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Degree of Approximation of Conjugate Fourier Series of Functions in the Besov space by Matrix Mean

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Abstract

he paper studies the degree of approximation of functions by their Fourier series in the Besov space by matrix mean and this generalizing many known results.

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1 Introduction

Let f be a 2π periodic function and let $f \in L_p[0,2\pi], p \geq 1$. The fourier series of f at x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

The conjugate series of ((1.1)) is given by

$$\sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx). \tag{1.2}$$

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Let $\widetilde{S}_n(x)$ be the nth partial sum of conjugate series ((1.2)) given by [Zygmund [6], Vol I, page-49]

$$\widetilde{S}_n(x) = \frac{-1}{\pi}\int_{-\pi}^{\pi}\psi_x(t)\widetilde{D}_n(t)dt \tag{1.3}$$

where

$$\psi_x(t) = f(x+t) - f(x-t) \tag{1.4}$$

and the conjugate Dirichlet kernel is defined by

$$\widetilde{D}_n(t) = \sum_{k=1}^{n}\sin kt = \frac{\cos\frac{t}{2} - \cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} \tag{1.5}$$

Let $A = (a_{n,k})$ be an infinite matrix.

We assume that elements of the matrix $A = (a_{n,k})$ satisfy the following regularity conditions

$$||A|| = \sup_n \sum_{k=0}^{\infty}|a_{n,k}| < \infty \tag{1.6}$$

$$(a_{n,k}) \to 0 \text{ as } n \to \infty \text{ and } k \text{ is fixed} \tag{1.7}$$

and

$$\sum_{k=0}^{\infty} a_{n,k} = 1 \text{ for each } n = 0,1,2\cdots. \tag{1.8}$$

Let $\tilde{t}_n(x)$ denote the $A = (a_{n,k})$ transformation of the conjugate series ((1.2)); that is

$$\tilde{t}_n(x) = \sum_{k=0}^{\infty} a_{n,k}\widetilde{S}_k(x) \tag{1.9}$$

$$(i) \quad \mathscr{I}^{0}_{p,b}f(z) = f(z),\ f \in A(p);$$

$$(ii) \quad \mathscr{I}^{-1}_{p,p-1}f(z) = a_p\, z^p + \sum_{n=1}^{\infty}\frac{n+p}{p}\, a_{n+p}\, z^{n+p} = \frac{zf'(z)}{p},\ f \in A(p);$$

$$(iii) \quad \mathscr{I}^{-1}_{1,0}f(z) = a_1\, z + \sum_{n=1}^{\infty}(n+1)\, a_{n+1}\, z^{n+1} = zf'(z),\ f \in A(1).$$

We use the following notations throughout.

$$\widetilde{K}_n(t) = \sum_{k=0}^{\infty} a_{n,k} \widetilde{D}_k(t) \tag{1.10}$$

$$H_n(t) = \sum_{k=0}^{\infty} a_{n,k} \frac{\cos(k+\frac{1}{2})t}{2\sin\frac{1}{2}t} \tag{1.11}$$

$$\tilde{f}(x;\varepsilon) = -\frac{1}{\pi}\int_{\varepsilon}^{\pi} \psi_x(t)\frac{1}{2}\cot\frac{1}{2}t\,dt, \varepsilon > 0 \tag{1.12}$$

$\tilde{f}(x) = \lim_{\varepsilon\to 0} \tilde{f}(x;\varepsilon)$, whenever the limit exists.

$$\widetilde{T}_n(x) = \tilde{t}_n(x) - \tilde{f}(x;\frac{\pi}{n}) \tag{1.13}$$

It is known ([6]) that for any integrable f the function $\tilde{f}$ exists almost everywhere. It is easy to see that

$$\widetilde{K}_n(t) = \frac{1}{2}\cot\frac{1}{2}t - H_n(t) \tag{1.14}$$

$$\begin{aligned}
\tilde{t}_n(x) &= -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\widetilde{K}_n(t)dt \\
&= -\frac{1}{\pi}\int_0^{\frac{\pi}{n}} \psi_x(t)\widetilde{K}_n(t)dt - \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} \psi_x(t)\widetilde{K}_n(t)dt \\
&= -\frac{1}{\pi}\int_0^{\frac{\pi}{n}} \psi_x(t)\widetilde{K}_n(t)dt - \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} \psi_x(t)\left\{\frac{1}{2}\cot\frac{1}{2}t - H_n(t)\right\}dt \\
&= \frac{-1}{\pi}\int_0^{\frac{\pi}{n}} \psi_x(t)\widetilde{K}_n(t)dt + \tilde{f}(x;\frac{\pi}{n}) + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} \psi_x(t)H_n(t)dt
\end{aligned}$$

(1.15)

Using ((1.3)) in ((1.9)) and there after making use of the notations given in ((1.10)) to ((1.14)), we get

$$\widetilde{t}_n(x) = \frac{-1}{\pi}\int_0^{\frac{\pi}{n}} \psi_x(t)\widetilde{K}_n(t)dt + \tilde{f}(x,\frac{\pi}{n}) + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} \psi_x(t)H_n(t)dt \tag{1.16}$$

Using ((1.13)) in ((1.16)), we have

$$\widetilde{T}_n(x) = \frac{-1}{\pi}\int_0^{\frac{\pi}{n}} \psi_x(u)\widetilde{K}_n(u)du + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} \psi_x(u)H_n(u)du \tag{1.17}$$

2 Definitions and Notations

Modulus of Continuity:

Let $A = R, R + [a,b] \subset R$ or T (which usually taken to be R with identification of points modulo 2π).

The modulus of continuity $w(f,t) = w(t)$ of a function f on A can be defined as

$$w(t) = w(f,t) = \sup_{\substack{|x-y| \leq t, \\ x,y \in A}} |f(x) - f(y)|, t \geq 0.$$

Modulus of Smoothness:

The k^{th} order modulus of smoothness [2] of a function $f : A \to R$ is defined by

$$w_k(f,t) = \sup_{0<h\leq t} \{\sup |\Delta_h^k(f,x)| : x, x+kh \in A\},\ t \geq 0 \tag{2.1}$$

where

$$\Delta_h^k(f,x) = \sum_{i=0}^{k} (-1)^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} f(x+ih), k \in N. \tag{2.2}$$

For $k = 1, w_1(f,t)$ is called the modulus of continuity of f. The function w is continuous at $t = 0$ if and only if f is uniformly continuous on A, that is $f \in \tilde{c}(A)$. The k^{th} order modulus of smoothness of $f \in L_p(A), 0 < p < \infty$ or of $f \in \tilde{c}(A), if p = \infty$ is defined by

$$w_k(f,t)_p = \sup_{0<h\leq t} ||\Delta_h^k(f,\cdot)||_p, t \geq 0 \tag{2.3}$$

if $p \geq 1, k = 1$, then $w_1(f,t)_p = w(f,t)_p$ is a modulus of continuity (or integral modulus of continuity). If $p = \infty, k = 1$ and f is continuous then $w_k(f,t)_p$ reduces to modulus of continuity $w_1(f,t)$ or $w(f,t)$.

Lipschitz Space:

If $f \in \tilde{c}(A)$ and

$$w(f,t) = O(t^{\alpha}), 0 < \alpha \leq 1 \tag{2.4}$$

then we write $f \in Lip\alpha$. If $w(f,t) = O(t)$ as $t \to 0+$ (in particular (1.9) holds for $\alpha > 1$) then f reduces to a constant.

If $f \in L_p(A), 0 < p < \infty$ and

$$w(f,t)_p = O(t^{\alpha}), 0 < \alpha \leq 1 \tag{2.5}$$

then we write $f \in Lip(\alpha, p)$, $0 < p < \infty$, $0 < \alpha \leq 1$.

The case $\alpha > 1$ is of no interest as the function reduces to a constant, whenever

$$w(f,t)_p = O(t) \text{ as t} \to 0+ \tag{2.6}$$

We note that if $p = \infty$ and $f \in c(A)$, then $Lip(\alpha, p)$ class reduces to $Lip\ \alpha$ class.

Generalized Lipschitz Space:

Let $\alpha > 0$ and suppose that $k = [\alpha] + 1$. For $f \in L_p(A)$, $0 < p < \infty$, if

$$w_k(f,t) = O(t^{\alpha}), t > 0 \tag{2.7}$$

then we write

$$f \in Lip^*(\alpha, p),\ \alpha > 0,\ 0 < p \leq \infty \tag{2.8}$$

and say that f belongs to generalized Lipschitz space. The seminorm is then

$$|f|_{Lip^*(\alpha,L_p)} = \sup_{t>0}(t^{-\alpha} w_k(f,t)_p).$$

It is known ([2], p-52) that the space $Lip^*(\alpha, L_p)$ contains $Lip(\alpha, L_p)$. For $0 < \alpha < 1$ the spaces coincide, (for $p = \infty$, it is necessary to replace L_∞ by $\tilde{c}$ of uniformly continuous function on A). For $0 < \alpha < 1$ and $p = 1$ the space $Lip^*(\alpha, L_p)$ coincide with $Lip\alpha$.

For $\alpha = 1, p = \infty$, we have

$$Lip(1, \tilde{c}) = Lip\ 1 \tag{2.9}$$

but

$$Lip^*(1, \tilde{c}) = z \tag{2.10}$$

is the Zygmund space [5] which is characterized by ((2.7)) with $k = 2$.

Hölder ($\mathbf{H}_\alpha$) Space:

For $0 < \alpha \leq 1$, let

$$H_\alpha = \{f \in C_{2\pi} : w(f,t) = O(t^\alpha)\}. \tag{2.11}$$

It is known [3] that H_α is a Banach Space with the norm $||\cdot||_\alpha$ defined by

$$||f||_\alpha = ||f||_c + \sup_{t>0} t^{-\alpha} w(t),\ 0 < \alpha \leq 1 \tag{2.12}$$

$$||f||_0 = ||f||_c$$

and

$$H_\alpha \subseteq H_\beta \subseteq C_{2\pi},\ 0 < \beta \leq \alpha \leq 1 \tag{2.13}$$

$\mathbf{H}_{(\alpha,p)}$ Space:

For $0 < \alpha \leq 1$, let

$$H_{(\alpha,p)} = \{f \in L_p[0,2\pi] : 0 < p \leq \infty, w(f,t)_p = O(t^\alpha)\} \tag{2.14}$$

and introduce the norm $||\cdot||_{(\alpha,p)}$ as follows

$$||f||_{(\alpha,p)} = ||f||_p + \sup_{t>0} t^{-\alpha} w(f,t)_p,\ 0 < \alpha \leq 1. \tag{2.15}$$

$$||f||_{(0,p)} = ||f||_p.$$

It is known [1] that $H_{(\alpha,p)}$ is a Banach space for $p \geq 1$ and a complete p-normed space for $0 < p < 1$.

Also

$$H_{(\alpha,p)} \subseteq H_{(\beta,p)} \subseteq L_p, 0 < \beta \leq \alpha \leq 1. \tag{2.16}$$

Note that $H_{(\alpha,\infty)}$ is the space H_α defined above.

For study of degree of approximation problems the natural way to proceed to consider with some restrictions on some modulus of smoothness as prescribed in H_α and $H_{(\alpha,p)}$ spaces. As we have seen above only a constant function satisfies Lipschitz condition for $\alpha > 1$. However for generalized Lipschitz class there is no such restriction on α. We required a finer scale of smoothness than is provided by Lipschitz class. For each $\alpha > 0$ Besov developed a remarkable technique for restricting modulus of smoothness by introducing a third parameter q (in addition to p on α) and applying $\alpha \cdot q$ norms (rather than α, ∞ norms) to the modulus of smoothness $w_k(f,\cdot)_p$ of f.

Besov space:

Let $\alpha > 0$ be given and let $k = [\alpha + 1]$. For $0 < p, q \leq \infty$, the Besov space ([2], p-54)$B_q^\alpha(L_p)$ is defined as follows:

$$B_q^\alpha(L_p) = \{f \in L_p : |f|_{B_q^\alpha(L_p)} = ||w_k(f,\cdot)||_{(\alpha,q)} \ is\ finite\}$$

where

$$||w_k(f,\cdot)||_{(\alpha,q)} = \begin{cases} \int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t})^{\frac{1}{q}}, & 0 < q < \infty \\ \sup\limits_{t>0} t^{-\alpha} w_k(f,t)_p, & q = \infty. \end{cases}$$

It is known ([2], p-55) that $||w_k(f,\cdot)||_{(\alpha,q)}$ is a seminorm if $1 \leq p, q \leq \infty$ and a quasi-seminorm in other cases.

The Besov norm for $B_q^\alpha(L_p)$ is

$$||f||_{B_q^\alpha(L_p)} = ||f||_p + ||w_k(f,\cdot)||_{(\alpha,q)} \tag{2.17}$$

We know ([2], p-56, [4], p-236) the following inclusion relations.

For fixed α and p

$$B_q^\alpha(L_p) \subset B_{q_1}^\alpha(L_p), q < q_1.$$

For fixed p and q

$$B_q^\alpha(L_p) \subset B_q^\beta(L_p), \beta < \alpha.$$

For fixed α and q

$$B_q^\alpha(L_p) \subset B_q^\alpha(L_{p_1}), p_1 < p.$$

Special cases of Besov space:

For $q = \infty, B_\infty^\alpha(L_p), \alpha > 0, p \geq 1$ is same as $Lip^*(\alpha, L_p)$ the generalized Lipschitz space and the corresponding norm $||\cdot||_{B_\infty^\alpha(L_p)}$ is given by

$$||f||_{B_\infty^\alpha(L_p)} = ||f||_p + \sup_{t>0} t^{-\alpha} w_k(f,t)_p \tag{2.18}$$

for every $\alpha > 0$ with $k = [\alpha] + 1$.

For the special case when $0 < \alpha < 1$, $B_\infty^\alpha(L_p)$ space reduces to $H_{(\alpha,p)}$ space due to Das et al. [1] and the corresponding norm is given by

$$||f||_{B_\infty^\alpha(L_p)} = ||f||_{(\alpha,p)} = ||f||_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, 0 < \alpha < 1. \tag{2.19}$$

For $\alpha = 1$, the norm is given by

$$||f||_{B_\infty^1(L_p)} = ||f||_p + \sup_{t>0} t^{-\alpha} w_2(f,t)_p. \tag{2.20}$$

Note that $||f||_{B_\infty^1(L_p)}$ is not same as $||f||_{(1,p)}$ and the space $B_\infty^1(L_p)$ includes the space $H(1,p),\ p \geq 1$. If we further specialize by taking $p = \infty$, B_∞^α, $0 < \alpha < 1$, coincides with H_α space due to Prossodorf [3] and the norm is given by

$$||f||_{B_\infty^\alpha(L_\infty)} = ||f||_\alpha = ||f||_c + \sup_{t>0} t^\alpha w(f,t),\ 0 < \alpha < 1. \tag{2.21}$$

For $\alpha = 1,\ p = \infty$, the norm is given by

$$||f||_{B_\infty^1(L_\infty)} = ||f||_c + \sup_{t>0} t^{-1} w_2(f,t),\ \alpha = 1 \tag{2.22}$$

which is different from $||f||_1$ and $B_\infty^1(L_\infty)$ includes the H_1 space.

3 Main Result:

We prove the following theorem.

Theorem : Let the matrix $A = (a_{n,k})$ be a regular matrix , which satisfy the condition $a_{n,k} \geq 0$ and $a_{n,k} \geq a_{n,k+1}$.

Let $0 < \alpha < 2$ and $0 \leq \beta < \alpha$. If $f \in B_q^{\alpha}(L_p), p \geq 1$ and $1 < q \leq \infty$ and let $\tilde{t}_n(x)$ be the $A = (a_{n,k})$ transformation of the conjugate series, then

Case 1:(For $1 < q < \infty$)

$$||\widetilde{T}_n(\cdot)||_{B_q^{\beta}(L_p)} = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O(a_{n0})\left\{\sum_{k=1}^{n}\left(\frac{1}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}}$$

Case 2:(For $q = \infty$)

$$||\widetilde{T}_n(\cdot)||_{B_{\infty}^{\beta}(L_p)} = O\left(\frac{1}{n^{\alpha-\beta}}\right) + O(a_{n0})\sum_{k=1}^{n}\frac{1}{k^{\alpha-\beta}}$$

4 Additional Notations and Lemmas:

We need the following additional notations for the proof of the theorem.

$$\psi(x,t,u) = \begin{cases} \psi_{x+t}(u) - \psi_x(u) & \\ \psi_{x+u}(t) - \psi_x(t), & 0 < \alpha < 1 \\ \psi_{x+t}(u) + \psi_{x-t}(u) - 2\psi_x(u) & \\ \psi_{x+u}(t) + \psi_{x-u}(t) - 2\psi_x(t), & 1 \leq \alpha < 2 \end{cases} \tag{4.23}$$

For $k = [\alpha] + 1$, we have for $p \geq 1$

$$w_k(f,t)_p = \begin{cases} w_1(f,t)_p, & 0 < \alpha < 1 \\ w_2(f,t)_p, & 1 \leq \alpha < 2 \end{cases} \tag{4.24}$$

Let

$$\widetilde{T}_n(x,t) = \begin{cases} \tilde{T}_n(x+t) - \tilde{T}_n(x) & 0 < \alpha < 1 \\ \tilde{T}_n(x+t) + \tilde{T}_n(x-t) - 2\tilde{T}_n(x) & 1 \leq \alpha < 2 \end{cases} \tag{4.25}$$

Using ((4.25)) and definition of $w_k(f,t)_p$, we have

$$w_k(\tilde{T}_n,t)_p = ||\widetilde{T}_n(\cdot,t)||_p \tag{4.26}$$

Using ((1.17)) and ((4.23)) respectively for the expressions $\tilde{T}_n(x)$ and $\psi(x,t,u)$, we have

$$\widetilde{T}_n(x,t) = \frac{-1}{\pi}\int_0^{\frac{\pi}{n}} \psi(x,t,u)\widetilde{K}_n(u)du + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} \psi(x,t,u)H_n(u)du \tag{4.27}$$

We need the following Lemmas to prove the theorem.

Lemma 1 *Let* $1 \le p \le \infty$ *and* $0 < \alpha < 2$. *If* $f \in L_p[0,2\pi]$, *then for* $0 < t,u \le \pi$

(i) $||\psi(\cdot,t,u)||_p \le 2w_k(f,t)_p$

(ii) $||\psi(\cdot,t,u)||_p \le 2w_k(f,u)_p$

(iii) $||\psi_\cdot(u)||_p \le 2w_k(f,u)_p$,

where $k = [\alpha]+1$.

Proof: We first consider the case $0 < \alpha < 1$.

Clearly $k = 1$ and we can express by virtue of ((4.23))

$$\psi(x,t,u) = \begin{cases} \psi_{x+t}(u) - \psi_x(u) \\ \psi_{x+u}(t) - \psi_x(t) \end{cases}$$

as follows:

$$\begin{aligned}
\psi(x,t,u) &= \begin{cases} \{f(x+t+u) - f(x+t-u)\} - \{f(x+u) - f(x-u)\} \\ \{f(x+t+u) - f(x+u-t)\} - \{f(x+t) - f(x-t)\} \end{cases} \\
&= \begin{cases} \{f(x+t+u) - f(x+u)\} - \{f(x-u+t) - f(x-u)\} & (4.6) \\ \{f(x+t+u) - f(x+t)\} - \{f(x-t+u) - f(x-t)\} & (4.7) \end{cases}
\end{aligned}$$

Applying Minkowski's inequality to (4.6), we get for $p \ge 1$

$$\begin{aligned}
||\psi(\cdot,t,u)||_p &\le ||f(\cdot+t+u) - f(\cdot+u)||_p + ||f(\cdot+t-u) - f(\cdot-u)||_p \\
&\le 2w_1(f,t)_p, \;\; 0 < \alpha < 1
\end{aligned}$$

Similarly applying Minkowski's inequality to (4.7), we get for $p \geq 1$

$$||\psi(\cdot,t,u)||_p \leq 2w_1(f,u)_p.$$

When $1 \leq \alpha < 2$, clearly $k = 2$ and we can write

$$\psi(x,t,u) = \begin{cases} \{f(x+t+u)-f(x+t-u)\}+\{f(x-t+u)+f(x-t-u)\} \\ -2\{f(x+u)-f(x-u)\} \\ \{f(x+t+u)-f(x-t+u)\}+\{f(x+t-u)+f(x-t-u)\} \\ -2\{f(x+t)-f(x-t)\} \end{cases}$$

$$= \begin{cases} \{f(x+t+u)+f(x+u-t)-2f(x+u)\} \\ -\{f(x-u+t)+f(x-t-u)-2f(x-u)\} \quad (4.8) \\ \{f(x+t+u)+f(x+t-u)-2f(x+t)\} \\ -\{f(x-t+u)+f(x-t-u)-2f(x-t)\} \quad (4.9) \end{cases}$$

Applying Minkowski's inequality to (4.8), we obtain for $p \geq 1$

$$\begin{aligned} ||\psi(\cdot,t,u)||_p &\leq ||f(\cdot+t+u)+f(\cdot+u-t)-2f(\cdot+u)||_p \\ &+ ||f(\cdot-u+t)+f(\cdot-t-u)-2f(\cdot-u)||_p \\ &\leq 2w_2(f,t)_p \end{aligned}$$

Similarly, applying Minkowski's inequality to (4.9), we obtain for $p \geq 1$

$$||\psi(\cdot,t,u)||_p \leq 2w_2(f,u)_p$$

and this completes the proof of part (i) and (ii).

The proof of (iii) follows from

$$\psi_{\cdot}(u) = \{f(\cdot+u)-f(\cdot)\}-\{f(\cdot-u)-f(\cdot)\}.$$

Lemma 2 *Let* $0 < \alpha < 2$, $0 \leq \beta < \alpha$. *If* $f \in B_q^\alpha(L_p), p \geq 1,\ 1 < q < \infty$, *then*

(i) $\int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| \left(\int_0^u \frac{||\psi(\cdot,t,u)||_p^q}{t^{\beta q}} \frac{dt}{t}\right)^{\frac{1}{q}} du = O(1) \left\{\int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta}|\widetilde{K}_n(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$

(ii) $\int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| \left(\int_u^{\pi} \frac{||\psi(\cdot,t,u)||_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} du = O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta+\frac{1}{q}} |\widetilde{K}_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$

(iii) $\int_0^{\frac{\pi}{n}} |H_n(u)| \left(\int_0^{u} \frac{||\psi(\cdot,t,u)||_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} du = O(1) \left\{ \int_{\frac{\pi}{n}}^{\pi} \left(u^{\alpha-\beta} |H_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$

(iv) $\int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_u^{\pi} \frac{||\psi(\cdot,t,u)||_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} du = O(1) \left\{ \int_{\frac{\pi}{n}}^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}} |H_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$

Proof: Applying Lemma 1(i), we get

$$\begin{aligned}
\int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| \left(\int_0^{u} \frac{||\psi(\cdot,t,u)||_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du &= O(1) \int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| \left(\int_0^{u} \left(\frac{w_k(f,t)_p}{t^{\alpha}} \right)^q t^{(\alpha-\beta)q} \frac{dt}{t} \right)^{\frac{1}{q}} du \\
&= O(1) \int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| u^{(\alpha-\beta)} \left(\int_0^{u} \frac{w_k(f,t)_p}{t^{\alpha}} \frac{dt}{t} \right)^{\frac{1}{q}} du \\
&= O(1) \int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| u^{(\alpha-\beta)} du
\end{aligned}$$

the inner integral being finite as $f \in B_q^{\alpha}(L_p)$. Applying Hölders inequality

$$\begin{aligned}
&= O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(|\widetilde{K}_n(u)| u^{(\alpha-\beta)} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \left(\int_0^{\pi} 1^q du \right)^{\frac{1}{q}} \\
&= O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(|\widetilde{K}_n(u)| u^{(\alpha-\beta)} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}
\end{aligned}$$

Applying Lemma 1(ii), we get

$$\begin{aligned}
\int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| \left(\int_u^{\pi} \frac{||\psi(\cdot,t,u)||_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du &= O(1) \int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| \left\{ \int_u^{\pi} \left(\frac{w_k(f,u)_p}{t^{\beta}} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} du \\
&= O(1) \int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| w_k(f,u)_p du \left(\int_u^{\pi} \frac{dt}{t^{\beta q+1}} \right)^{\frac{1}{q}} \\
&= O(1) \int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| w_k(f,u)_p u^{-\beta} du \\
&= O(1) \int_0^{\frac{\pi}{n}} \left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}} \right) u^{\alpha-\beta+\frac{1}{q}} |\widetilde{K}_n(u)| du
\end{aligned}$$

Applying Hölder's inequality

$$\begin{aligned}
&= O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(\frac{w_k(f,u)_p}{u^{\alpha}} \right)^q \frac{du}{u} \right\}^{\frac{1}{q}} \left\{ \int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta+\frac{1}{q}} |\widetilde{K}_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
&= O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta+\frac{1}{q}} |\widetilde{K}_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}
\end{aligned}$$

As the first integral on the above is finite by hypothesis. Third part and 4th part of the Lemma follows from above replacing $\widetilde{K}_n(u)$ by $H_n(u)$.

Lemma 3 *Let* $0 < \alpha < 2$. *Suppose that* $0 \le \beta < \alpha$. *If* $f \in B_q^\alpha(L_p),\ p \ge 1$ *and* $q = \infty$, *then*

$$\sup_{0<t,u\le\pi} t^{-\beta}||\psi(\cdot,t,u)||_p = O(u^{\alpha-\beta})$$

Proof: For $0 < t \le u \le \pi$, applying Lemma 1(i), we have

$$\begin{aligned} \sup_{\substack{t,\\ 0<t\le u\le\pi}} t^{-\beta}||\psi(\cdot,t,u)||_p &= \sup_{\substack{t,\\ 0<t\le u\le\pi}} t^{\alpha-\beta}(t^{-\alpha}||\psi(\cdot,t,u)||_p) \\ &\le 4u^{\alpha-\beta}\sup_t(t^{-\alpha}w_k(f,t)_p) \\ &= O(u^{\alpha-\beta}), \quad \text{by the hypothesis.} \end{aligned}$$

Next for $0 < u \le t \le \pi$, applying Lemma 1(ii), we get

$$\begin{aligned} \sup_{\substack{t,\\ 0<u\le t\le\pi}} t^{-\beta}||\psi(\cdot,t,u)||_p &\le 4w_k(f,u)_p \sup_{\substack{t,\\ 0<u\le t\le\pi}} t^{-\beta} \\ &\le 4u^{\alpha-\beta}\sup_u(u^{-\alpha}w_k(f,u)_p) \\ &= O(u^{\alpha-\beta}), \quad \text{by the hypothesis} \end{aligned}$$

and this completes the proof.

Lemma 4 *Let the matrix* $A = (a_{n,k})$ *and kernel* $\widetilde{K}_n(u)$ *and* $H_n(u)$ *of the conjugate Fourier series be defined as in ((1.10)) and ((1.11)).*

Let there exist a positive non-decreasing sequence (μ_n)*, then for* $0 < u \le \pi$

$$\widetilde{K}_n(u) = O\left(\frac{1}{u}\right).$$

$$H_n(u) = O\left(\frac{a_{n0}}{u^2}\right)$$

Proof. From ((1.10)), we have

$$\begin{aligned} \widetilde{K}_n(u) = \sum_{k=0}^{\infty} a_{n,k}\widetilde{D}_k(u) &\le O\left(\frac{1}{u}\right)\left(\sum_{k=0}^{\infty}|a_{n,k}|\right)\left(\text{since } \widetilde{D}_k(u) = O\left(\frac{1}{u}\right)\right) \\ &\le O\left(\frac{1}{u}\right)(\text{ by regularity condition (1.8)}) \end{aligned}$$

Now, from ((1.11)), we have

$$\begin{aligned} H_n(u) &= \sum_{k=0}^{\infty} a_{n,k} \frac{\cos(k+\frac{1}{2})u}{\sin\frac{u}{2}} = R\left[\frac{e^{iu/2}}{\sin\frac{u}{2}} \sum_{k=0}^{\infty} a_{n,k} e^{iku}\right] \\ &= R\left[\frac{e^{iu/2}}{\sin\frac{u}{2}} \left(\sum_{k=0}^{\mu_n} + \sum_{k=\mu_n+1}^{\infty}\right) a_{n,k} e^{iku}\right] = A + B \end{aligned}$$

where

$$\begin{aligned} A &= R\left[\frac{e^{iu/2}}{\sin\frac{u}{2}} \sum_{k=0}^{\mu_n} a_{n,k} e^{iku}\right] \\ &= O\left(\frac{1}{u}\right) \left|\sum_{k=0}^{\mu_n} a_{n,k} e^{iku}\right| (\text{ since } a_{nk} \geq 0 \text{ and } \downarrow) \end{aligned}$$

Applying Abel's transformation, we have

$$\begin{aligned} &= O\left(\frac{1}{u}\right)\left[\left|\sum_{k=0}^{\mu_n} (a_{n,k} - a_{n,k+1}) \sum_{r=0}^{k} e^{iru} + a_{n,\mu_n} \sum_{r=0}^{\mu_n} e^{iru}\right|\right] \\ &= O\left(\frac{1}{u}\right)\left[\sum_{k=0}^{\mu_n} |a_{n,k} - a_{n,k+1}| \left|\sum_{r=0}^{k} e^{iru}\right| + a_{n,\mu_{n+1}} \left|\sum_{r=0}^{\mu_n} e^{iru}\right|\right] \\ &= O\left(\frac{1}{u}\right)\left[\sum_{k=0}^{\mu_n} |a_{n,k} - a_{n,k+1}| + |a_{n,\mu_{n+1}}|\right] \\ &= O\left(\frac{1}{u}\right)\left[\sum_{k=0}^{\mu_n} (a_{n,k} - a_{n,k+1}) + a_{n,\mu_{n+1}}\right] \\ &= O\left(\frac{a_{n0}}{u^2}\right) \end{aligned}$$

$$\begin{aligned} B &= R\left[\frac{e^{iu/2}}{\sin\frac{u}{2}} \sum_{k=\mu_n+1}^{\infty} a_{n,k} e^{iku}\right] = O\left(\frac{1}{u}\right) a_{n\mu_n} \left|\sum_{k=\mu_n+1}^{\infty} e^{iku}\right| \\ &= O\left(\frac{1}{u^2}\right) a_{n\mu_n+1} = O\left(\frac{a_{n0}}{u^2}\right) \end{aligned}$$

Hence

$$H_n(u) = O\left(\frac{a_{n0}}{u^2}\right) + O\left(\frac{a_{n0}}{u^2}\right) = O\left(\frac{a_{n0}}{u^2}\right)$$

□

5 Proof of Theorem

Case 1: For $(1 < q < \infty)$

We first consider the case $1 < q < \infty$.

We have for $p \geq 1$ and $0 \leq \beta < \alpha < 2$, by use of Besov norm defined in ((2.17)) for $B_q^{\alpha}(L_p)$ is

$$||\widetilde{T}_n(\cdot)||_{B_q^{\beta}(L_p)} = ||\widetilde{T}_n(\cdot)||_p + ||w_k(\widetilde{T}_n, \cdot)||_{\beta,q} \tag{5.28}$$

Applying Lemma 1(iii) in equation ((1.17)), we have

$$\begin{aligned} ||\widetilde{T}_n(\cdot)||_p &\leq \frac{1}{\pi}\int_0^{\frac{\pi}{n}} ||\psi_{\cdot}(u)||_p|\widetilde{K}_n(u)|du + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} ||\psi_{\cdot}(u)||_p|H_n(u)|du \\ &= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)|w_k(f,u)_p du + \int_{\frac{\pi}{n}}^{\pi} |H_n(u)|w_k(f,u)_p du\right] \end{aligned}$$

Applying Hölder's inequality, we have

$$\begin{aligned} ||\widetilde{T}_n(\cdot)||_p &\leq \frac{2}{\pi}\left[\left\{\int_0^{\frac{\pi}{n}} \left(|\widetilde{K}_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \left\{\int_0^{\frac{\pi}{n}} \left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right)^q du\right\}^{\frac{1}{q}}\right. \\ &\quad \left. + \left\{\int_{\frac{\pi}{n}}^{\pi} \left(|H_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \left\{\int_{\frac{\pi}{n}}^{\pi} \left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right)^q du\right\}^{\frac{1}{q}}\right] \\ &= O(1)\left[\left\{\int_0^{\frac{\pi}{n}} \left(|\widetilde{K}_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} + \left\{\int_{\frac{\pi}{n}}^{\pi} \left(|H_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\right] \\ &= O(1)\left[I+J\right], \quad (say) \end{aligned} \tag{5.29}$$

By using Lemma 4 in I of ((5.29)), we have

$$\begin{aligned} I &= \left\{\int_0^{\frac{\pi}{n}} (|\widetilde{K}_n(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(1)\left\{\int_0^{\frac{\pi}{n}} \left(u^{\alpha+\frac{1}{q}-1}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(1)\left\{\int_0^{\frac{\pi}{n}} \left(u^{\frac{q\alpha}{q-1}-1}\right) du\right\}^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{n^{\alpha}}\right) \end{aligned} \tag{5.30}$$

Applying Lemma 4 in J of ((5.29)), we have

$$
\begin{aligned}
J &= \left\{\int_{\frac{\pi}{n}}^{\pi}\left(|H_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O\left(a_{n0}\right)\left\{\int_{\frac{\pi}{n}}^{\pi}\left(u^{\alpha+\frac{1}{q}-2}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O\left(a_{n0}\right)\left\{\sum_{k=1}^{n-1}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}u^{(\alpha+\frac{1}{q}-2)\cdot\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O\left(a_{n0}\right)\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}u^{(\alpha+\frac{1}{q}-2)\cdot\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O\left(a_{n0}\right)\left\{\sum_{k=1}^{n}\left(\frac{1}{k^{(\alpha-\frac{1}{q})}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (5.31)
\end{aligned}
$$

Using ((5.30))and ((5.31)) and we have from ((5.29)),

$$
||\widetilde{T}_n(\cdot)||_p = O\left(\frac{1}{n^{\alpha}}\right)+O\left(a_{n0}\right)\left\{\sum_{k=1}^{n}\left(\frac{1}{k^{\alpha-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (5.32)
$$

By using Besov space, we have

$$\begin{aligned}
||w_k(\widetilde{T}_n,\cdot)||_{\beta,q} &= \left\{\int_0^\pi \left(t^{-\beta} w_k(\widetilde{T}_n,t)_p\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&\leq \left[\int_0^\pi \left(\frac{||\widetilde{T}_n(\cdot,t)||_p}{t^\beta}\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} \\
&= \left[\int_0^\pi \left\{\int_0^\pi |\widetilde{T}_n(x,t)|^p dx\right\}^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}} \\
&= \left[\int_0^\pi \left\{\int_0^\pi \left|\frac{-1}{\pi}\int_0^{\frac{\pi}{n}} \psi(x,t,u)\widetilde{K}_n(u)du + \frac{1}{\pi}\int_{\frac{\pi}{n}}^\pi \psi(x,t,u)H_n(u)du\right|^p dx\right\}^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}} \\
&\leq \frac{1}{\pi}\left[\int_0^\pi \frac{dt}{t^{\beta q+1}} \left\{\int_0^\pi \left|\int_0^{\frac{\pi}{n}} \psi(x,t,u)\widetilde{K}_n(u)du + \int_{\frac{\pi}{n}}^\pi \psi(x,t,u)H_n(u)du\right|^p dx\right\}^{\frac{q}{p}}\right]^{\frac{1}{q}} \\
&= \frac{1}{\pi}\left[\int_0^\pi \frac{dt}{t^{\beta q+1}} \left|\left|\int_0^{\frac{\pi}{n}} \psi(\cdot,t,u)\widetilde{K}_n(u)du + \int_{\frac{\pi}{n}}^\pi \psi(\cdot,t,u)H_n(u)du\right|\right|_p^q\right]^{\frac{1}{q}} \\
&\leq \frac{1}{\pi}\left[\int_0^\pi \left(\frac{||\int_0^{\frac{\pi}{n}} \psi(\cdot,t,u)\widetilde{K}_n(u)du||_p + ||\int_{\frac{\pi}{n}}^\pi \psi(\cdot,t,u)H_n(u)du||_p}{t^{\beta+\frac{1}{q}}}\right)^q dt\right]^{\frac{1}{q}},
\end{aligned}$$

by Minkowski's inequality.

Again applying Minkowski's inequality, we get

$$\begin{aligned}
||w_k(\widetilde{T}_n,\cdot)||_{\beta,q} &\leq \frac{1}{\pi}\left[\int_0^\pi \left(\frac{||\int_0^{\frac{\pi}{n}} \psi(\cdot,t,u)\widetilde{K}_n(u)du||_p}{t^{\beta+\frac{1}{q}}}\right)^q dt\right]^{\frac{1}{q}} \frac{1}{\pi}\left[\int_0^\pi \left(\frac{||\int_{\frac{\pi}{n}}^\pi \psi(\cdot,t,u)H_n(u)du||_p}{t^{\beta+\frac{1}{q}}}\right)^q dt\right]^{\frac{1}{q}} \\
&= O(1)[I'+J'], \quad (say) \qquad (5.33)
\end{aligned}$$

$$\begin{aligned}
I' &= \left[\int_0^\pi \left(\frac{||\int_0^{\frac{\pi}{n}} \psi(\cdot,t,u)\widetilde{K}_n(u)du||_p}{t^{\beta+\frac{1}{q}}}\right)^q dt\right]^{\frac{1}{q}} \\
&= \left\{\int_0^\pi \left(\int_0^\pi \left|\int_0^{\frac{\pi}{n}} \psi(x,t,u)\widetilde{K}_n(u)du\right|^p dx\right)^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right\}^{\frac{1}{q}}
\end{aligned}$$

(5.34)

By generalized Minkowski's inequality, we get

$$\begin{aligned} I' &= \left[\int_0^{\pi}\left\{\int_0^{\frac{\pi}{n}}\left(\int_0^{\pi}\left|\psi(x,t,u)\widetilde{K}_n(u)\right|^p dx\right)^{\frac{1}{p}} du\right\}^q \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}} \\ &= \left[\int_0^{\pi}\left\{\int_0^{\frac{\pi}{n}} \frac{||\psi(x,t,u)||_p|\widetilde{K}_n(u)|du}{t^{\beta+\frac{1}{q}}}\right\}^q dt\right]^{\frac{1}{q}} \end{aligned}$$

Again applying generalized Minkowski's inequality, we get

$$\begin{aligned} I' &\leq \int_0^{\frac{\pi}{n}}\left(\int_0^{\pi}\frac{||\psi(x,t,u)||_p^q|\widetilde{K}_n(u)|^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}} du \\ &= \int_0^{\frac{\pi}{n}}|\widetilde{K}_n(u)|du\left(\int_0^{\pi}\frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}} \\ &\leq \int_0^{\frac{\pi}{n}}|\widetilde{K}_n(u)|du\left(\left(\int_0^{u}+\int_u^{\pi}\right)\frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}} \\ &\leq \int_0^{\frac{\pi}{n}}|\widetilde{K}_n(u)|\left(\int_0^{u}\frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}} du+\int_0^{\frac{\pi}{n}}|\widetilde{K}_n(u)|\left(\int_u^{\pi}\frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}} du \end{aligned}$$

(by the inequality $(x+y)^r \leq x^r+y^r, 0<r\leq 1.$)

$$I' = I_1'+I_2', \quad \text{(say)} \tag{5.35}$$

Applying Lemma 2(i), we get

$$I_1' = O(1)\left\{\int_0^{\frac{\pi}{n}}\left(|\widetilde{K}_n(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

Applying Lemma 4, we get

$$\begin{aligned} I_1' &= O(1)\left\{\int_0^{\frac{\pi}{n}}\left(u^{\alpha-\beta-1}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(1)\left\{\int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta-1)}du\right\}^{1-\frac{1}{q}} \\ &= O(1)\left\{\int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta-\frac{1}{q})-1}du\right\}^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) \end{aligned} \tag{5.36}$$

$$I_2' = \int_0^{\frac{\pi}{n}} |\widetilde{K}_n(u)| \left(\int_u^{\pi} \frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du$$

Applying Lemma 2(ii)

$$I_2' = O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(|\widetilde{K}_n(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

Applying Lemma 4, we get

$$\begin{aligned} I_2' &= O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta+\frac{1}{q}-1} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\ &= O(1) \left\{ \int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta-1+\frac{1}{q})} du \right\}^{1-\frac{1}{q}} \\ &= O(1) \left\{ \int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta)-1} du \right\}^{1-\frac{1}{q}} = O\left(\frac{1}{n^{\alpha-\beta}} \right) \end{aligned} \tag{5.37}$$

From ((5.35)), ((5.36)) and ((5.37)), we get

$$I' = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}} \right) \tag{5.38}$$

$$\begin{aligned} J' &= \left[\int_0^{\pi} \left(\frac{|| \int_{\frac{\pi}{n}}^{\pi} \psi(\cdot,t,u) H_n(u) du ||_p}{t^{\beta+\frac{1}{q}}} \right)^q dt \right]^{\frac{1}{q}} \\ &= \left\{ \int_0^{\pi} \left(\int_0^{\pi} \left| \int_{\frac{\pi}{n}}^{\pi} \psi(x,t,u) H_n(u) du \right|^p dx \right)^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}} \right\}^{\frac{1}{q}} \end{aligned}$$

Proceeding as above as in I'.

$$\begin{aligned} J' &\leq \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_0^{\pi} \frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du \\ &= \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\left(\int_0^u + \int_u^{\pi} \right) \frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du \\ &\leq \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_0^u \frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du + \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_u^{\pi} \frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du \\ &\quad \text{(by the inequality } (x+y)^r \leq x^r + y^r, 0 < r \leq 1.) \end{aligned}$$

$$= J_1' + J_2', \quad \text{(say)} \tag{5.39}$$

Now,

$$J_1' = \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_0^u \frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du$$

Applying Lemma 2(iii), we get

$$J_1' = O(1) \left\{ \int_{\frac{\pi}{n}}^{\pi} \left(|H_n(u)| u^{\alpha-\beta} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

Applying Lemma 4, we get

$$\begin{aligned} J_1' &= O(a_{n0}) \left\{ \int_{\frac{\pi}{n}}^{\pi} \left(u^{\alpha-\beta-2} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\ &= O(a_{n0}) \left\{ \int_{\frac{\pi}{n}}^{\pi} u^{\frac{q}{q-1}(\alpha-\beta-2)} du \right\}^{1-\frac{1}{q}} \\ &= O(a_{n0}) \left\{ \sum_{k=1}^{n} \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} u^{\frac{q}{q-1}(\alpha-\beta-2)} du \right\}^{1-\frac{1}{q}} \end{aligned}$$

Let $g(u) = \left(u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}}$ and $G(u)$ is a primitive of $g(u)$, then

$$\begin{aligned} \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} \left(u^{\alpha-\beta-2} \right)^{\frac{q}{q-1}} du &= \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} g(u) du \\ &= G\left(\frac{\pi}{k}\right) - G\left(\frac{\pi}{k+1}\right) \\ &= \left(\frac{\pi}{k} - \frac{\pi}{k+1} \right) g(c) \text{ for some } \frac{\pi}{k+1} < c < \frac{\pi}{k} \\ &= O(1) \left(\frac{1}{k^{\alpha-\beta-\frac{2}{q}}} \right)^{\frac{q}{q-1}} \end{aligned}$$

(5.40) $$J_1' = O(a_{n0}) \left\{ \sum_{k=1}^{n} \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} \left(\frac{1}{k^{\alpha-\beta-\frac{2}{q}}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}$$

Now,

$$J_2' = \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_u^{\pi} \frac{||\psi(x,t,u)||_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du$$

Applying Lemma 2(iv), we get

$$J_2' = O(a_{n0}) \left\{ \int_{\frac{\pi}{n}}^{\pi} \left(|H_n(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

Applying Lemma 4, we get

$$\begin{aligned} J_2' &= O(a_{n0})\left\{\int_{\frac{\pi}{n}}^{\pi}\left(u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}} \\ &= O(a_{n0})\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} u^{\frac{q}{q-1}(\alpha-\beta-2+\frac{1}{q})}du\right\}^{1-\frac{1}{q}} \end{aligned}$$

Proceeding as in J_1', we have

$$J_2' = O(a_{n0})\left\{\sum_{k=1}^{n}\left(\frac{1}{k^{\alpha-\beta-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.41}$$

$$J' = O(a_{n0})\left\{\sum_{k=1}^{n}\left(\frac{1}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.42}$$

From ((5.35)), ((5.38)) and ((5.42)), we get

$$||w_k(\widetilde{T}_n,\cdot)||_{\beta,q} = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right)+O(a_{n0})\left\{\sum_{k=1}^{n}\left(\frac{1}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.43}$$

From ((5.28)),((5.40)) and ((5.43)), for $1<q<\infty$, $o\le\beta<\alpha<2, f\in B_q^{\alpha}(L_p), p\ge 1$, we have

$$||\widetilde{T}_n(\cdot)||_{B_q^{\beta}(L_p)} = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right)+O(a_{n0})\left\{\sum_{k=1}^{n}\left(\frac{1}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.44}$$

This completes the proof of Case 1.

Case 2 ($q=\infty$)

Now, we consider the case $q=\infty$.

$$||\widetilde{T}_n(\cdot)||_{B_\infty^{\beta}(L_p)} = ||\widetilde{T}_n(\cdot)||_p+||w_k(\widetilde{T}_n,\cdot)||_{\beta,\infty} \tag{5.45}$$

$$
\begin{aligned}
||w_k(\widetilde{T}_n,\cdot)||_{\beta,\infty} &= \sup_{t>0}\frac{||\widetilde{T}_n(\cdot,t)||_p}{t^\beta} \\
&= \sup_{t>0}\frac{t^{-\beta}}{\pi}\left\{\int_0^\pi\left|-\int_0^{\frac{\pi}{n}}\psi(x,t,u)\widetilde{K}_n(u)du+\int_{\frac{\pi}{n}}^\pi\psi(x,t,u)H_n(u)du\right|^p dx\right\}^{\frac{1}{p}}
\end{aligned}
$$

Applying Minkowski's inequality, we have

$$
||w_k(\widetilde{T}_n,\cdot)||_{\beta,\infty} \leq \sup_{t>0}\frac{t^{-\beta}}{\pi}\left\{\left(\int_0^\pi\left|\int_0^{\frac{\pi}{n}}\psi(x,t,u)\widetilde{K}_n(u)du\right|^p dx\right)^{\frac{1}{p}}+\left(\int_0^\pi\left|\int_{\frac{\pi}{n}}^\pi\psi(x,t,u)H_n(u)du\right|^p dx\right)^{\frac{1}{p}}\right\}
$$

Applying Generalized Minkowski's inequality, we have

$$
\begin{aligned}
||w_k(\widetilde{T}_n,\cdot)||_{\beta,\infty} &= \sup_{t>0}\frac{t^{-\beta}}{\pi}\left\{\int_0^{\frac{\pi}{n}}\left(\int_0^\pi|\psi(x,t,u)|^p|\widetilde{K}_n(u)|dx\right)^{\frac{1}{p}}du+\int_{\frac{\pi}{n}}^\pi\left(\int_0^\pi|\psi(x,t,u)|^p|H(u)|^p dx\right)^{\frac{1}{p}}du\right\} \\
&= \sup_{t>0}\frac{t^{-\beta}}{\pi}\left\{\int_0^{\frac{\pi}{n}}||\psi(x,t,u)||_p|\widetilde{K}_n(u)|du+\int_{\frac{\pi}{n}}^\pi||\psi(x,t,u)||_p|H(u)|du\right\} \\
&\leq \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{n}}|\widetilde{K}_n(u)|\left(\sup_{t>0}\frac{||\psi(x,t,u)||_p}{t^\beta}\right)du+\int_{\frac{\pi}{n}}^\pi|H_n(u)|\left(\sup_{t>0}\frac{||\psi(x,t,u)||_p}{t^\beta}\right)du\right\}
\end{aligned}
$$

Using Lemma 3, we have

$$
\begin{aligned}
||w_k(\widetilde{T}_n,\cdot)||_{\beta,\infty} &\leq O(1)\int_0^{\frac{\pi}{n}}|\widetilde{K}_n(u)|u^{\alpha-\beta}du+O(1)\int_{\frac{\pi}{n}}^\pi|H_n(u)|u^{\alpha-\beta}du \\
&= O(1)[I''+J''], \quad (say) \qquad (5.46)
\end{aligned}
$$

Using Lemma 4 in I'' and J'', we have

$$
\begin{aligned}
I'' &= O(1)\int_0^{\frac{\pi}{n}}|\widetilde{K}_n(u)|u^{\alpha-\beta}du \\
&= O(1)\int_0^{\frac{\pi}{n}}u^{\alpha-\beta-1}du = O\left(\frac{1}{n^{\alpha-\beta}}\right) \qquad (5.47)
\end{aligned}
$$

$$
\begin{aligned}
J'' &= O(1)\int_{\frac{\pi}{n}}^\pi|H_n(u)|u^{\alpha-\beta}du \\
&= O(a_{n0})\int_{\frac{\pi}{n}}^\pi u^{\alpha-\beta-2}du \\
&= O(a_{n0})\sum_{k=1}^{n-1}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}u^{\alpha-\beta-2}du \\
&= O(a_{n0})\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}u^{\alpha-\beta-2}du \\
&= O(a_{n0})\left\{\sum_{k=1}^{n}\frac{1}{k^{\alpha-\beta}}\right\} \qquad (5.48)
\end{aligned}
$$

From ((5.46)), ((5.47)) and ((5.48)), we have

$$||w_k(\widetilde{T}_n,\cdot)||_{\beta,\infty} = O\left(\frac{1}{n^{\alpha-\beta}}\right) + O\left(a_{n0}\right)\sum_{k=1}^{n}\left(\frac{1}{k^{\alpha-\beta}}\right) \tag{5.49}$$

Now,

$$||\widetilde{T}_n(\cdot)||_p \leq \frac{1}{\pi}\int_0^{\frac{\pi}{n}} ||\psi(u)||_p|\widetilde{K}_n(u)|du + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} ||\psi(u)||_p|H_n(u)|du$$

Applying Lemma 1(iii), we have

$$\begin{aligned} ||\widetilde{T}_n(\cdot)||_p &\leq \frac{2}{\pi}\int_0^{\frac{\pi}{n}} w_k(f,u)_p|\widetilde{K}_n(u)|du + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} w_k(f,u)_p|H_n(u)|du \\ &= O(1)\int_0^{\frac{\pi}{n}} u^{\alpha}|\widetilde{K}_n(u)|du + O(1)\int_{\frac{\pi}{n}}^{\pi} u^{\alpha}|H_n(u)|du \\ &= O(1)\int_0^{\frac{\pi}{n}} u^{\alpha-1}du + O(a_{n0})\int_{\frac{\pi}{n}}^{\pi} u^{\alpha-2} \\ &= I''' + J''', \quad (say) \end{aligned} \tag{5.50}$$

Using Lemma 4 in I''' and J''', we have

$$\begin{aligned} I''' &= O(1)\int_0^{\frac{\pi}{n}} u^{\alpha-1}du \\ &= O\left(\frac{1}{n^{\alpha}}\right) \end{aligned} \tag{5.51}$$

$$\begin{aligned} J''' &= O(a_{n0})\int_{\frac{\pi}{n}}^{\pi} u^{\alpha-2} \\ &= O(a_{n0})\left\{\sum_{k=1}^{n-1}\left(\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} u^{\alpha-2}du\right)\right\} \\ &= O(a_{n0})\left\{\sum_{k=1}^{n}\left(\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} u^{\alpha-2}du\right)\right\} \\ &= O(a_{n0})\sum_{k=1}^{n}\frac{1}{k^{\alpha}} \end{aligned} \tag{5.52}$$

From ((5.50)), ((5.51)) and ((5.52)), we have

$$||\widetilde{T}_n(\cdot)||_p = O\left(\frac{1}{n^{\alpha}}\right) + O(a_{n0})\sum_{k=1}^{n}\left(\frac{1}{k^{\alpha}}\right) \tag{5.53}$$

From ((5.49)) and ((5.53)), for $q = \infty$, $0 \leq \beta < \alpha < 2, f \in B_q^\alpha(L_p), p \geq 1$, we have

$$||\widetilde{T}_n(\cdot)||_{B^\beta_\infty(L_p)} = O\left(\frac{1}{n^{\alpha-\beta}}\right) + O(a_{n0}) \sum_{k=1}^{n} \frac{1}{k^{\alpha-\beta}}$$

This completes the proof of Case 2.

Combining the Case 1 and Case 2, we obtain the proof of the theorem.

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