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Fuzzy Topological, Algebraic and Geometrical Concepts

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This paper is a brief survey on fuzzy topological, algebraic and geometrical concepts.

1 Introduction

The concept of fuzzy set was introduced by Zadeh [66] in the year 1965. A fuzzy subset of a set is a mapping $\mu : X \to [0,1]$ and an ordinary (crisp) subset is special case of fuzzy subset where $\mu : X \to \{0,1\}$. μ is called membership function and we will not distinguish between a fuzzy subset and its membership function.

This paper is a review of fuzzy topological, algebraic and geometrical concepts. Paper is divided into three parts. Part I deals with fuzzy topology, Part II with fuzzy algebraic concepts and finally Part III deals with some geometrical concepts.

2 Fuzzy Topological Concepts

The concepts of fuzzy set theory has been applied to fuzzy topological spaces (fts). The present author has used this concept to introduce almost compact, strongly compact, completely connected and super-connected fts. The purpose of the present section is to give a brief review of fts including some published and unpublished works of the author. Let X be a nonempty set. A fuzzy topology is a family J of fuzzy sets in X which satisfies the following conditions:

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(i) $\varnothing, X \in J$,

(ii) $A, B \in J \Rightarrow A \cap B \in J$,

(iii) $A_i \in J$ for each $i \in I \Rightarrow \bigcup_{i \in I} A \in J$.

J is called a fuzzy topology for X and the pair (X, J) is called a fuzzy topological space. Every member of J is called an open fuzzy set. The concept of fuzzy topological spaces have been discussed by several authors (see, for example [1], [4]-[6], [8]-[10], [12]-[22], [26, 27], [33], [38], [39], [42]-[45], [49], [55]-[64]).

2.1 Remarks

1. Every topology on a set X is a fuzzy topology but the converse is not true since every set is fuzzy set but not conversely.

2. The indiscrete fuzzy topology contains only $\varnothing$ and X and thus it is the same as indiscrete topology.

3. The discrete fuzzy topology on X consists of all fuzzy subsets of X.

Definition 2.1. Let (X, J) be a fuzzy topological spaces.

(1) A fuzzy subset of X is called closed if its complement if open.

(2) Let A be a fuzzy subset of X, the largest open fuzzy set contained in A is called the interior of A and is denoted by A°.

(3) The smallest closed fuzzy set contained A is called the closure of A and is denoted by $\overline{A}$.

(4) Let X and Y be ftss. A map $f : X \to Y$ is called fuzzy continuous if the inverse image of every fuzzy open set is fuzzy open.

2.2 Remarks

Fuzzy topologies can also be introduced through Kuratowsky closure and in particular from fuzzy neighbourhood systems.

The closure map is defined to be a function $a : L^X \to L^X$ with the following axioms:

$$a(\varnothing) = \varnothing,\ a(X) = X,$$

$$a(A) \supset A,\ a(A \cup B) = a(A) \cup a(B),$$

$$a^2(A) = a(A)$$

with the help of the closure map we define the interior map $i : L^X \to L^X$ as

$$i(A) = (a(A^c))^c.$$

A fuzzy set A is said to be closed iff $a(A) = A$ and open if $i(A) = A$.

If a and b are closure maps, a map $f : (X_a) \to (Y_b)$ is continuous iff

$$A \subset X \Rightarrow f(a(A)) \subset b(f(A)).$$

For fuzzy topologies it is equivalent to say that inverse image of every closed fuzzy set is closed.

Continuity etc. The following concepts have been introduced and discussed in Nanda ([39], [42]-[45]).

Let A be a fuzzy subset in a fuzzy topological space X. A is regular open if $A = \left(\overline{A}\right)^\circ$, and regularly closed if $A = \overline{(A^\circ)}$. $f : X \to Y$ is almost continuous (see Nanda [39]) and Azad [4] if the inverse image of every regularly open fuzzy set is open. f is almost open (afoN) iff the image of every regularly open set is open in Y. Ganguly and Saha [16] have also introduced this concept. According to them f is almost fuzzy open (afoG) if for each $A \in J$, $f^{-1}\left(\overline{A}\right) \subseteq f^{-1}(A)$. A is pre-open if $A \subset \left(\overline{A}\right)^\circ$, semi-open if $A \subset \overline{(A^\circ)}$.

$f : X \to Y$ is pre-continuous if the inverse image of every open fuzzy set in Y is pre-open in X, semicontinuous if the inverse image of every open fuzzy set is semi-open. f is fuzzy weakly continuous

(Azad [4]) iff for every open set B of Y, $f^{-1}(B) \subseteq f^{-1}(\overline{B})$. X is strongly compact if every pre-open cover of X has a finite subcover.

We note quote the following results from [39] and [16].

Theorem 2.2. *If $f : X \to Y$ is afoN and fuzzy semi-continuous, then f is afoG.*

Theorem 2.3. *If $f : X \to Y$ is a afoG and if the image of each fuzzy semi-closed set is fuzzy semi-closed, then f is afoN.*

Theorem 2.4. *If $f : X \to Y$ is fuzzy almost continuous and afoN, then*

(i) *the inverse image $f^{-1}(A)$ of each fuzzy regular open set A of Y is a fuzzy regular open set in X,*

(ii) *then inverse image $f^{-1}(B)$ of each fuzzy regular closed set B of Y is a fuzzy regular closed set in X.*

Theorem 2.5. *If $f : X \to Y$ is a fuzzy almost continuous and afoG, then the inverse image $f^{-1}(A)$ of each fuzzy regular open (closed) set of A of Y is fuzzy regular open (closed) in X.*

Theorem 2.6. *If $f : X \to Y$ is a afoG and fuzzy weakly continuous then f is almost continuous.*

Let (X,J) be a fuzzy topological spaces. Two fuzzy sets A and B in (X,J) are said to be Q-separated if there are closed fuzzy sets F and H such that $F \supset A$, $G \supset B$, $F \cap B = \varnothing$, $H \cap A = \varnothing$.

A fuzzy set D in (X,J) is called disconnected if there are nonempty fuzzy sets A and B in the subspace (D_0, J_{D_0}) such that A and B are Q-separated and $A \cup B = D$. A fuzzy set is called connected if it is not disconnected.

A fts (X,J) is called locally connected at a point $x \in X$ if for any fuzzy neighbourhood U of x, $\exists$ a connected fuzzy neighbourhood V of x such that $V \subset U$. X is said to be locally connected if it is locally connected at every point.

X is completely connected if it is a fuzzy continuous image of some connected and locally connected fts. X is called super-connected if every fuzzy open set is connected and X is Principal if every fuzzy point has a smallest fuzzy neighbourhood.

2.3 Remarks

(1.) Every connected and locally connected fts is completely connected.

(2.) Every completely connected fts is connected.

We have the following results (see Nanda [42, 43]:

Theorem 2.7. *Let $\{X_i\}$ be a family of completely connected fts. If $\cap X_i \neq \varnothing$, then $\cup X_i$ completed connected.*

Theorem 2.8. *Every principal superconnected fts is fuzzy path-connected.*

Let D denote the set of all closed bounded intervals $A = \left[\underset{\sim}{A}, \tilde{A}\right]$ on the real line $\mathbb{R}$. For $A, B \in D$ define

$$A \leq B \text{ iff } \underset{\sim}{A} \leq \underset{\sim}{B} \text{ and } \tilde{A} \leq \tilde{B}$$
$$D(A,B) = \max\left(\left|\underset{\sim}{A} - \underset{\sim}{B}\right|, \left|\tilde{A} - \tilde{B}\right|\right).$$

It is easy to see that d defines a metric on D and (D,d) is a complete metric space. Also $\leq$ is a partial order in D.

A fuzzy number is fuzzy subset of the real number $\mathbb{R}$ which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support. In other words, if $X \in L(\mathbb{R})$ then for any $\alpha \in [0,1]$, X^α is compact where

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha, & \text{if } \alpha \in (0,1], \\ t : X(t) \geq 0 & \text{if } \alpha = 0. \end{cases}$$

Define a map $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$\bar{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d\left(X^\alpha, Y^\alpha\right).$$

For $X, Y \in L(\mathbb{R})$, define

$$X \leq Y \text{ iff } X^\alpha \leq Y^\alpha \text{ for any } \alpha \in [0,1].$$

A subset E of $L(\mathbb{R})$ is said to be bounded above if there exists a fuzzy number C, called an upper bound of E, such that $X \leq C$ for every $X \in E$. C is called least upper bound (l.u.b. or sup) of E if C is an upper bound and is the smallest of all upper bounds. A lower bound and the greatest lower bound (g.l.b. or inf) are defined similarly. E is said to be bounded if it is both bounded above and bounded below.

We now quote the following definition which will be needed in the sequel.

Definition 2.9. A sequence $X = \{X_n\}$ of fuzzy numbers is a function X from the set N of all positive integers into $L(\mathbb{R})$. The fuzzy number X_n denotes the value of the function $n \in \mathbb{N}$ and is called the n-th term of the sequence.

Definition 2.10. A sequence $X = \{X_n\}$ of fuzzy numbers is said to be convergent to the fuzzy number X_0, written as $\lim_n X_n = X_0$, if for every $\varepsilon > 0$ there exists a positive number n_0 such that

$$\bar{d}(X_n, X_0) < \varepsilon \text{ for } n > n_0.$$

Let c denote the set of all convergent sequence of fuzzy numbers. $X = \{X_n\}$ is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\bar{d}(X_n, X_m) < \varepsilon \text{ for } n, m > n_0.$$

Let C denote the set of all Cauchy sequences of fuzzy numbers.

Definition 2.11. A sequence $X = \{X_n\}$ of fuzzy numbers is said to be bounded if the set $\left\{\lim_n X_n : n \in \mathbb{N}\right\}$ of fuzzy numbers is bounded. Let m denote the set of all bounded sequences of fuzzy numbers.

It is straightforward to see that $c \subset C$ and that $c \subset m$.

It is known or can be easily seen that $L(\mathbb{R})$ is a complete metric space with the metric $\bar{d}$. For if $\left\{X^i\right\}$ is a Cauchy sequence in $L(\mathbb{R})$, then $\left\{(X^\alpha)^i\right\}$ is a Cauchy sequence in D for each α, $0 \leq \alpha \leq 1$, and hence $(X^\alpha)^i = X^\alpha$. Now $\lim_i X^i = X$ and $X \in L(\mathbb{R})$.

We now introduce the ℓ_p-spaces $()1 \leq p \leq \infty$ of sequences of fuzzy numbers. We have

$$\ell_p = \left\{ X = \{X_n\} : \sum_n \left[\bar{d}(X_n, 0)\right]^p < \infty \right\}.$$

We have the following result.

Theorem 2.12. *c and m are complete metric spaces each with the metric ρ defined by*

$$\rho(X,Y) = \sup_n \bar{d}(X_n, Y_n)$$

where $X = \{X_n\}$ and $Y = \{Y_n\}$ are convergent or bounded sequences of fuzzy numbers.

Theorem 2.13. *ℓ_p is a complete metric space with the metric h defined by*

$$h(X,Y) = \left(\sum_n \left[\bar{d}(X_n, Y_n)\right]^p \right)^{1/p}$$

where $X = \{X_n\}$ and $Y = \{Y_n\}$ are sequences of fuzzy numbers in ℓ_p.

3 Fuzzy Algebraic Concepts

The concept of fuzzy set has been applied in algebra by various authors to develop fuzzy algebraic concepts (see, for examples [46, 47, 2, 3, 23, 24, 25, 28, 30, 29, 31, 32, 34, 35, 37, 39, 43, 50, 65, 67].

The study of fuzzy vector spaces is warranted at the least by their potential applications. This concept over real and complex number field was introduced in Katsaras and Liu [22]. This concept of fuzzy fields and fuzzy vector space over fuzzy field over was introduced by Nanda [40]. Also fuzzy vector spaces have been discussed in Malik and Mordeson [31], Lubczonok [28] and Muganda [37]. In Muganda [37] it is shown that if A is a fuzzy subspace of V over F, then A has a basis over F. But Mordeson [34] has shown that this result does not hold without the assumption that A has the sup property. This whether or not A has a basis in general, still remains an open question. Also fuzzy vector spaces under triangular norm has been discussed by Das [11]; where as t-fuzzy subfields and t-fuzzy vector spaces have been discussed in Osman [46]. Fuzzy algebras have been considered by Nanda [43].

We first quote the definitions and then state some significant results.

Definition 3.1. *Let X be a group and G is a fuzzy set in X with membership function μ_G. G is called a fuzzy group in X iff*

(*i*) $\mu_G(xy) \geq \min\{\mu_G(x), \mu_G(y)\}$,

(*ii*) $\mu_G\left(x^{-1}\right) \geq \mu_G(x)$.

Definition 3.2. *Let X be a ring and R is a fuzzy set in X with membership function μ_R. R is called a fuzzy ring in X iff*

(*i*) $\mu_R(x+y) \geq \min\{\mu_R(x), \mu_R(y)\}$,

(*ii*) $\mu_R(-x) \geq \mu_R(x)$,

(*iii*) $\mu_R(xy) \geq \min\{\mu_R(x), \mu_R(y)\}$.

R is a fuzzy left (right) ideal iff

(*i*) $\mu_R(xy) \geq \mu_R(y)$,

(*ii*) $[\mu_R(xy) \geq \mu_R(x)]$.

Definition 3.3 (Nanda [40])**.** *Let X be a field and F be a fuzzy subset of X with membership function μ_Y. F is a field iff*

(*i*) $\mu_Y(x+y) \geq \min\{\mu_Y(x), \mu_Y(y)\}$,

(*ii*) $\mu_Y(-x) \geq \mu_Y(x)$,

(*iii*) $\mu_Y(xy) \geq \min\{\mu_Y(x), \mu_Y(y)\}$,

(*iv*) $\mu_Y(x^{-1}) \geq \mu_Y(x)$,

(*v*) $\mu_Y(0)1, \mu_Y(1) = 0$.

Definition 3.4 (Nanda [40]). *Let X be a field and F be a fuzzy subset of X with membership function μ_Y. Let Y be a linear space over field and V a fuzzy subset of Y with membership function μ_Y. V is a linear space in Y iff*

(i) $\mu_Y(x+y) \geq \min\{\mu_Y(x), \mu_Y(y)\}$,

(ii) $\mu_Y(\lambda x) \geq \min\{\mu_F(\lambda), \mu_Y(x)\}$, $\lambda \in F$,

(iii) $\mu_Y(0) = 1$.

If F is an ordinary field or in particular if $F = X$, then (ii) is replaced by

$(ii)'$ $\mu_Y(\lambda x) \geq \mu_Y(x)$.

Definition 3.5 (Nanda [44]). *Let X be a field and F be a fuzzy field in X with membership function μ_F. Let Y be an algebra over X and A a fuzzy subset of Y with membership function μ_A. A is called fuzzy algebra iff*

(i) $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(ii) $\mu_A(\lambda x) \geq \min\{\mu_F(\lambda), \mu_A(x)\}$,

(iii) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$.

If F is an ordinary field or in particular if $F = X$, then (ii) is replaced by

$(ii)'$ $\mu_A(\lambda x) \geq \mu_A(x)$.

Definition 3.6 (Nanda [44]). *Let X be a ring and R be a fuzzy ring in X with membership function μ_R. Let Y be module over R and M a fuzzy subset of Y with membership function μ_M. M is called fuzzy module iff*

(i) $\mu_M(x+y) \geq \min\{\mu_M(x), \mu_M(y)\}$,

(ii) $\mu_M(\lambda x) \geq \min\{\mu_F(\lambda), \mu_M(x)\}$,

(iii) $\mu_M(0) = 1$.

If R is an ordinary ring, then (ii) *is replaced by*

$(ii)'$ $\mu_M(\lambda x) \geq \mu_M(x)$.

Definition 3.7. *A fuzzy set A is a set X, with membership function* μ_A*, is said to have sup property, if for any subset* T *of* X*, there exists* $t_0 \in T$ *such that*

$$\mu_A(t_0) = \sup_{t \in T} \mu_A(t).$$

The results

Theorem 3.8. *Let* X *and* Y *be fields,* f *a homomorphism from* X *into* Y *and* F *a fuzzy field in* Y*. Then* $f^{-1}(F)$ *is a fuzzy field in* X*.*

Theorem 3.9. *If* F *is a fuzzy field in* X*, then* $f(F)$ *need not be a fuzzy field in* Y*. If* F *has sup property, then* $f(F)$ *is a fuzzy field in* Y*.*

Theorem 3.10. *If L is a complete lattice, then the intersection of a family of fuzzy fields is a fuzzy field.*

Remark 3.11. *It should be noted that results similar to Theorem 1, 2 and 3 hold for fuzzy linear spaces, fuzzy algebras, fuzzy (algebra) ideal and fuzzy modules.*

Theorem 3.12. *Let* $X = Q$ *(the field of rational), and* F *any fuzzy field in* X *with membership function* μ_F*. Then*

$$\mu_F(x) = 1 \textit{ for all } x \in X.$$

Theorem 3.13. *Let A and B be fuzzy subspaces of a linear space* V*. Then* $A \times B$ *is a fuzzy subspace of* $V \times V$

Theorem 3.14. *Let B_1 and B_2 be fuzzy subalgebras (fuzzy ideals) of an algebra A. Then $B_1 \times B_2$ is a fuzzy subalgebra (fuzzy ideal) of $A \times A$.*

Theorem 3.15. *Let A and B be fuzzy subsets of a linear space V such that $A \times B$ is a fuzzy subspace of $V \times V$. Then*

(i) *either $\mu_A(x) \leq \mu_A(0)$ for all $x \in V$ or $\mu_B(x) \leq \mu_B(0)$ for all $x \in V$;*

(ii) *if $\mu_A(x) \leq \mu_A(0)$ for all $x \in V$ then either $\mu_A(x) \leq \mu_B(0)$ for all $x \in V$ or $\mu_B(x) \leq \mu_B(0)$ for all $x \in V$;*

(iii) *if $\mu_B(x) \leq \mu_B(0)$ for all $x \in V$ then either $\mu_A(x) \leq \mu_A(0)$ for all $x \in V$ or $\mu_B(x) \leq \mu_A(0)$ for all $x \in V$;*

(iv) *either A or B is a fuzzy subspace of V,*

(v) *both A or B need not be a fuzzy subspace of V.*

Theorem 3.16. *Similar results hold for algebras.*

We also consider a similar problem for fuzzy subfields of a field F. The situation her is different since $F \times F$ is not a field.

Theorem 3.17. *Let A be a fuzzy subset of a field F.*

(i) *If A is a fuzzy subfield of F; then A_t is subfield of F for all $t \in [0, \mu_A(e)]$,*

(ii) *If A_t is a fuzzy subfield of F for all $t \in I_m(\mu_A)$, then A is a fuzzy subfield of F.*

Theorem 3.18. *Let A be a fuzzy subset of a linear space V.*

(i) *If A is a fuzzy subspace of V; then A_t is subspace of F for all $t \in [0, \mu_A(e)]$,*

(ii) *If A_t is a fuzzy subspace of V for all $t \in I_m(\mu_A)$, then A is a fuzzy subspace of V.*

Theorem 3.19. *Let B be a fuzzy subset of an algebra A.*

(i) *If B is a fuzzy subalgebra (fuzzy ideal) of A; then B_t is subalgebra (ideal) of A for all $t \in [0, \mu_B(e)]$,*

(ii) *if B_t is a fuzzy subalgebra (ideal) of A for all $t \in I_m(\mu_B)$, then B is a fuzzy subalgebra (fuzzy ideal) of A.*

4 Fuzzy Geometry Concepts

In pattern recognition and image processing one often needs to measure the geometrical concepts like the length, breadth, height, width, area, perimeter and diameter etc. of regions in images. This was known long long ago if the region is crisply defined. But it was not known until recently how to measure geometrical concepts if the region were fuzzy.

The height, width, diameter, perimeter and area etc. of two dimensional fuzzy sets were discussed in Rosenfeld [51, 52], Rosenfeld and Haler [53]. Bogomolny which generalizes that "the area of a fuzzy set is less than or equal to its height times width".

More recently Pal and Ghosh [48] introduced some further new concepts like length, breadth, major axis, minor axis, center of gravity, density etc. fuzzy sets.

The purpose of this paper is to establish an inequality, which in particular says that "the area of a fuzzy set is less than or equal to its length times the breadth". Another result is also established which generalizes the well known fact that the perimeter is equal to π times the diameter. This sharpens the inequality established by Mukherjee [36] because from the definitions length and breadth which are respectively less than or equal to height and width.

Let $h(\mu)$, $w(\mu)$ and $A(\mu)$ denote respectively the height, width and area of a fuzzy set μ. These concepts are defined as follows (see [51, 52, 53, 54])

$$h(\mu) = \int_R \left[\max_x \mu(x,y) \right] dy,$$
$$w(\mu) = \int_R \left[\max_y \mu(x,y) \right] dx,$$
$$A(\mu) = \iint_R \mu(x,y)\, dx\, dy.$$

More recently Pal and Ghosh [48] introduced the concepts of length and breadth. Let $l(\mu)$ and $b(\mu)$ respectively denote the length and breadth of fuzzy set μ. Then (see [48])

$$l(\mu) = \max_x \int_R \mu(x,y)\, dy,$$
$$b(\mu) = \max_y \int_R \mu(x,y)\, dx.$$

Observe that

$$l(\mu) \leq h(\mu) \text{ and } b(\mu) \leq w(\mu).$$

The concept of height and width was generalized by Mukherjee [36] in the following way.

For $p > 0$ and for a unit vector $\bar{\alpha}$, the $\bar{\alpha}$-p-width of a fuzzy set μ is given by

$$w\frac{p}{\alpha}(\mu) = \int_R^{\max_{\text{set } X}} \mu^{\frac{1}{p}}\left(t\bar{\alpha} + s\bar{b}\right) dt,\ t \in R.$$

Then

$$h^p(\mu) = w\frac{p}{e_2}(\mu) \text{ and } w^p(\mu) = w^p_{e_1}(\mu)$$

where $\bar{e}(\mu) = h(\mu)$ and $w^p(\mu) = w(\mu)$.

Mukherjee [36] proved that

Theorem 4.1. *$A(\mu) \leq h^p(\mu) w^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p,q > 0$.*

This generalizes the earlier known result

$$A\left(\mu^2\right) = \iint_{R^2} \mu^2(x,y) dx\, dy \leq h(\mu) w(\mu).$$

We now introduce the following definition.

For $p > 0$ and for a unit vector $\bar{\alpha}$, $\bar{\alpha}$-p-breadth of a fuzzy set μ is defined by

$$B\frac{p}{\alpha}(\mu) = \max_{x\in R} \int_R \mu^{\frac{1}{p}} \left(t\bar{\alpha} + s\bar{b}\right) dt,\ t \in R.$$

For the p-length and p-breadth are defined respectively by

$$t^p = (\mu) = B\frac{p}{e_2}(\mu) \text{ and } b^p(\mu) = B\frac{p}{e_1}(\mu)$$

where $\overline{e_1} = (1,0)$ and $\overline{e_2} = (0,1)$.

Observe that if $p = 1$, then

$$l^p(\mu) = l(\mu) \text{ and } b^p(\mu) = b(\mu).$$

In this note we prove that

Theorem 4.2.

$$A(\mu) \leq l^p(\mu) b^q(\mu)\, for\ \frac{1}{p} + \frac{1}{q} = 1, p, q > 0.$$

As a special case we have

$$A\left(\mu^2\right) \leq l(\mu) b(\mu),$$

Theorem 4.2 generalizes Theorem 4.1 since $B\frac{p}{\alpha}(\mu) \leq w\frac{p}{\alpha}(\mu)$. We now introduce length, breadth and height of three dimensional fuzzy sets.

Definition 4.3.

$$\begin{aligned}
&\textit{Length} \qquad l_1(\mu) = \int_R \max_x \max_y \mu(x,y,z) dz \\
&\textit{Breadth} \qquad b_1(\mu) = \int_R \max_z \max_x \mu(x,y,z) dy \\
&\textit{Height} \qquad h_1(\mu) = \int_R \max_y \max_z \mu(x,y,z) dx
\end{aligned}$$

Definition 4.4.

$$\begin{aligned}
&\textit{Length} \qquad l_2(\mu) = \max_x \int_R \max_y \mu(x,y,z)dz\\
&\textit{Breadth} \qquad b_2(\mu) = \max_z \int_R \max_x \mu(x,y,z)dy\\
&\textit{Height} \qquad h_2(\mu) = \max_y \int_R \max_z \mu(x,y,z)dx
\end{aligned}$$

Definition 4.5.

$$\begin{aligned}
&\textit{Length} \qquad l'_2(\mu) = \max_y \int_R \max_x \mu(x,y,z)dz\\
&\textit{Breadth} \qquad b'_2(\mu) = \max_x \int_R \max_z \mu(x,y,z)dy\\
&\textit{Height} \qquad h'_2(\mu) = \max_z \int_R \max_y \mu(x,y,z)dx
\end{aligned}$$

Definition 4.6.

$$\begin{aligned}
&\textit{Length} \qquad l_3(\mu) = \max_x \max_y \int_R \mu(x,y,z)dz\\
&\textit{Breadth} \qquad b_3(\mu) = \max_z \max_x \int_R \mu(x,y,z)dy\\
&\textit{Height} \qquad h_3(\mu) = \max_y \max_z \int_R \mu(x,y,z)dx
\end{aligned}$$

It may be noted that the following inequalities hold.

$$l_3 \le l_2 \le l_1,\ b_3 \le b_2 \le b_1,\ h_3 \le h_2 \le h_1.$$

If we put

$$B^p_{\bar{a}} = \max_y \max_z \int_R \mu(x\bar{a} + y\bar{b} + z\bar{c})dx,$$

$$l^p_3 = B^p_{\bar{e}_1},\ b^p_3 = B^p_{\bar{e}_2}, h^p_3 = B^p_{\bar{e}_3}$$

where $\bar{e}_1 = (1,0,0)$, $\bar{e}_2 = (0,1,0)$, $\bar{e}_3 = (0,0,1)$.

We define the volume of a fuzzy set by

$$V(\mu) = \iiint_{R^3} \mu(x,y,z)\, dx\, dy\, dz.$$

Now we have the following

Theorem 2.

$$V(\mu) \le l_3^p b_3^q h_3^r \text{ where } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \text{ and } p,q,r > 0.$$

References

1. I. W. Aldarton, Function spaces in Fuzzy Topology, FSS, 32(1) (1989), 115.

2. J. M. Authony and H. Sherwood, Fuzzy groups redefined, JMAA 69 (1979), 279 - 305.

3. J. M. Authony and H. Sherwood, A characterisation of Fuzzy subgroups, FSS 7 (1982), 279 - 305.

4. K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, JMAA, 82 (1981). 14. 32.

5. K. K. Azad, Fuzzy Hausdorff spaces and fuzzy parfect mapping, JMAA, 82 (1981), 297 - 305.

6. R. Badard, Fuzzy pretopological spaces and their applications, JMAA, 81 (1981), 378 - 90.

7. A. Bogomolny, On the perimeter and area of fuzzy sets, FSS, 23 (1987), 257 - 269.

8. C. L. Chane, Fuzzy topological spaces, JMAA. 24 (1968), 182 - 90.

9. F. Conrad, Fuzzy topological concepts, JMAA, 74 (1980), 432 - 40.

10. P. Das, Note on semi-connectedness, Indian J. Mech. Math., 12(1) (1974), 31 - 34.

11. P. S. Das, Fuzzy vector spaces under triangular norm, FSS 3 (1988), 73 - 85.

12. A. A. Fora, Separation axioms for fuzzy spaces, FSS 33 (1989), 59.

13. M. H. Chania and A. S. Mashhour, Characterisation of fuzzy topologies by *S*-neighbourhood systems, FSS 9 (1983), 211 - 213.

14. D. H. Foster, Fuzzy topological groups, JMAA 67, (1979), 549 - 64.

15. J. A. Goguen, The fuzzy tychonoff theorem, JMAA, 43 (1973), 734 - 42.

16. S. Ganguly and S. Saha, A note on semi-open sets in fuzzy topological spaces, FSS 18 (1986) 83 - 96.

17. B. Ghosh, Semicontinuous and semiclosed mapping and semi-connectedness in fuzzy setting, FSS 35 (1990), 345 -

18. J. S. Golan, Making modules fuzzy, FSS, 32 (1989), 91.

19. B. Hutton, Normality in fuzzy topological spaces, JMAA, 50 (1975), 74 - 79.

20. B. Hurron, Uniformities in fuzzy topological spaces, JMAA, 58 (1977), 559 - 571.

21. B. Hutton and I. Reilly, Seperation axiom in fuzzy topological spaces, FSS 3 (1980), 13 - 104.

22. A. K. Katsaras and D. B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, JMAA 58 (1977), 135 - 46.

23. R. Kumar, Fuzzy primary ideals: some ring theorctic analogues, Bull. Cal. Math. Soc., (to apear).

24. W.J. Liu, Fuzzy invariant subgroups and fuzzy ideals, FSS 8 (1982), 135 - 139.

25. W.J. Liu, Operations on fuzzy ideals, FSS 11 (1982), 31 - 41.

26. R. Lowen, Initial and final fuzzy topologies and the fuzzy Tychonoff Theorem, JMAA 58 (1977), 11 - 21.

27. R. Lowen, Fuzzy neighbourhood spaces, FSS 7 (1982), 165 - 69.

28. P. Lubezonok, Fuzzy vector spaces, FSS 38 (1990), 329 - 343, F-7.

29. D. S. Malik and J. N. Mordeson, Fuzzy prime ideals of a ring, FSS 37 (1990) 93.

30. D. S. Malik, Fuzzy ideals of Artinlan rings, FSS 37 (1990) 111.

31. D. S. Malik and J. N. Mordeson, Fuzzy vector spaces, Inform. Soc., 55 (1991), 271 - 281.

32. D. S. Malik and J. N. Mordeson, Fuzzy subgroups of abelian groups, Chinese J. Math.

33. A. S. Mashhour, R. Badard and A. A. Ramadan, Fuzzy preproximity spaces, FSS 35 (1990), 333 -

34. J. N. Mordeson, Bases of fuzzy vector spaces, Inform. Soc., (Private Communications).

35. J. N. Mordeson, Fuzzy field extensions, FSS, (Private Communications).

36. R. N. Mukherjee, A note on area and perimeter of fuzzy sets (Private communication)

37. G. C. Muganda, Fuzzy linear and affine spaces, FSS 19 (1986) 365 - 373.

38. M. N. Mukherjee and S. P. Sinha, On some weaker forms of fuzzy continuous and fuzzy open mappings on fuzzy topological spaces, FSS 32(1) (1984), 103 -

39. S. Nanda, On fuzzy topological spaces, FSS 19 (1986), 193 - 97.

40. S. Nanda, Fuzzy fields and fuzzy linear space, FSS 19 (1986), 89 - 94.

41. S. Nanda, Fuzzy modules over fuzzy rings, Bull. Cal. Math. Soc. 81 (1989), 197 - 200.

42. S. Nanda, Completely connected FTS, FSS

43. S. Nanda, Fuzzy algebra over fuzzy fields, FSS 37 (1990), 99 - 103.

44. S. Nanda, Principal superconnected FTS, FSS 35 (1990), 397 - 99.

45. S. Nanda, Strongly compact FTS, FSS 42 (1991).

46. M. T. Abu Osman, On the direct product of fuzzy subgroups, FSS, 12 (1984), 84 - 89.

47. M. T. Abu Osman, On t-fuzzy subfield and t-fuzzy subgroups, FSS, 33 (1989), 111.

48. S. K. Pal and A. Ghosh, Fuzzy geometry in image analysis, FSS 48 (1992), 23 - 40.

49. S. E. Rodabaugh, The Housdorff separation axiom for fuzzy topological spaces, Topology Appl. 11 (1980), 319 - 334.

50. A. Rosenfeld, fuzzy groups, JMAA 35 (1971), 512 - 17.

51. A. Rosenfeld, The diameter of fuzzy sets, Fuzzy sets and systems, 13 (1984), 241 - 246.

52. A. Rosenfeld, Fuzzy geometry of image subsets, Pattern Recog. Lett., 2 (1984), 311 - 317.

53. A. Rosenfeld and S. Haler, The perimeter of fuzzy sets, Pattern Recog. 18 (1985), 125 - 130.

54. A. Rosenfeld, The diameter of a fuzzy sets, Fuzzy sets and systems, 13 (1994), 241 - 246.

55. R. Rowen, Fuzzy topological spaces and fuzzy compactness, JMAA 56 (1976), 62 - 633.

56. R. Srivastava, S. N. Lal and A, K. Srivastava, Fuzzy T_1-Topological spaces, JMAA 102 (1984), 442 - 448.

57. R. Srivastava, S. N. Lal and A. K. Srivastava, On fuzzy T_1-Topological spaces, JMAA, 136 (1988), 124 - 130.

58. R. Srivastava and A. K. Srivastava, On Fuzzy Hausdorffness concepts, FSS 17 (1985) 67 - 71.

59. A. Srivastava, R_1 Fuzzy Topological spaces, JMAA 127 (1987), 151 - 154.

60. R. A. K. Srivastava and D. M. Ali, A comparison of some F T_2 concepts, FSS 23 (1987), 289 - 294.

61. M. Sarkar, On fuzzy topological spaces, JMAA 79 (1981), 383 - 394.

62. C. K. Wong, Converting properties of FTS, JMAA 43 (1973).

63. C. K. Wong, Fuzzy topology-product and quotient theorems, JMAA, 45 (1974), 512.

64. C. K. Wong, Fuzzy points and local properties of fuzzy topology, JMAA 46 (1974), 316 - 326.

65. W. M. Wu, Fuzzy Normal fuzzy subgroups, Fuzzy Math 1 (1981), 21 - 30. 46 (1974), 316 - 326.

66. L. A. Zadeh, Fuzzy sets, Information and control 8 (1965), 338 - 353.

67. L. A. Zadeh, Similarity relations and fuzzy ordering, Information science 3 (1971), 177 - 220.