex error function Erf (z), is defined as in (4) (*or* in (2)), and it satisfies the inequality:

$$\left| \sqrt{\pi}\,\mathrm{Erf}'(z) - 2 \right| < 6\rho\,, \tag{3.7}$$

then

$$\left| \sqrt{\pi}\,\mathrm{Erf}(z) - 2z \right| < 2\rho|z| \tag{3.8}$$

for some ρ $(\rho > 0)$ and for any $z \in \mathbb{U}$.

Proof. By the help of the complex error function Erf (z) given as in the form (4), let us define a function $w(z)$ in the form:

$$\begin{aligned} w(z) &= \frac{\sqrt{\pi}}{2}\,\frac{\mathrm{Erf}(z)}{z} - 1 \qquad (3.9)\\ &= -\frac{1}{3}z^2 + \frac{1}{10}z^4 - \frac{1}{42}z^6 + \cdots \quad \big(z \in \mathbb{U}\big). \end{aligned}$$

Then, it is easy seen that $w(z) \in \mathscr{A}(2)$ and it is analytic in $\mathbb{U}$ and also satisfies the condition of Lemma 1, which is $w(0) = 0$. In view of the steps used in the proof of Theorem 1, the proof of Theorem 2 can be then obtained. The detail is here omitted.

Theorem 3. If the complex error function Erf (z), is given as in (4) (*or* in (2)) and satisfies the inequality:

$$\left| \sqrt{\pi}\left(z\mathrm{Erf}''(z) + \mathrm{Erf}'(z)\right) - 2 \right| < 6\rho\,, \tag{3.10}$$

then

$$\left| \sqrt{\pi}\,\mathrm{Erf}'(z) - 2 \right| < 2\rho \tag{3.11}$$

for some ρ $(\rho > 0)$ and for any $z \in \mathbb{U}$.

Proof. Under favour of the complex error function $\mathrm{Erf}\,(z)$ given as in the form (4), let a function $w(z)$ be in the form:

$$w(z) = \frac{\sqrt{\pi}}{2}\,\mathrm{Erf}'\,(z) - 1 \tag{3.12}$$
$$= -z^2 + \frac{1}{2}z^4 - \frac{1}{6}z^6 + \cdots \quad \big(z \in \mathbb{U}\big).$$

Then, $w(z)$ belongs to the class $\in \mathscr{A}(2)$, is analytic in $\mathbb{U}$ and also satisfies the condition of Lemma 1, $w(0) = 0$. In consideration of the steps used in the proof of Theorem 1, the proof of Theorem 2 can be easy obtained. The detail is again omitted.

Theorem 4. If the complex error function $\mathrm{Erf}\,(z)$, is defined as in (4) (*or* in (2)) and satisfies the inequality:

$$\sqrt{\pi}\left|z^2\,\mathrm{Erf}''\,(z) - 2z\,\mathrm{Erf}'\,(z) + 2\mathrm{Erf}\,(z)\right| < 2\rho|z|^2, \tag{3.13}$$

then

$$\sqrt{\pi}\left|z\,\mathrm{Erf}'\,(z) - \mathrm{Erf}\,(z)\right| < 2\rho|z|^2 \tag{3.14}$$

for some ρ $(\rho > 0)$ and for any z in $\mathbb{U}$.

Proof. With the help of the complex error function $\mathrm{Erf}\,(z)$ given as in the form (4), we also define a function $w(z)$ in the form:

$$w(z) = \frac{\sqrt{\pi}}{2}\left(\frac{\mathrm{Erf}\,(z)}{z}\right)' \tag{3.15}$$
$$= -\frac{2}{3}z + \frac{2}{5}z^3 - \frac{1}{7}z^5 + \cdots \quad \big(z \in \mathbb{U}\big).$$

Then, the function $w(z)$ is in the class $\mathscr{A}(1)$, analytic in $\mathbb{U}$ and also satisfies the related condition of Lemma 1, which is $w(0) = 0$. In the light of the steps used in the proof of Theorem 1, the proof can be easily constituted. The detail is here omitted.

4 Concluding Remarks

By focusing on the information concerning the complex error functions given in (7) through (15) in the first section, using all the main results obtained in the second section and considering real *or* imaginary parts of all possible results, which were obtained in the related sections, a large number of possible implications crelating to the complex error functions can be also obtained. Some of the results may also be useful to be considered for application in the papers in [1,2,16,19,24,27,30,40-44] for the approach theory. We present only two instances below:

(i) By means of the well-known properties presented in (8)-(15) and considering all main results obtained in the second chapter, it can redetermine several new results like all theorems (Theorems 1-4). Let us exemplify these with two examples.

By using the property given by (10) and taking into the related theorems, one can easy obtain several results for the complex complementary error function defined as in (7). For example, by letting

$$\mathrm{Erf}(z) := 1 - \mathrm{Erfc}(z)$$

in the Theorem 1, the following proposition can be first revealed.

Proposition 1. For the complementary complex error function $\mathrm{Erfc}(z)$, given as in (7) (*or* in (5)), the following implications:

$$\sqrt{\pi}\left|z\mathrm{Erfc}'(z) + \mathrm{Erfc}(z) - 1\right| < 4\rho$$

$$\Rightarrow \sqrt{\pi}\left|\mathrm{Erfc}(z) - 1\right| < 2\rho$$

$$\Rightarrow \sqrt{\pi} - 2\rho \leq \sqrt{\pi}\,\Re e\Big(\mathrm{Erfc}(z)\Big) \leq \sqrt{\pi} + 2\rho$$

are satisfied for any z in $\mathbb{U}$ and for some ρ $(\rho > 0)$.

By using the property given by (13) and taking into account the related theorems, one can also obtain several results for the confluent hypergeometric function of the first kind. For example, through the instrumentality of the well-known property:

$$\frac{d}{dz}\left({}_1F_1\left(\alpha;\gamma;z\right)\right)=\frac{\alpha}{\gamma}\,{}_1F_1\left(1+\alpha;1+\gamma;z\right)$$

$$\left(\alpha,\gamma\neq 0,-1,-2,-3,\cdots\right)$$

and by setting

$$\mathrm{Erf}\left(z\right):=\frac{2}{\sqrt{\pi}}\,z\,{}_1F_1\left(1/2;3/2;-z^2\right)$$

in the theorem 2, the following proposition can be next revealed.

Proposition 2. For any z belonging to $\mathbb{U}$ and for some ρ $(\rho>0)$, if the following inequality:

$$\left|3\,{}_1F_1\left(1/2;3/2;-z^2\right)+z\,{}_1F_1\left(3/2;5/2;-z^2\right)-3\right|<9\rho$$

is satisfied, then the inequality:

$$\left|{}_1F_1\left(1/2;3/2;-z^2\right)-1\right|<\rho\quad\left(\rho>0;z\in\mathbb{U}\right)$$

is satisfied.

(ii) By taking into all theorems (Theorems 1-4) and the well-known properties in the complex plane, several corollaries can be also deduced. For instance, in view of Theorem 1 and by using the well-known *reverse triangle inequality,* the following proposition can be easily presented.

Proposition 3. For the complex error function $\mathrm{Erf}(z)$, given as in (4) (*or* in (2)), the following implication:

$$\sqrt{\pi}\left|\,|z|\left|\mathrm{Erf}'(z)\right|-\left|\mathrm{Erf}(z)\right|\,\right|<4\rho\;\Rightarrow\;\sqrt{\pi}\left|\mathrm{Erf}(z)\right|<2\rho$$

holds for any z in $\mathbb{U}$ and for some ρ $(\rho>0)$.

Finally, some possible applications of the main results and also their applications to a great number of numerical results analysis in 2D or 3D can be determined or discussed by the researchers interested in error functions (in the complex plane) . All possible investigations and their analyses, omitted here, are left to the relevant researchers. For some numerical results and their implications, in particular, it will be sufficient to take into account the computational results in the papers [1,2,36] in the references.

References

1. S. M. Abrarov and B. M. Quine, Accurate approximations for the complex error function with small imaginary argument, *J. Math. Research* **7** (1) (2015) 44-53

2. S. M. Abrarov and B. M. Quine, A rational approximation of the Dawsons integral for efficient computation of the complex error function, *Appl. Math. Comput.* **321** (2018) 526543

3. H. Alzer, Functional inequalities for the error function II, *Aequationes Math.* **78** (12) (2009) 113121.

4. H. Alzer, Error function inequalities, *Adv. Comput. Math.* **33** (3) (2010) 349379.

5. B. H. Armstrong and B.W. Nicholls, Emission, absorption and transfer of radiation in heated atmospheres, Pergamon Press, New York, 1972.

6. M. V. Berry, Uniform asymptotic smoothing of Stokess discontinuities, *Proc. Roy. Soc. London Ser. A* **1989**, 422 (1862), 7-21.

7. L. Carlitz, The inverse of the error function, *Pacific J. Math.* **13** (1963) 459470.

8. C. Chiarella and A. Reichel, On the evaluation of integrals related to the error function, *Math. Comp.* **22**, (1968), 137-143.

9. M. A. Chuev, An efficient method of analysis of the hyperfine structure of gamma resonance spectra using the Voigt profile, *Doklady Phys.* **56** (2011) 318-322.

10. W.J. Cody, K.A. Paciorek and H.C. Thacher, Chebyshev approximations for Dawsons integral, Math. Comput. **24** (1970) 171178.

11. P. L. Duren, *Univalent Functions,* Grundlehren der Mathematishen Wissenschaften 259, Springer-Verlag, 1983

12. D.E. Dominici, The inverse of the cumulative standard normal probability function, *Integral Transforms Spec. Funct.* **14** (4) (2003) 281292.

13. D. Emerson, *Interpreting astronomical spectra,* John Wiley & Sons Ltd, 1996.

14. A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions,* McGraw-Hill Book Co., Inc., 1953.

15. V. N. Faddeyeva and N. M. Terentev, *Tables of the probability integral* $w(z) = e^{-z^2}\left(1 + 2i/\sqrt{\pi}\int_0^z e^{-t^2}dt\right)$ *for complex argument,* Pergamon Press, Oxford, 1961

16. W. Gautschi, Efficient computation of the complex error function, *SIAM J. Numer. Anal.* **7** (1970) 187-198.

17. A. W. Goodman, *Univalent Functions.* Vols. I and II, Polygonal Publishing Company, 1983.

18. G. V. Gordeyev, Plasma oscillations in a magnetic Field, *J. Exp. Theor. Phys.* **6** (1952) 660-669.

19. R. G. Hart, A close approximation related to the error function, *Math. Comp.* **20** (1966) 600602.

20. G. Herden, *The role of error-functions in order to obtain relatively optimal classification, Classification and related methods of data analysis,* North-Holland, Amsterdam, 1988.

21. H. Irmak, Some novel applications in relation with certain equations and inequalities in the complex plane, *Math. Commun.* **23**, 1 (2018), 9-14.

22. H. Irmak and P. Agarwal, Some comprehensive inequalities consisting of Mittag-Leffler type functions in the Complex Plane, *Math. Model. Nat. Phenom.* **12**, 3 (2017), 65-71.

23. H. Irmak and P. Agarwal, Comprehensive Inequalities and Equations Specified by the Mittag-Leffler Functions and Fractional Calculus in the Complex Plane. In: Agarwal P., Dragomir S., Jleli M., Samet B. (eds) Advances in Mathematical Inequalities and Applications. Trends in Mathematics. Birkhuser, Singapore, 2018.

24. H. Irmak, Various results in relation with the hypergeometric equations and the hypergeometric functions in thecomplex plane, Stud. Univ. Babeş-Bolyai Math. (Accepted).

25. H. Irmak, Ö. F. Çetin, Some theorems involving inequalities on p-valent functions, *Turkish J. Math.* **23** (3) (1999), 453-459.

26. I. S. Jack, Functions starlike and convex of order α, J. London Math. Soc. **3** (1971), 469-474.

27. J. C. Jaeger, Some applications of the repeated integrals of the error function, *Quart. Appl. Math.* **4** (1946) 100-103.

28. F. Matta and A. Reichel, Uniform computation of the error function and other related functions, *Math. Comp.* **25** (1971) 339-344.

29. J. H. McCabe, A continued fraction expansion with a truncation error estimate for Dawson's integral, *Math. Comp.* **127** (1974) 811-816.

30. S. J. McKenna, A method of computing the complex probability function and other related functions over the whole complex plane, *Astrophys. Space Sci.* **107** (1) (1984) 71-83.

31. M. Mori, A method for evaluation of the error function of real and complex variable with high relative cccuracy, *Publ. RIMS, Kyoto Univ.* **19** (1983) 1081-1094

32. S. S. Miller and P. T. Mocanu, Second-order differential inequalities in the complex plane, *J. Math. Anal. Appl.* **65** (1978) 289-305.

33. Z. Nehari, Conformal mapping, McGraw-Hill Book Company, New York, Toronto, and London, 1952.

34. F. W. J. Oliver, D.W. Lozier, R. F. Boisvert and C. W. Clark, *NIST Handbook of Mathematical Functions,* Cambridge University Press, 2010.

35. F. W. J. Olver, *Asymptotics and special functions,* AKP Classics, A K Peters Ltd., Wellesley, MA, 1997.

36. G. P. M. Poppe and C. M. J. Wijers, More efficient computation of the complex error function, *ACM Transact. Math. Software* **16**, (1990), 38-46.

37. A. Rc, Error estimates in simple quadrature with Voigt functions, *Math. Comp.* **21**, (1967), 647-651.

38. G. B. Rybicki, Dawson's integral and the sampling theorem, *Comp. Phys.* **3**, (1989), 85-87.

39. F. Schreier, The Voigt and complex error function: A comparison of computational methods, *J. Quant. Spectrosc. Radiat. Transfer* **48** (4-5)(1992) 743-762.

40. H. Takahasi and M. Mori, Error estimation in the numerical integration of analytic functions, *Rep. Comput. Centre Univ. Tokyo* **3** (1970) 41-108.

41. C. Tellambura and A. Annamalai, Efficient computation of erfc(x) for Large Arguments, *IEEE Trans. Commun.* **48** (2000) 529-532.

42. H. Wallman, Transient response and the central limit theorem of probability, *Proc. Symposia Appl. Math.* **2**, (1950), 91-105.

43. J.A.C Weideman, Computation of the complex error function, *SIAM J. Numer. Anal.* **31** (5) (1995), 1497-1518.

44. Z.-H. Yang and Y.-M. Chu, On approximating the error function, *J. Inequal. Appl.* **2016** 2016, Article 311, 17 pages.