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On Exponentially Convex Functions

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Abstract

In this paper, we define and introduce some new concepts of the exponentially convex functions. We investigate several properties of the exponentially convex functions and discuss their relations with convex functions. Optimality conditions are characterized by a class of variational inequalities. Several interesting results characterizing the exponentially convex functions are obtained. Results obtained in this paper can be viewed as significant improvement of previously known results

1 Introduction

Convex functions and convex sets have played an important and fundamental part in the development of various fields of pure and applied sciences. Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. Some of these developments have made mutually enriching contacts with other fields. Ideas explaining these concepts led to the developments of new and powerful techniques to solve a wide class of linear and nonlinear problems. The development of convexity theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts

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on the qualitative behaviour of solutions (regarding its existence, uniqueness and regularity) to important classes of problems; on the other hand, it also enables us to develop highly efficient and powerful new numerical methods to solve nonlinear problems, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In recent years, various extensions and generalizations of convex functions and convex sets have been considered and studied using innovative ideas and techniques. It is known that more accurate and inequalities can be obtained using the logarithmically convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions, the origin of exponentially convex functions can be traced back to Bernstein [6]. Avriel [4] introduced and studied the concept of r-convex functions. For further properties of the r-convex functions, see Zhao et al[24] and the references therein. which have important applications in information theory, big data analysis, machine learning and statistic. See, for example, [2, 3, 18, 19, 21, 22, 23, 24] and the references therein.

Motivated and inspired by the ongoing research in this interesting, applicable and dynamic field, we again consider the concept of exponentially convex functions. We discuss the basic properties of the exponentially convex functions. It is has been shown that the exponentially convex(concave) have nice nice properties which convex functions enjoy. Several new concepts have been introduced and investigated. We show that the local minimum of the exponentially convex functions is the global minimum. The optimal conditions of the differentiable exponentially convex functions can be characterized by a class of variational inequalities, which is itself an interesting outcome of our main results. The difference (sum) of the exponentially convex function and exponentially affine convex function is again a exponentially convex function. The ideas and techniques of this paper may be starting point for further research in these areas.

2 Preliminary Results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively. Let $F: K \to R$ be a continuous function.

Definition 2.1. [7].*The set K in H is said to be convex set, if*

$$u + t(v-u) \in K, \qquad \forall u, v \in K, t \in [0,1].$$

Definition 2.2. *A function F is said to be convex, if*

$$F((1-t)u+tv) \leq (1-t)F(u)+tF(v), \quad \forall u,v \in K, \quad t \in [0,1]. \tag{2.1}$$

We now consider the concept of the exponentially convex function, which is mainly due to Noor and Noor [14, 15] and Rashid et al[21, 22] as:

Definition 2.3. *[3] A function F is said to be exponentially convex function, if*

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)}+te^{F(v)}, \quad \forall u,v \in K, \quad t \in [0,1].$$

We remark that Definition 2.5 can be rewritten in the following equivalent way, which is due to Antczak [3].

Definition 2.4. *A function F is said to be exponentially convex function, if*

$$F((1-t)a+tb) \leq \log[(1-t)e^{F(a)}+te^{F(b)}], \quad \forall a,b \in K, \quad t \in [0,1]. \tag{2.2}$$

A function is called the exponentially concave function f, if $-f$ is exponentially convex function. It is obvious that two concepts are equivalent. This equivalent have been used to discuss various aspects of the exponentially convex functions. It is worth mentioning that one can also deduce the concept of exponentially convex functions from r-convex functions, which were considered by Avriel [4] and Bernstein [6].

For the applications of the exponentially convex functions in the mathematical programming and information theory, see Antczak [3],Alirezaei and Mathar[2] and Pal et al[23].. For the applications of the exponentially concave function in the communication and information theory, we have the following example.

Example [2]: The error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

becomes an exponentially concave function in the form $erf(\sqrt{x}), \quad x \geq 0$, which describes the bit/symbol erorr probability of communication systems depending on the square root of the underlying signal-to-noise ratio. This shows that the exponentially concave functions can play important part in communication theory and information theory.

Definition 2.5. *[3] A function F is said to be exponentially affine convex function, if*

$$e^{F((1-t)u+tv)} = (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u,v \in K, \quad t \in [0,1].$$

Definition 2.6. *The function F on the convex set K is said to be exponentially quasi convex, if*

$$e^{F(u+t(v-u))} \leq \max\{e^{F(u)}, e^{F(v)}\}, \quad \forall u,v \in K, t \in [0,1].$$

From the above definitions, we have

$$\begin{aligned} e^{F(u+t(v-u))} &\leq (1-t)e^{F(u)} + te^{F(v)}) \\ &\leq \max\{e^{F(u)}, e^{F(v)}\}. \end{aligned}$$

This shows that every exponentially convex function is a exponentially quasi-convex function. However, the converse is not true.

Let $K = I = [a,b]$ be the interval. We now define the exponentially convex functions on I.

Definition 2.7. *Let $I = [a,b]$. Then F is exponentially convex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & b \\ e^{F(a)} & e^{F(x)} & e^{F(b)} \end{vmatrix} \geq 0; \quad a \leq x \leq b.$$

One can easily show that the following are equivalent:

1. F is exponentially convex function.

2. $e^{F(x)} \leq e^{F(a)} + \frac{e^{F(b)}-e^{F(a)}}{b-a}(x-a)$.

3. $\frac{e^{F(x)}-e^{F(a)}}{x-a} \leq \frac{e^{F(b)}-e^{F(a)}}{b-a}$.

4. $(b-x)e^{F(a)} + (a-b)e^{F(x)} + (x-a)e^{F(b)}) \geq 0$.

5. $\frac{F(a)}{(b-a)(a-x)} + \frac{e^{F(x)}}{(x-b)(a-x)} + \frac{e^{F(b}}{(b-a)(x-b)} \leq 0,$

where $x = (1-t)a + tb \in [0,1]$.

3 Main Results

In this section, we consider some basic properties of generalized strongly convex functions.

Theorem 3.1. *Let F be a strictly exponentially convex function. Then any local minimum of F is a global minimum.*

Proof. Let the exponentially convex function F have a local minimum at $u \in K$. Assume the contrary, that is, $F(v) < F(u)$ for some $v \in K$. Since F is exponentially convex, so

$$e^{F(u+t(v-u))} < te^{F(v)} + (1-t)e^{F(u)}, \quad \text{for} \quad 0 < t < 1.$$

Thus

$$e^{F(u+t(v-u))} - e^{F(u)} < t[e^{F(v)} - e^{F(u)}] < 0,$$

from which it follows that

$$e^{F(u+t(v-u))} < e^{F(u)},$$

for arbitrary small $t > 0$, contradicting the local minimum. $\square$

Theorem 3.2. *If the function F on the convex set K is exponentially convex, then the level set $L_\alpha = \{u \in K : e^{F(u)} \leq \alpha, \quad \alpha \in R\}$ is a convex set.*

Proof. Let $u, v \in L_\alpha$. Then $e^{F(u)} \leq \alpha$ and $e^{F(v)} \leq \alpha$. Now, $\forall t \in (0,1), \quad w = v + t(u-v) \in K$, since K is a convex set. Thus, by the exponentially convexity of F, we have

$$\begin{aligned} Fe^{(v+t(u-v))} &\leq (1-t)e^{F(v)} + te^{F(u)} \\ &\leq (1-t)\alpha + t\alpha = \alpha, \end{aligned}$$

from which it follows that $v + t(u-v) \in L_\alpha$ Hence L_α is convex set. $\square$

Theorem 3.3. *The function F is exponentially convex, if and only if*

$$epi(F) = \{(u, \alpha) : u \in K : e^{F(u)} \leq \alpha, \alpha \in R\}$$

is a convex set.

Proof. Assume that$F$ is exponentially convex. Let $(u, \alpha), \quad (v, \beta) \in epi(F)$. Then it follows that $e^{F(u)} \leq \alpha$ and $e^{F(v)} \leq \beta$. Thus, $\forall t \in [0,1], \quad u, v \in K$, we have

$$\begin{aligned} e^{F(u+t(v-u))} &\leq (1-t)e^{F(u)} + te^{F(v)} \\ &\leq (1-t)\alpha + t\beta, \end{aligned}$$

which implies that

$$(u + t(v-u), (1-t)\alpha + t\beta) \in epi(F).$$

Thus $epi(F)$ is a convex set. Conversely, let $epi(F)$ be a convex set. Let $u, v \in K$. Then $(u, e^{F(u)}) \in epi(F)$ and $(v, e^{F(v)}) \in epi(F)$. Since $epi(F)$ is a convex set, we must have

$$(u+t(v-u), (1-t)e^{F(u)} + te^{F(v))} \in epi(F),$$

which implies that

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(u)}.$$

This shows that F is exponentially convex function. □

Theorem 3.4. *The function F is exponentially quasi convex, if and only if, the level set $L_\alpha = \{u \in K, \alpha \in R : e^{F(u)} \leq \alpha\}$ is a convex set.*

Proof. Let $u, v \in L_\alpha$. Then $u, v \in K$ and $\max(e^{F(u)}, e^{F(v)}) \leq \alpha$. Now for $t \in (0,1), w = u + t(v-u) \in K$, We have to prove that $u + t(v-u) \in L_\alpha$. By the exponentially quasi convexity of F, we have

$$e^{F(u+t(v-u))} \leq \max\left(e^{F(u)}, e^{F(v)}\right) \leq \alpha,$$

which implies that $u + t(v-u) \in L_\alpha$, showing that the level set L_α is indeed a convex set.

Conversely, assume that L_α is a convex set. Then for any $u, v \in L_\alpha, t \in [0,1]$, $u + t(v-u) \in L_\alpha$. Let $u, v \in L_\alpha$ for

$$\alpha = max(e^{F(u)}, e^{F(v)}) \quad \text{and} \quad e^{F(v)} \leq e^{F(u)}.$$

Then from the definition of the level set L_α, it follows that

$$e^{F(u+t\ (v,u))} \leq \max\left(e^{F(u)}, e^{F(v)}\right) \leq \alpha.$$

Thus F is an exponentially quasi convex function. This completes the proof. □

Theorem 3.5. *Let F be an exponentially convex function.. Let $\mu = \inf_{u \in K} F(u)$. Then the set $E = \{u \in K : e^{F(u)} = \mu\}$ is a convex set of K. If F is strictly exponentially , then E is a singleton.*

Proof. Let $u,v \in E$. For $0<t<1$, let $w=u+t(v-u)$. Since F is a exponentially convex function, then

$$F(w)=e^{F(u+t(v-u))} \leq (1-t)e^{F(u)}+te^{F(v)}$$
$$= t\mu+(1-t)\mu=\mu,$$

which implies that to $w\in E$. and hence E is a convex set. For the second part, assume to the contrary that $F(u)=F(v)=\mu$. Since K is a convex set, then for $0<t<1, u+t(v-u)\in K$. Further, since F is strictly exponentially convex,

$$e^{F(u+t(v-u))} < (1-t)e^{F(u)}+te^{F(v)}$$
$$= (1-t)\mu+t\mu=\mu.$$

This contradicts the fact that $\mu=\inf_{u\in K}F(u)$ and hence the result follows. □

Theorem 3.6. *If F is an exponentially convex function such that*
$e^{F(v)}<e^{F(u)}, \forall u,v\in K$, *then F is a strictly exponentially quasi convex function.*

Proof. By the exponentially convexity of the function F,
$\forall u,v\in K, t\in[0,1]$, we have

$$e^{F(u+t(v-u))}\leq(1-t)e^{F(u)}+te^{F(v)}<e^{F}(u),$$

since $e^{F(v)}<e^{F(u)}$, which shows that the function F is strictly exponentially quasi convex. □

We now derive some properties of the differentiable exponentially convex functions.

Theorem 3.7. *Let F be a differentiable function on the convex set K. Then the function F is exponentially convex function, if and only if,*

$$e^{F(v)}-e^{F(u)})\geq\langle e^{F(u)}F'(u),v-u\rangle,\quad \forall v,u\in K. \tag{3.1}$$

Proof. Let F be a exponentially convex function. Then

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)}, \qquad \forall u,v \in K,$$

which can be written as

$$e^{F(v)} - e^{F(u)} \geq \{\frac{e^{F(u+t(v-u))} - e^{F(u)}}{t}\}.$$

Taking the limit in the above inequality as $t \to 0$, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u), v-u)\rangle,$$

which is (3.1), the required result.

Conversely, let (3.1) hold. Then

$\forall u,v \in K, t \in [0,1]$, $v_t = u + t(v-u) \in K$, we have

$$\begin{aligned} e^{F(v)} - e^{F(v_t)} &\geq \langle e^{F(v_t)}F'(v_t), v - v_t)\rangle \\ &= (1-t)\langle e^{F(v_t)}F'(v_t), v-u\rangle. \end{aligned} \tag{3.2}$$

In a similar way, we have

$$\begin{aligned} e^{F(u)} - e^{F(v_t)} &\geq \langle e^{F(v_t)}F'(v_t), u - v_t)\rangle \\ &= -t\langle e^{F(v_t)}F'(v_t), v-u\rangle. \end{aligned} \tag{3.3}$$

Multiplying (3.2) by t and (3.3) by $(1-t)$ and adding the resultant, we have

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)},$$

showing that F is a exponentially convex function. □

Remark 3.1. *From* (3.1)*, we have*

$$e^{F(v)-F(u)} - 1 \geq \langle F'(u), v-u\rangle, \quad \forall v,u \in K,$$

which can be written as

$$F(v) - F(u) \geq \log\{1 + \langle F'(u), v-u\rangle\} \quad \forall v,u \in K, \tag{3.4}$$

Changing the role of u and v in (3.4)*, we also have*

$$F(u) - F(v) \geq \log\{1 + \langle F'(v), u-v\rangle\} \quad \forall v,u \in K, \tag{3.5}$$

Adding (3.4) *and* (3.5)*, we have*

$$\langle F'(u) - F'(v), u-v\rangle \geq (\langle F'(u)u-v\rangle)(\langle F'(v), u-v\rangle)$$

which express the monotonicity of the differential $F'(.)$ of the exponentially convex function.

Theorem 3.7 enables us to introduce the concept of the exponentially monotone operators, which appears to be new ones.

Definition 3.1. *The differential $F'(.)$ is said to be exponentially monotone, if*

$$\langle e^{F(u)}F'(u) - e^{F(v)}F'(v), u-v\rangle \geq 0, \quad \forall u,v \in H.$$

Definition 3.2. *The differential $F'(.)$ is said to be exponentially pseudo-monotone, if*

$$\langle e^{F(u)}F'(u), v-u\rangle \geq 0, \quad \Rightarrow \langle e^{F(v)}F'(v), v-u\rangle \geq 0, \quad \forall u,v \in H.$$

From these definitions, it follows that exponentially monotonicity implies exponentially pseudo-monotonicity, but the converse is not true.

Theorem 3.8. *Let F be differentiable on the convex set K. Then* (3.1) *holds, if and only if, F' satisfies*

$$\langle e^{F(u)}F'(u) - e^{F(v)}F'(v), u-v\rangle \geq 0, \quad \forall u,v \in K. \tag{3.6}$$

Proof. Let F be a exponentially convex function on the convex set K. Then, from Theorem 3.1, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u), v-u\rangle, \quad \forall u,v \in K. \tag{3.7}$$

Changing the role of u and v in (3.7), we have

$$e^{F(u)} - e^{F(v)} \geq \langle e^{F(v)}F'(v), u-v)\rangle, \quad \forall u,v \in K. \tag{3.8}$$

Adding (3.7) and (3.8), we have

$$\langle e^{F(u)}F'(u) - e^{F(v)}F'(v), u-v\rangle \geq 0,$$

which shows that F' is exponentially monotone.

Conversely, from (3.6), we have

$$\langle e^{F(v)}F'(v), u-v\rangle \leq \langle e^{F(u)}F'(u), u-v)\rangle. \tag{3.9}$$

Since K is an convex set, $\forall u,v \in K, \quad t \in [0,1]\ v_t = u+t(v-u) \in K$.

Taking $v = v_t$ in (3.9), we have

$$\begin{aligned}\langle e^{F(v_t)}F'(v_t), u-v_t\rangle &\leq \langle e^{F(u)}F'(u), u-v_t\rangle \\ &= -t\langle e^{F(u)}F'(u), v-u\rangle,\end{aligned}$$

which implies that

$$\langle e^{F(v_t)}F'(v_t), v-u\rangle \geq \langle e^{F(u)}F'(u), v-u\rangle. \tag{3.10}$$

Consider the auxiliary function

$$g(t) = e^{F(u+t(v-u))},$$

from which, we have

$$g(1) = e^{F(v)}, \quad g(0) = e^{F(u)}.$$

Then, from (3.10), we have

$$g'(t) \quad = \quad \langle e^{F(v_t)}F'(v_t, v-u\rangle \geq \langle e^{F(u)}F'(u), v-u\rangle. \tag{3.11}$$

Integrating (3.11) between 0 and 1, we have

$$g(1)-g(0) = \int_0^1 g'(t)dt \geq \langle e^{F(u)}F'(u), v-u\rangle.$$

Thus it follows that

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u), v-u\rangle,$$

which is the required (3.1). □

We now give a necessary condition for exponentially pseudo-convex function.

Theorem 3.9. *Let F' be exponentially pseudomonotone. Then F is a exponentially pseudo-convex function.*

Proof. Let F' be a exponentially pseudomonotone. Then, $\forall u, v \in K$,

$$\langle e^{F(u)}F'(u), v-u\rangle \geq 0.$$

implies that

$$\langle e^{F(v)}F'(v), v-u\rangle \geq 0. \tag{3.12}$$

Since K is an convex set, $\forall u, v \in K$, $t \in [0,1]$, $v_t = u + t(v-u) \in K$.

Taking $v = v_t$ in (3.12), we have

$$\langle e^{F(v_t)}F'(v_t), v-u\rangle \geq 0. \tag{3.13}$$

Consider the auxiliary function

$$g(t) = e^{F(u+t(v-u))} = e^{F(v_t)}, \quad \forall u, v \in K, t \in [0,1],$$

which is differentiable, since F is differentiable function. Then, using (3.13), we have

$$g'(t) = \langle e^{F(v_t)}F'(v_t), v-u)\rangle \geq 0.$$

Integrating the above relation between 0 to 1, we have

$$g(1) - g(0) = \int_0^1 g'(t)dt \geq 0,$$

that is,

$$e^{F(v)} - e^{F(u)} \geq 0,$$

showing that F is a exponentially pseudo-convex function. □

Definition 3.3. *The function F is said to be sharply exponentially pseudo convex, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned} \langle e^{F(u)}F'(u), v-u\rangle &\geq 0 \\ &\Rightarrow \\ F(v) &\geq e^{F(v+t(u-v))}, \quad \forall u, v \in K, t \in [0,1]. \end{aligned}$$

Theorem 3.10. *Let F be a sharply exponentially pseudo convex function on K. Then*

$$\langle e^{F(v)}F'(v), v-u\rangle \geq 0, \quad \forall u, v \in K.$$

Proof. Let F be a sharply exponentially pesudo convex function on K. Then

$$e^{F(v)} \geq e^{F(v+t(u-v))}, \quad \forall u, v \in K, t \in [0,1].$$

from which we have

$$\begin{aligned} 0 &\leq \lim_{t\to 0}\{\frac{e^{F(v+t(u-v))} - e^{F(v)}}{t}\} \\ &= \langle e^{F(v)}F'(v), v-u\rangle, \end{aligned}$$

the required result. □

Definition 3.4. *A function F is said to be a pseudo convex function, if there exists a strictly positive bifunction $b(.,.)$, such that*

$$\begin{aligned} e^{F(v)} &< e^{F(u)} \\ &\Rightarrow \\ e^{F(u+t(v-u))} &< e^{F(u)} + t(t-1)b(v,u), \forall u,v \in K, t \in [0,1]. \end{aligned}$$

Theorem 3.11. *If the function F is exponentially convex function such that $e^{F(v)} < e^{F(u)}$, then the function F is exponentially pseudo convex.*

Proof. Since $e^{F(v)} < e^{F(u)}$ and F is exponentially convex function, then $\forall u,v \in K, \quad t \in [0,1]$, we have

$$\begin{aligned} e^{F(u+t\ l(v,u))} &\leq e^{F(u)} + t(e^{F(v)} - e^{F(u)}) \\ &< e^{F(u)} + t(1-t)(e^{F(v)} - e^{F(u)}) \\ &= e^{F(u)} + t(t-1)(e^{F(u)} - e^{F(v)})) \\ &< e^{F(u)} + t(t-1)b(u,v), \end{aligned}$$

where $b(u,v) = e^{F(u)} - e^{F(v)} > 0$, the required result. This shows that the function F is exponentially convex function. □

We now discuss the optimality condition for the differentiable exponentially convex functions, which is the main motivation of our next result.

Theorem 3.12. *Let F be a differentiable exponentially convex function. Then $u \in K$ is the minimum of the function F, if and only if, $u \in K$ satisfies the inequality*

$$\langle e^{F(u)}F'(u), v-u \rangle \geq 0, \quad \forall u,v \in K. \tag{3.14}$$

Proof. Let $u \in K$ be a minimum of the function F. Then

$$F(u) \leq F(v), \forall v \in K.$$

from which, we have

$$e^{F(u)} \leq e^{F(v)}, \forall v \in K. \tag{3.15}$$

Since K is a convex set, so, $\forall u, v \in K, \quad t \in [0,1]$,

$$v_t = (1-t)u + tv \in K.$$

Taking $v = v_t$ in (3.15), we have

$$\begin{aligned} 0 &\leq \lim_{t \to 0} \{ \frac{e^{F(u+t(v-u))} - e^{F(u)}}{t} \} \\ &= \langle e^{F(u)} F'(u), v-u \rangle. \end{aligned} \tag{3.16}$$

Since F is differentiable exponentially convex function, so

$$e^{F(u+t(v-u))} \leq e^{F(u)} + t(e^{F(v)} - e^{F(u)}, \quad u, v \in K, t \in [0,1],$$

from which, using (3.16), we have

$$\begin{aligned} e^{F(v)} - e^{F(u)} &\geq \lim_{t \to 0} \{ \frac{e^{F(u+t(v-u))} - e^{F(u)}}{t} \} \\ &= \langle e^{F(u)} F'(u), v-u \rangle \geq 0, \end{aligned}$$

from which , we have

$$e^{F(v)} - e^{F(u)} \geq 0,$$

which implies that

$$F(u) \leq F(v), \quad \forall v \in K.$$

This shows that $u \in K$ is the minimum of the differentiable exponentially convex function, the required result. □

Remark 3.2. *The inequality of the type* (3.14) *is called the exponentially variational inequality and appears to be new one. For the applications, formulations, numerical methods and other aspects of variational inequalities, see Noor [12, 13].*

We now show that the difference of exponentially convex function and exponentially affine convex function is again an exponentially convex function.

Theorem 3.13. *Let f be a exponentially affine convex function. Then F is a exponentially convex function, if and only if, $g = F - f$ is a exponentially convex function.*

Proof. Let f be exponentially affine convex function. Then

$$e^{f((1-t)u+tv)} = (1-t)e^{f(u)} + te^{f(v)}, \quad \forall u, v \in K, \quad t \in [0,1]. \tag{3.17}$$

From the exponentially convexity of F, we have

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u, v \in K, \quad t \in [0,1]. \tag{3.18}$$

From (3.17) and (3.18), we have

$$e^{F((1-t)u+tv)} - e^{f((1-t)u+tv)} \leq (1-t)(e^{F(u)} - e^{f(u)}) + t(e^{F(v)} - e^{f(v)}), \tag{3.19}$$

from which it follows that

$$\begin{aligned} e^{g((1-t)u+tv)} &= e^{F((1-t)u+tv)} - e^{f((1-t)u+tv)} \\ &\leq (1-t)(e^{F(u)} - e^{f(u)}) + t(e^{F(v)} - e^{f(v)}), \end{aligned}$$

which show that $g = F - f$ is an exponentially convex function.

The inverse implication is obvious. □

4 Conclusion

In this paper, we have introduced and studied a new class of convex functions, which is called the exponentially convex function. It has been shown that exponentially convex functions enjoy several properties which convex functions have. We have shown that the minimum of the differentiable exponentially convex functions can be characterized by a new class of variational inequalities, which is called the exponential variational inequality. One can explore the applications of the exponentially variational inequalities This may stimulate further research.

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