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New Higher Order Fractional Programming for Sufficient Optimality Conditions Based on Hybrid $V-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-Sonvexities

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Abstract

This research first deals with a new comprehensive second order generalization to exponential type invexities, which encompasses most of the existing generalized sonvexity concepts (including [25] and [41]) in the literature, and then a wide range of parametric sufficient optimality conditions leading to the solvability for multiobjective fractional programming problems are established. These results are new and application-oriented to other fields of mathematical programming.

To the best of our knowledge, the obtained results seem to be most advanced on generalized higher order invexities.

Keywords: Generalized invexity, Minimax fractional programming, optimal solutions, Parametric sufficient optimality conditions.

2010 AMS Subject Classification: 90C30, 90C32, 90C34

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1 Introduction

Recently, Zalmai [41] presented a generalization to the exponential type invexities, and applied to a class of global parametric sufficient optimality criteria using various assumptions for semiinfinite discrete minimax fractional programming problems. This is followed by Verma [30] who introduced the second order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities to the context of parametric sufficient optimality conditions in semiinfinite discrete minimax fractional programming, while Zalmai and Zhang [42] have established a set of necessary efficiency conditions and a fairly large number of global nonparametric sufficient efficiency results under various frameworks for generalized $(\eta, \rho)-$invexity for semi-infinite discrete minimax fractional programming problems.

Verma [25] also constructed a general framework for a class of $(\rho, \eta, \theta)-$invex functions to examine some parametric sufficient efficiency conditions for multiobjective fractional programming problems for weakly $\varepsilon-$efficient solutions. Motivated by the recent advances on first order $B-(p,r)-$invexities and other generalizations to the context of multiobjective fractional programming problems, we first introduce the second order $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-invexities - a major generalization to Antczak type first order $B-(\tilde{p},\tilde{r})-$invexities - well-explored and well-cited in the literature, second we establish some parametric sufficient optimality conditions for multiobjective fractional programming to achieve optimal solutions to multiobjective fractional programming problems, and then we further establish some generalized sufficiency results. The results established in this paper generalize the results on exponential type first order $B-(\tilde{p},\tilde{r})-$invexities.

Next, we consider under the general framework of the second order B-(b, c, ρ, η, ω, θ, $\tilde{p}$, $\tilde{r}$, $\tilde{s}$)-invexities of functions the following minimax fractional programming problem:

(P)

$$Minimize\ max_{1\leq i\leq p}\frac{f_i(x)}{g_i(x)}$$

subject to $x \in Q = \{x \in X : H_j(x) \leq 0, j \in \{1, 2, \cdots, m\}\}$,

where X is a nonempty open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), f_i and g_i for $i \in \{1, \cdots, p\}$ and H_j for $j \in \{1, \cdots, m\}$ are real-valued functions defined on X such that $f_i(x) \geq 0$, $g_i(x) > 0$ for $i \in \{1, \cdots, p\}$ and for all $x \in Q$. Here Q denotes the feasible set of (P).

Semiinfinite fractional programming problems serve a significant useful purpose, especially in terms of applications to game theory, statistical analysis, engineering design (including design of control systems, design of earthquakes-resistant structures, digital filters, and electronic circuits), random graphs, boundary value problems, wavelet analysis, environmental protection planning, decision and management sciences, optimal control problems, continuum mechanics, robotics, and data envelopment analysis. For more details, we refer the reader [1- 44].

2 Hybrid Sonvexities

Next, we first present the second order $V-(b, c, \rho, \eta, \omega, \theta, \tilde{p}, \tilde{r}, \tilde{s})$-invexities - a generalization of the second order $B-(b, \rho, \eta, \theta, \tilde{p}, \tilde{r}, \tilde{s})$-invexities, and then establish some results on optimal solutions to (P).

Definition 2.1. *The function f is said to be second order $V-(b, c, \rho, \eta, \omega, \theta, \tilde{p}, \tilde{r}, \tilde{s})$-invex at $x^* \in X$ if there exist functions $\eta, \omega : X \times X \to \mathbb{R}^n$, functions $b : X \times X \to [0, \infty)$, $c : X \times X \to (0, \infty)$, and real numbers $\tilde{r}, \tilde{p}, \tilde{s}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,*

$$\begin{aligned} & b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \\ \geq \ & c(x, x^*)\Big(\frac{1}{\tilde{p}}\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\rangle \\ + \ & \frac{1}{2\tilde{s}}\langle \omega(x, x^*), \nabla^2 f(x^*) z\rangle\Big) \\ + \ & \rho(x, x^*)\|\theta(x, x^*)\|^2 \ \textit{for } \tilde{p} \neq 0, \tilde{r} \neq 0 \textit{ and } \tilde{s} \neq 0, \end{aligned}$$

$$\begin{aligned} & b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \\ \geq\ & c(x,x^*)\Big(\big\langle\nabla f(x^*),\eta(x,x^*)\big\rangle \\ +\ & \frac{1}{2}\langle\omega(x,x^*),\nabla^2 f(x^*)z\rangle\Big) \\ +\ & \rho(x,x^*)\|\theta(x,x^*)\|^2 \ \textit{for } \tilde{p}=0,\ \tilde{s}=0 \textit{ and } \tilde{r}\neq 0, \end{aligned}$$

$$\begin{aligned} & b(x,x^*)\Big([f(x)-f(x^*)]\Big) \\ \geq\ & c(x,x^*)\frac{1}{\tilde{p}}\Big(\big\langle\nabla f(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle \\ +\ & \frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1,\nabla^2 f(x^*)z\rangle\Big) \\ & \rho(x,x^*)\|\theta(x,x^*)\|^2 \ \textit{for } \tilde{p}\neq 0,\ \tilde{s}\neq 0 \textit{ and } \tilde{r}=0, \end{aligned}$$

$$\begin{aligned} & b(x,x^*)\Big([f(x)-f(x^*)]\Big)\geq c(x,x^*)\Big(\big\langle\nabla f(x^*),\eta(x,x^*)\big\rangle \\ +\ & \frac{1}{2}\langle\omega(x,x^*),\nabla^2 f(x^*)z\rangle\Big) \\ +\ & \rho(x,x^*)\|\theta(x,x^*)\|^2 \ \textit{for } \tilde{p}=0,\ \tilde{s}=0 \textit{ and } \tilde{r}=0. \end{aligned}$$

Definition 2.2. *The function f is said to be second order $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s}\Big)$-pseudoinvex with respect to η, b and c at $x^*\in X$ if there exist functions $\eta,\omega: X\times X\to\mathbb{R}^n$, functions $b: X\times X\to[0,\infty)$, $c: X\times X\to(0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x\in X$ and $z\in\mathbb{R}^n$,*

$$\begin{aligned} & c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle\nabla f(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle \\ +\ & \frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1,\nabla^2 f(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2\geq 0 \\ \Rightarrow\ & b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big)\geq 0\ \textit{for } \tilde{p}\neq 0,\ \tilde{s}\neq 0 \textit{ and } \tilde{r}\neq 0, \end{aligned}$$

$$c(x,x^*)\Big(\big\langle\nabla f(x^*),\eta(x,x^*)\big\rangle+\frac{1}{2}\langle\omega(x,x^*),\nabla^2 f(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2\geq 0$$

$$\Rightarrow\quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big)\geq 0 \text{ for } \tilde{p}=0,\ \tilde{s}=0 \text{ and } \tilde{r}\neq 0,$$

$$c(x,x^*)\Big(\frac{1}{\tilde{p}}\Big(\big\langle\nabla f(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1,\nabla^2 f(x^*)z\rangle\Big)$$

$$+\quad \rho(x,x^*)\|\theta(x,x^*)\|^2\geq 0$$

$$\Rightarrow\quad b(x,x^*)\Big([f(x)-f(x^*)]\Big)\geq 0 \text{ for } \tilde{p}\neq 0,\ \tilde{s}\neq 0 \text{ and } \tilde{r}=0,$$

$$c(x,x^*)\Big(\big\langle\nabla f(x^*),\eta(x,x^*)\big\rangle+\frac{1}{2}\langle\omega(x,x^*),\nabla^2 f(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2\geq 0$$

$$\Rightarrow\quad b(x,x^*)\Big([f(x)-f(x^*)]\Big)\geq 0 \text{ for } \tilde{p}=0,\ \tilde{s}=0 \text{ and } \tilde{r}=0.$$

Definition 2.3. *The function f is said to be second order $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-quasiinvex with respect to η, ω, b and c at $x^*\in X$ if there exist functions $\eta,\omega: X\times X\to\mathbb{R}^n$, functions $b: X\times X\to[0,\infty)$, $c: X\times X\to(0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x\in X$ and $z\in\mathbb{R}^n$,*

$$
\begin{aligned}
& b(x,x^*)\Big(\frac{1}{\tilde r}\big(e^{\tilde r[f(x)-f(x^*)]}-1\big)\Big)\le 0\\
\Rightarrow\quad & c(x,x^*)\Big(\frac{1}{\tilde p}\big\langle \nabla f(x^*), e^{\tilde p\eta(x,x^*)}-1\big\rangle\\
+\quad & \frac{1}{2\tilde s}\langle e^{\tilde s\omega(x,x^*)}-1, \nabla^2 f(x^*)z\rangle\Big)\\
+\quad & \rho(x,x^*)\|\theta(x,x^*)\|^2\le 0 \; \textit{for } \tilde p\neq 0,\ \tilde s\neq 0 \textit{ and } \tilde r\neq 0,
\end{aligned}
$$

$$
\begin{aligned}
& b(x,x^*)\Big(\frac{1}{\tilde r}\big(e^{\tilde r[f(x)-f(x^*)]}-1\big)\Big)\le 0\\
\Rightarrow\quad & c(x,x^*)\Big(\big\langle \nabla f(x^*), \eta(x,x^*)\big\rangle\\
+\quad & \frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle\Big)\\
+\quad & \rho(x,x^*)\|\theta(x,x^*)\|^2\le 0
\end{aligned}
$$

for $\tilde p = 0$, $\tilde s = 0$ *and* $\tilde r \neq 0$,

$$b(x,x^*)\Big([f(x)-f(x^*)]\Big)\leq 0$$

$$\Rightarrow \quad c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle\nabla f(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle\Big)$$

$$+ \quad \frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1,\nabla^2 f(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2\leq 0$$

for $\tilde{p}\neq 0$, $\tilde{s}\neq 0$ and $\tilde{r}=0$,

$$b(x,x^*)\Big([f(x)-f(x^*)]\Big)\leq 0$$

$$\Rightarrow \quad c(x,x^*)\Big(\big\langle\nabla f(x^*),\eta(x,x^*)\big\rangle+\frac{1}{2}\langle\omega(x,x^*),\nabla^2 f(x^*)z\rangle\Big)$$

$$+ \quad \rho(x,x^*)\|\theta(x,x^*)\|^2\leq 0$$

for $\tilde{p}=0$, $\tilde{s}=0$ and $\tilde{r}=0$.

Definition 2.4. *The function f is said to be second order strictly $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-pseudoinvex with respect to η, ω and b at $x^*\in X$ if there exist functions $\eta,\omega: X\times X\to\mathbb{R}^n$, functions $b: X\times X\to[0,\infty)$, $c: X\times X\to(0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x\in X$ and $z\in\mathbb{R}^n$,*

$$c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle$$

$$+\frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1, \nabla^2 f(x^*)z\rangle\Big)$$

$$+ \quad \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) > 0$$

for $\tilde{p} \neq 0$, $\tilde{s} \neq 0$ *and* $\tilde{r} \neq 0$,

$$c(x,x^*)\Big(\big\langle \nabla f(x^*), \eta(x,x^*)\big\rangle + \frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) > 0 \text{ for } \tilde{p}=0, \tilde{s}=0 \text{ and } \tilde{r}\neq 0,$$

$$c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle + \frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1, \nabla^2 f(x^*)z\rangle$$

$$+ \quad \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x)-f(x^*)]\Big) > 0 \text{ for } \tilde{p}\neq 0, \tilde{s}\neq 0 \text{ and } \tilde{r}=0,$$

$$c(x,x^*)\Big(\big\langle \nabla f(x^*), \eta(x,x^*)\big\rangle + \frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x)-f(x^*)]\Big) > 0 \text{ for } \tilde{p}=0,\ \tilde{s}=0 \text{ and } \tilde{r}=0.$$

Definition 2.5. *The function f is said to be second order strictly $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-quasiinvex with respect to η, ω and b and c at $x^* \in X$ if there exist functions $\eta,\omega : X\times X \to \mathbb{R}^n$, functions $b : X\times X \to [0,\infty)$, $b : X\times X \to [0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x\in X$ and $z\in\mathbb{R}^n$,*

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \leq 0$$

$$\Rightarrow \quad c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle$$

$$+ \quad \frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1, \nabla^2 f(x^*)z\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 < 0$$

for $\tilde{p}\neq 0$, $\tilde{s}\neq 0$ and $\tilde{r}\neq 0$,

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big)\leq 0$$

$$\Rightarrow \quad c(x,x^*)\Big(\big\langle \nabla f(x^*),\eta(x,x^*)\big\rangle$$

$$+ \quad \frac{1}{2}\langle \omega(x,x^*),\nabla^2 f(x^*)z\rangle\Big)$$

$$+ \quad \rho(x,x^*)\|\theta(x,x^*)\|^2<0 \, for \; \tilde{p}=0, \; \tilde{s}=0 \; and \; \tilde{r}\neq 0,$$

$$b(x,x^*)\Big(\big[f(x)-f(x^*)\big]\Big)\leq 0$$

$$\Rightarrow \quad c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle \nabla f(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle$$

$$+ \quad \frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1,\nabla^2 f(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2<0$$

for $\tilde{p}\neq 0$, $\tilde{s}\neq 0$ *and* $\tilde{r}=0$,

$$b(x,x^*)\Big([f(x)-f(x^*)]\Big)\leq 0$$

$$\Rightarrow \quad c(x,x^*)\Big(\big\langle \nabla f(x^*),\eta(x,x^*)\big\rangle+\frac{1}{2}\langle \omega(x,x^*),\nabla^2 f(x^*)z\rangle\Big)$$

$$+ \quad \rho(x,x^*)\|\theta(x,x^*)\|^2<0$$

for $\tilde{p}=0$, $\tilde{s}=0$ *and* $\tilde{r}=0$.

Definition 2.6. *The function f is said to be second order prestrictly $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-pseudoinvex with respect to η, ω and b at $x^*\in X$ if there exist functions $\eta,\omega : X\times X\to\mathbb{R}^n$, functions $b : X\times X\to[0,\infty)$, $c : X\times X\to(0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x\in X$ and $z\in\mathbb{R}^n$,*

$$c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle \nabla f(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle$$

$$+\frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1,\nabla^2 f(x^*)z\rangle\Big)$$

$$+ \quad \rho(x,x^*)\|\theta(x,x^*)\|^2>0$$

$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big)\geq 0$$

for $\tilde{p}\neq 0$, $\tilde{s}\neq 0$ *and* $\tilde{r}\neq 0$,

$$c(x,x^*)\Big(\big\langle \nabla f(x^*), \eta(x,x^*)\big\rangle + \frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$

$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \geq 0 \ \textit{for}\ \tilde{p}=0,\ \tilde{s}=0\ \textit{and}\ \tilde{r}\neq 0,$$

$$c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle + \frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1, \nabla^2 f(x^*)z\rangle$$

$$+ \quad \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x)-f(x^*)]\Big) \geq 0 \ \textit{for}\ \tilde{p}\neq 0,\ \tilde{s}\neq 0\ \textit{and}\ \tilde{r}=0,$$

$$c(x,x^*)\Big(\big\langle \nabla f(x^*), \eta(x,x^*)\big\rangle + \frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x)-f(x^*)]\Big) \geq 0 \ \textit{for}\ \tilde{p}=0,\ \tilde{s}=0\ \textit{and}\ \tilde{r}=0.$$

Next, we recall the following result (Verma [28]) that is crucial to developing the results for the next section based on second order $B-(b,c,\rho,\eta,\theta,\tilde{p},\tilde{r},\tilde{s})$-invexities.

Theorem 2.1. *Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1\leq i\leq p} f_i(x^*)/g_i(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z,t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z,s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an optimal solution of (P), if the second order generalized Abadie constraint* *qualification holds at x^*, and if for any critical direction y, the set cone*

$$\{\Big(\nabla G_j(x^*,t),\langle y,\nabla^2 G_j(x^*,t)y\rangle\Big) : t\in \hat{T}_j(x^*), j\in \underline{q}\}$$

$$+\quad span\{\Big(\nabla H_k(x^*,s),\langle y,\nabla^2 H_k(x^*,s)y\rangle\Big) : s\in S_k, k\in \underline{r}\},$$

$$\text{where } \hat{T}_j(x^*)\equiv\{t\in T_j\ :\ G_j(x^*,t)=0\},$$

is closed, then there exist $u^*\in U\equiv\{u\in\mathbb{R}^p : u\geq 0,\ \sum_{i=1}^p u_i=1\}$ and integers ν_0^* and ν^*, with $0\leq\nu_0^*\leq\nu^*\leq n+1$, such that there exist ν_0^* indices j_m, with $1\leq j_m\leq q$, together with ν_0^* points $t^m\in\hat{T}_{j_m}(x^*)$, $m\in\underline{\nu_0^*}$, $\nu^*-\nu_0^*$ indices k_m, with $1\leq k_m\leq r$, together with $\nu^*-\nu_0^*$ points $s^m\in S_{k_m}$ for $m\in\underline{\nu^*}\backslash\underline{\nu_0^*}$, and ν^* real numbers v_m^*, with $v_m^*>0$ for $m\in\underline{\nu_0^*}$, with the property that

$$\sum_{i=1}^p u_i^*[\nabla f_i(x^*)-\lambda^*(\nabla g_i(x^*)]+\sum_{m=1}^{\nu_0^*} v_m^*[\nabla G_{j_m}(x^*,t^m)$$

$$+\quad\sum_{m=\nu_0^*+1}^{\nu^*} v_m^*\nabla H_k(x^*,s^m)=0, \tag{2.1}$$

$$\langle y,\Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)-\lambda^*\nabla^2 g_i(x^*)]+\sum_{m=1}^{\nu_0^*} v_m^*\nabla^2 G_{j_m}(x^*,t^m)$$

$$+\quad\sum_{m=\nu_0^*+1}^{\nu^*} v_m^*\nabla^2 H_k(x^*,s^m)\Big]y\rangle\geq 0, \tag{2.2}$$

where $\hat{T}_{j_m}(x^*)=\{t\in T_{j_m} : G_{j_m}(x^*,t)=0\}$, $U=\{u\in\mathbb{R}^p : u\geq 0,\sum_{i=1}^p u_i=1\}$, and $\underline{\nu^*}\backslash\underline{\nu_0^*}$ is the complement of the set $\underline{\nu_0^*}$ relative to the set $\underline{\nu^*}$.

3 Second Order Sufficient Optimality Conditions

Now, we first present our main result on sufficient optimality conditions and the second order $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-invexities to the context of optimality solutions to (P).

Let $E_i(x;x^*,u^*)\ \forall i\in\{1,\cdots,p\}$ be defined by $\Sigma_{i=1}^p u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]$, and $B_j(x,v^*)\ \forall j\in\{1,\cdots,m\}$ be defined by $\Sigma_{j=1}^m v_j^* H_j(x)$.

Theorem 3.1. *Let $x^*\in Q$. Let f_i,g_i for $i\in\{1,\cdots,p\}$ with $\phi(x^*)=\frac{f_i(x^*)}{g_i(x^*)}\geq 0$, $g_i(x^*)>0$ and H_j for $j\in\{1,\cdots,m\}$ be twice continuously differentiable at $x^*\in Q$, and let there exist $u^*\in U=\{u\in\mathbb{R}^p: u>0,\Sigma_{i=1}^p u_i=1\}$ and $v^*\in\mathbb{R}_+^m$ such that*

$$\Sigma_{i=1}^p u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*)]+\Sigma_{j=1}^m v_j^*\bigtriangledown H_j(x^*)=0 \tag{3.1}$$

$$\Big\langle z,\Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)]+\sum_{j=1}^m v_j^*\nabla^2 H_j(x^*)\Big]z\Big\rangle\geq 0, \tag{3.2}$$

and

$$v_j^* H_j(x^*)=0,\ j\in\{1,\cdots,m\}. \tag{3.3}$$

Suppose, in addition, that any one of the following assumptions holds:

(i) $E_i(.;x^*,u^*)\ \forall i\in\{1,\cdots,p\}$ are second order $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-pseudoinvex with respect to η, ω, b and c at $x^*\in X$ if there exist functions $\eta,\omega: X\times X\to\mathbb{R}^n$, functions $b: X\times X\to[0,\infty)$, $c: X\times X\to(0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x\in X$ and $z\in\mathbb{R}^n$, and $B_j(.,v^*)$ $\forall j\in\{1,\cdots,m\}$ are second order $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-quasiinvex with respect to η, b and c

at $x^* \in X$ if there exist functions $\eta, \omega : X \times X \to \mathbb{R}^n$, functions $b : X \times X \to [0,\infty)$, $c : X \times X \to (0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $z \in \mathbb{R}^n$, $\rho(x,x^*) \geq 0$, and $b(x,x^*) > 0$.

(ii) *$E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ are second order $B-(b,c,\rho_1,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-pseudoinvex with respect to η, ω, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, functions $b : X \times X \to [0,\infty)$, $c : X \times X \to (0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$, and $B_j(.,v^*)$ $\forall j \in \{1,\cdots,m\}$ are second order $B-(b,c,\rho_2,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-quasiinvex with respect to η, b and c at $x^* \in X$ if there exist functions $\eta, \omega : X \times X \to \mathbb{R}^n$, functions $b : X \times X \to [0,\infty)$, $c : X \times X \to (0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $z \in \mathbb{R}^n$, $b(x,x^*) > 0$ and $\rho_1(x,x^*), \rho_2(x,x^*) \geq 0$ with $\rho_2(x,x^*) \geq \rho_1(x,x^*)$.*

(iii) *$E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ are second order prestrictly $B-(b,c,\rho_1,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-pseudoinvex with respect to η, b and c at $x^* \in X$ if there exist functions $\eta, \omega : X \times X \to \mathbb{R}^n$, functions $b : X \times X \to [0,\infty)$, $c : X \times X \to (0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$, and $B_j(.,v^*)$ $\forall j \in \{1,\cdots,m\}$ are second order strictly $B-(b,c,\rho_2,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-quasiinvex with respect to η, b and c at $x^* \in X$ if there exist functions $\eta, \omega : X \times X \to \mathbb{R}^n$, functions $b : X \times X \to [0,\infty)$, $c : X \times X \to (0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $z \in \mathbb{R}^n$, $b(x,x^*) > 0$, and $\rho_1(x,x^*), \rho_2(x,x^*) \geq 0$ with $\rho_2(x,x^*) \geq \rho_1(x,x^*)$.*

(iv) *$E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ are second order prestrictly $B-(b,c,\rho_1,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-quasi-invex with respect to η, ω b and c at $x^* \in X$ [if there exist functions $\eta, \omega : X \times X \to \mathbb{R}^n$, functions $b : X \times X \to [0,\infty)$, $c : X \times X \to (0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$, and $B_j(.,v^*)$ $\forall j \in \{1,\cdots,m\}$ are second order strictly $B-(b,c,\rho_2,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-pseudoinvex with respect to η, ω, b and c at $x^* \in X$ if there exist functions $\eta, \omega : X \times X \to \mathbb{R}^n$, functions*

$b: X\times X\to[0,\infty)$, $c: X\times X\to(0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x\in X$, $z\in\mathbb{R}^n$, $b(x,x^*)>0$, and $\rho_1(x,x^*),\rho_2(x,x^*)\geq 0$ with $\rho_2(x,x^*)\geq\rho_1(x,x^*)$.

(v) For each $i\in\{1,\cdots,p\}$, f_i is second order $B-(b,c,\rho_1,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})-$invex and $-g_i$ is second order $B-(b,c,\rho_2,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})-$invex at x^*. $H_j(.,v^*)$ $\forall j\in\{1,\cdots,m\}$ is $B-(b,c,\rho_3,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})-$quasi-invex at x^*, and $\Sigma_{j=1}^m v_j^*\rho_3+\rho^*\geq 0$ for $\rho^*=\Sigma_{i=1}^p u_i^*(\rho_1+\phi(x^*)\rho_2)$ and for $\phi(x^*)=\frac{f_i(x^*)}{g_i(x^*)}$ with $b(x,x^*)>0$.

Then x^* is an optimal solution to (P).

Proof. If (i) holds, and if $x\in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned}&\frac{1}{\tilde{p}}\langle\Sigma_{i=1}^p u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*)],e^{\tilde{p}\eta(x,x^*)}-1\rangle\\ +&\frac{1}{\tilde{p}}\langle\Sigma_{j=1}^m v_j^*\bigtriangledown H_j(x^*),e^{\tilde{p}\eta(x,x^*)}-1\rangle=0\forall x\in Q,\end{aligned}\tag{3.4}$$

$$\frac{1}{2\tilde{s}}\Big\langle e^{\tilde{s}\omega(x,x^*)}-1,\Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)]+\sum_{j=1}^m v_j^*\nabla^2 H_j(x^*)\Big]z\Big\rangle\geq 0.\tag{3.5}$$

Since $v^*\geq 0$, $x\in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x)\leq 0=\Sigma_{j=1}^m v_j^* H_j(x^*),$$

and so

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(x^*)]}-1\big)\Big)\leq 0$$

since $\tilde{r} \neq 0$ and $b(x,x^*) > 0$ for all $x \in Q$. In light of the $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})$-quasiinvexity of $B_j(.,v^*)$ at x^*, and $c(x,x^*) > 0$, it follows that

$$c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle\nabla H_j(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1,\nabla^2 H_j(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0,$$

and hence,

$$\begin{aligned} & c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle\Sigma_{j=1}^m \nabla H_j(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2\tilde{s}}\langle e^{\tilde{p}\omega(x,x^*)}-1,\Sigma_{j=1}^m\nabla^2 H_j(x^*)z\rangle\Big) \\ + \quad & \rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0. \end{aligned} \tag{3.6}$$

It follows from (3.4), (3.5) and (3.6) that

$$\begin{aligned} & c(x,x^*)\Big(\frac{1}{\tilde{p}}\langle\Sigma_{i=1}^p u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*)], e^{\tilde{p}\eta(x,x^*)}-1\rangle \\ + \quad & \frac{1}{2\tilde{s}}\Big\langle e^{\tilde{s}\omega(x,x^*)}-1,\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z]\Big\rangle\Big) \\ \geq \quad & \rho(x,x^*)\|\theta(x,x^*)\|^2. \end{aligned} \tag{3.7}$$

Since $\rho(x,x^*) \geq 0$, applying $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{s},\tilde{r})-$pseudo-invexity at x^* to (3.7), we have

$$\frac{1}{\tilde{r}}b(x,x^*)\big(e^{\tilde{r}[E_i(x,x^*,u^*)-E_i(x^*,x^*,u^*)]}-1\big) \geq 0. \tag{3.8}$$

Since $b(x,x^*) > 0$, (3.8) implies

$$\begin{aligned} & \Sigma_{i=1}^p u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \\ \geq \quad & \Sigma_{i=1}^p u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]) \\ = \quad & 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \geq 0. \tag{3.9}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there exists an $x \in Q$ such that

$$\lambda^* \leq \max_{1\leq i\leq p} \frac{\sum_{i=1}^{p} u_i^* f_i(x)}{\sum_{i=1}^{p} u_i^* g_i(x)} \leq \max_{u\in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

Hence, x^* is an optimal solution to (P).

The proof for (ii) is similar to that of (i), but we include for the sake of the completeness. If (ii) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned} & \frac{1}{\tilde{p}}\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)], e^{\tilde{p}\eta(x,x^*)} - 1\rangle \\ + \quad & \frac{1}{\tilde{p}}\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\rangle = 0 \forall x \in Q, \end{aligned} \tag{3.10}$$

$$\frac{1}{\tilde{p}}\Big\langle e^{\tilde{p}\omega(x,x^*)} - 1, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \tag{3.11}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

and so

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(x^*)]} - 1\big)\Big) \leq 0$$

since $\tilde{r} \neq 0$ and $b(x,x^*) > 0$ for all $x \in Q$. In light of the $B-(b,c,\rho_2,\eta,\theta,\tilde{p},\tilde{r},\tilde{s})$-quasiinvexity of $B_j(.,v^*)$ at x^*, it follows that

$$c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle \nabla H_j(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)} - 1, \nabla^2 H_j(x^*)z\rangle\Big) + \rho_2(x,x^*)\|\theta(x,x^*)\|^2 \leq 0,$$

and hence,

$$\begin{aligned} & c(x,x^*)\Big(\frac{1}{\tilde{p}}\Sigma_{j=1}^{m}\big\langle \nabla H_j(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle + \frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1, \Sigma_{j=1}^{m}\nabla^2 H_j(x^*)z\rangle\Big) \\ + & \rho_2(x,x^*)\|\theta(x,x^*)\|^2 \leq 0. \end{aligned} \tag{3.12}$$

It follows from (3.10), (3.11) and (3.12) that

$$\begin{aligned} & c(x,x^*)\Big(\frac{1}{\tilde{p}}\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*)], e^{\tilde{p}\eta(x,x^*)}-1\rangle \\ + & \frac{1}{2\tilde{s}}\Big\langle e^{\tilde{s}\omega(x,x^*)}-1, \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z]\Big\rangle\Big) \\ \geq & \rho_2(x,x^*)\|\theta(x,x^*)\|^2. \end{aligned} \tag{3.13}$$

Since $\rho_1(x,x^*), \rho_2(x,x^*) \geq 0$ with $\rho_2(x,x^*) \geq \rho_1(x,x^*)$, and $c(x,x^*) > 0$, applying $B-(b,c,\rho_1,\eta,\theta,\tilde{p},\tilde{r},\tilde{s})-$pseudo-invexity at x^* to (3.13), we have

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[E_i(x,x^*,u^*)-E_i(x^*,x^*,u^*)]}-1\big)\Big) \geq 0. \tag{3.14}$$

Since $b(x,x^*) > 0$, (3.13) implies

$$\begin{aligned} & \Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \\ \geq & \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]) \\ = & 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \geq 0. \tag{3.15}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there is an $x \in Q$ such that

$$\lambda^* \leq \max_{1 \leq i \leq p} \frac{\sum_{i=1}^{p} u_i^* f_i(x)}{\sum_{i=1}^{p} u_i^* g_i(x)} \leq \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

Hence, x^* is an optimal solution to (P).

Next, we start off the proof for (iii) as follows: if (iii) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned} & \frac{1}{\tilde{p}}\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)], e^{\tilde{p}\eta(x,x^*)} - 1\rangle \\ + \quad & \frac{1}{\tilde{p}}\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\rangle = 0 \forall x \in Q, \end{aligned} \tag{3.16}$$

$$\frac{1}{\tilde{p}}\Big\langle e^{\tilde{p}\omega(x,x^*)} - 1, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \tag{3.17}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

which implies

$$b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x) - H_j(x^*)]} - 1\big)\Big) \leq 0.$$

Then, in light of the strict $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})-$quasi-invexity of $B_j(.,v^*)$ at x^*, we have

$$c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle\nabla H_j(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2\tilde{s}}\langle e^{\tilde{s}z}-1,\nabla^2 H_j(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2<0. \tag{3.18}$$

It follows from (3.3), (3.16), (3.17) and (3.18) that

$$\begin{aligned} & \frac{1}{\tilde{p}}\Big(\langle\Sigma_{i=1}^{p}u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*)],e^{\tilde{p}\eta(x,x^*)}-1\rangle \\ + & \frac{1}{2\tilde{s}}\Big\langle e^{\tilde{s}\omega(x,x^*)}-1,\sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z]\Big\rangle\Big) \\ > & \rho(x,x^*)\|\theta(x,x^*)\|^2. \end{aligned} \tag{3.19}$$

As a result, since $\rho(x,x^*)\geq 0$, applying the prestrict $B-(b,c,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})-$pseudo-invexity at x^* to (3.19), we have

$$\Big(\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(g^*)})g_i(x)]-\Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big)\geq 0,$$

which implies

$$\begin{aligned} & \Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \\ \geq & \Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]) \\ = & 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \geq 0. \tag{3.20}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there is an $x \in Q$ such that

$$\lambda^* \leq \max_{1 \leq i \leq p} \frac{\sum_{i=1}^{p} u_i^* f_i(x)}{\sum_{i=1}^{p} u_i^* g_i(x)} \leq \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

Hence, x^* is an efficient solution to (P).

The proof applying (iv) is similar to that of (iii), but still we include it as follows: if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned} & \frac{1}{\tilde{p}}\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)], e^{\tilde{p}\eta(x,x^*)} - 1\rangle \\ + \quad & \frac{1}{\tilde{p}}\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\rangle = 0 \forall x \in Q, \end{aligned} \tag{3.21}$$

$$\frac{1}{\tilde{s}}\Big\langle e^{\tilde{s}\omega(x,x^*)} - 1, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \tag{3.22}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

which implies

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x) - H_j(x^*)]} - 1\big)\Big) \leq 0.$$

Then, in light of the equivalent form for the strict $B-(\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})-$pseudo-invexity of $B_j(.,v^*)$ at x^*, we have

$$c(x,x^*)\Big(\frac{1}{\tilde{p}}\big\langle\nabla H_j(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2\tilde{s}}\langle e^{\tilde{s}\omega(x,x^*)}-1,\nabla^2 H_j(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2<0.$$

It follows from (3.21) and (3.22) that

$$\begin{aligned}
&\frac{1}{\tilde{p}}\Big(\langle\Sigma_{i=1}^{p}u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*)],e^{\tilde{p}\eta(x,x^*)}-1\rangle\\
+\quad&\frac{1}{2\tilde{s}}\Big\langle e^{\tilde{s}\omega(x,x^*)}-1,\sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z]\Big\rangle\Big)\\
>\quad&\rho(x,x^*)\|\theta(x,x^*)\|^2. \qquad (3.23)
\end{aligned}$$

As a result, since $\rho(x,x^*)\geq 0$, applying the equivalent form for the prestrict $(b,\rho,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})-$quasi-invexity of $E_i(.;x^*,u^*)$ at x^* to (3.47), we have

$$\Big(\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(g^*)})g_i(x)]-\Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big)\geq 0,$$

which implies

$$\begin{aligned}
&\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\\
\geq\quad&\Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)])\\
=\quad&0.
\end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\geq 0. \qquad (3.24)$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there exists an $x \in Q$ such that

$$\lambda^* \leq \max_{1 \leq i \leq p} \frac{\sum_{i=1}^p u_i^* f_i(x)}{\sum_{i=1}^p u_i^* g_i(x)} \leq \max_{u \in U} \frac{\sum_{i=1}^p u_i f_i(x)}{\sum_{i=1}^p u_i g_i(x)}.$$

Hence, x^* is an optimal solution to (P).

Finally, to prove (v), we start with: since $x \in Q$, it follows that $H_j(x) \leq H_j(x^*)$, i.e., $H_j(x) - H_j(x^*) \leq 0$, which implies

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(x^*)]} - 1\big)\Big) \leq 0.$$

Then applying the $B-(b,c,\rho_3,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})-$quasi-invexity of H_j at x^* and $v^* \in R_+^m$, we have

$$\begin{aligned}
& c(x,x^*)\Big(\frac{1}{\tilde{p}}\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\rangle \\
+ \quad & \frac{1}{2\tilde{s}}\Big\langle e^{\tilde{s}\omega(x,x^*)} - 1, \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z\Big\rangle\Big) \\
\leq \quad & -\Sigma_{j=1}^m v_j^* \rho_3 \|\theta(x,x^*)\|^2.
\end{aligned}$$

Since $u^* \geq 0$ and $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, it follows from $B-(b,c,\rho_3,\eta,\omega,\theta,\tilde{p},\tilde{r},\tilde{s})-$invexity assumptions that

$$
\begin{aligned}
& b(x,x^*)\Big(\frac{1}{\tilde{r}}e^{\tilde{r}\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]}-1\Big) \\
= \quad & b(x,x^*)\Big(\frac{1}{\tilde{r}}e^{\tilde{r}\Sigma_{i=1}^{p}u_i^*\{[f_i(x)-f_i(x^*)]-(\frac{f_i(x^*)}{g_i(x^*)})[g_i(x)-g_i(x^*)]\}}-1\Big) \\
\geq \quad & c(x,x^*)\frac{1}{\tilde{p}}\Big(\Sigma_{i=1}^{p}u_i^*\{\langle\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*),e^{\tilde{p}\eta(x,x^*)}-1\rangle\}\Big) \\
+ \quad & c(x,x^*)\Big[\frac{1}{\tilde{s}}\Big(\frac{1}{2}\langle e^{\tilde{s}\omega(x,x^*)}-1,\Sigma_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z\rangle]\Big] \\
+ \quad & \Sigma_{i=1}^{p}u_i^*[\rho_1+\phi(x^*)\rho_2]\|\theta(x,x^*)\|^2 \\
\geq \quad & -c(x,x^*)\frac{1}{\tilde{p}}\big[\langle\Sigma_{j=1}^{m}v_j^*\bigtriangledown H_j(x^*),e^{\tilde{p}\eta(x,x^*)}-1\rangle \\
+ \quad & \frac{1}{\tilde{s}}\Big(\frac{1}{2}\Big\langle e^{\tilde{s}\omega(x,x^*)}-1,\Sigma_{j=1}^{m}v_j^*\nabla^2 H_j(x^*)z\Big\rangle\Big)\Big] \\
+ \quad & \Sigma_{i=1}^{p}u_i^*[\rho_1+\phi(x^*)\rho_2]\|\theta(x,x^*)\|^2 \\
\geq \quad & (\Sigma_{j=1}^{m}v_j^*\rho_3+\Sigma_{i=1}^{p}u_i^*[\rho_1+\phi(x^*)\rho_2])\|\theta(x,x^*)\|^2 \\
= \quad & (\Sigma_{j=1}^{m}v_j^*\rho_3+\rho^*)\|\theta(x,x^*)\|^2\geq 0,
\end{aligned}
$$

where $\phi(x^*)=\frac{f_i(x^*)}{g_i(x^*)}$ and $\rho^*=\Sigma_{i=1}^{p}u_i^*(\rho_1+\phi(x^*)\rho_2)$. □

Note that when functions f_i, g_i, H_j have first-order derivatives, the established results seem to be specialized to $B-(p,r)-$invexities frameworks introduced by Antczak [1-3] and later investigated by others.

Theorem 3.2. *Let $x^* \in Q$. Let f_i, g_i for $i \in \{1,\cdots,p\}$ with $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1,\cdots,m\}$ be differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \mathbb{R}^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \mathbb{R}_+^m$ such that*

$$\Sigma_{i=1}^p u_i^* [\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \triangledown g_i(x^*)] + \Sigma_{j=1}^m v_j^* \triangledown H_j(x^*) = 0 \tag{3.25}$$

and

$$v_j^* H_j(x^*) = 0,\ j \in \{1,\cdots,m\}. \tag{3.26}$$

Suppose, in addition, that any one of the following assumptions holds:

(i) *$E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ are $B-(b,\rho,\eta,\theta,\tilde{p},\tilde{r})$-pseudoinvex with respect to η, and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to (0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$, and $B_j(.,v^*)$ $\forall j \in \{1,\cdots,m\}$ are $B-(b,\rho,\eta,\theta,\tilde{p},\tilde{r})$-quasiinvex with respect to η, and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to (0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $z \in \mathbb{R}^n$, and $\rho(x,x^*) \geq 0$.*

(ii) *$E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ are $B-(b,\eta,\rho_1,\theta,\tilde{p},\tilde{r})$-pseudoinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to (0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$, and $B_j(.,v^*)$ $\forall j \in \{1,\cdots,m\}$ are $B-(b,\rho_2,\eta,\theta,\tilde{p},\tilde{r})$-*

quasiinvex with respect to η*, and* b *and at* $x^* \in X$ *if there exist a function* $\eta : X \times X \to \mathbb{R}^n$*, a function* $b : X \times X \to (0,\infty)$*, and real numbers* $\tilde{r}$ *and* $\tilde{p}$ *such that for all* $x \in X$*,* $z \in \mathbb{R}^n$*, and* $\rho_1(x,x^*), \rho_2(x,x^*) \geq 0$ *with* $\rho_2(x,x^*) \geq \rho_1(x,x^*)$*.*

(iii) $E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ *are strictly* $B-(b,\rho,\eta,\theta,\tilde{p},\tilde{r})-$*pseudo-invex at* x^**, and* $B_j(.,v^*)$ $\forall j \in \{1,\cdots,m\}$ *are prestrictly* $B-(b,\rho,\eta,\theta)-$*quasi-invex at* x^**.*

(iv) $E_i(.;x^*,u^*)$ $\forall i \in \{1,\cdots,p\}$ *are strictly* $B-(b,\rho,\eta,\theta,\tilde{p},\tilde{r})-$*pseudo-invex at* x^**, and* $B_j(.,v^*)$ $\forall j \in \{1,\cdots,m\}$ *are prestrictly* $B-(b,\rho,\eta,\theta,\tilde{p},\tilde{r})-$*quasi-invex at* x^* *with* $\rho(x,x^*) \geq 0$*.*

(v) For each $i \in \{1,\cdots,p\}$*,* f_i *is* $B-(b,\rho_1,\eta,\theta,\tilde{p},\tilde{r})-$*invex and* $-g_i$ *is* $B-(b,\rho_2,\eta,\theta)-$*invex at* x^**.* $H_j(.,v^*)$ $\forall j \in \{1,\cdots,m\}$ *is* $B-(\rho_3,\eta)-$*quasi-invex at* x^**, and* $\Sigma_{j=1}^m v_j^* \rho_3 + \rho^* \geq 0$ *for* $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1 + \phi(x^*)\rho_2)$ *and for* $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$*.*

Then x^* *is an optimal solution to (P).*

4 Concluding Remarks

We observe that the obtained results in this communication can be generalized to the case of multiobjective fractional subset programming with generalized invex functions, for instance based on the work of Mishra et al. [16] and Verma [29] to the case of the $\varepsilon-$ optimality and weak $\varepsilon-$optimality conditions to the context of minimax fractional programming problems.

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