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Higher - Order Parameter-Free Optimality Models for Discrete Fractional Programming

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Abstract

In this communication, we deal with establishing several sets of generalized parameter-free sufficient optimality conditions for a discrete minmax fractional programming problem using two partitioning schemes and various second-order $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$ - univexities. The obtained optimality results are application-oriented to other problems in mathematical programming in the interdisciplinary nature.

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1 Introduction and Preliminaries

In order to establish several sets of generalized parameter-free sufficient optimality conditions, we begin with the following discrete minmax fractional programming problem:

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$$(P) \qquad \text{Minimize } \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)}$$

subject to $\quad G_j(x) \leq 0,\ j \in \underline{q},\ \ H_k(x) = 0,\ k \in \underline{r},\ \ x \in X,$

where $\mathbb{F}$ is feasible set of (P), X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), $f_i,\ g_i$, $i \in \underline{p} \equiv \{1,2,\ldots,p\}$, $G_j,\ j \in \underline{q}$, and $H_k,\ k \in \underline{r}$, are real-valued functions defined on X, and for each $i \in \underline{p},\ g_i(x) > 0$ for all x satisfying the constraints of (P).

We aim at investigating some results on generalized second-order parameter-free sufficient optimality conditions for (P) based on various second-order ($\mathscr{F}$, β, ϕ, π, ω, ρ, θ, m) - univexity constraints. We shall apply two partitioning schemes due to Mond and Weir [3] and Yang [9], in conjunction with the new classes of generalized second-order invex functions, to formulate and discuss numerous sets of generalized second-order parameter-free sufficient optimality conditions for (P). To the best of our knowledge, all the second-order sufficient optimality results established in this paper are new in the area of discrete minmax fractional programming, while have a wide range of applications to mathematical programming including, but limited to, several types of optimization problems.

The rest of this paper is organized as follows: In the remainder of this section, we generalize some basic definitions and recall some auxiliary results which will be needed in the sequel. In Section 2, we state and prove various second-order parameter-free sufficient optimality results for (P) using a variety of generalized $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-sounivexity assumptions. Finally, in Section 3 we summarize our main results and also point out some further research opportunities arising from the principal problem investigated in the present paper.

We next present more generalized versions of the new classes of (strictly) $(\phi,\eta,\rho,\theta,m)$-sonvex, (strictly) $(\phi,\eta,\rho,\theta,m)$-pseudosonvex, and (prestrictly) $(\phi,\eta,\rho,\theta,m)$-quasisonvex functions introduced recently in [5]. Here we shall further generalize these functions by considering their univex counterparts. We shall use the word **sounivex** for second - order univex.

Let $f : X \to \mathbb{R}$ be a twice differentiable function.

Definition 1.1. *The function f is said to be* (strictly) ($\mathscr{F}$, β, ϕ, π, ω, ρ, θ, m)-sounivex *at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+ \equiv (0,\infty)$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\pi, \omega, \theta : X \times X \to \mathbb{R}^n$, a sublinear function $\mathscr{F}(x,x^*;\cdot) : \mathbb{R}^n \to \mathbb{R}$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,*

$$\phi\big(f(x) - f(x^*)\big)(>) \geq \mathscr{F}\big(x,x^*;\beta(x,x^*)\nabla f(x^*)\big) + \beta(x,x^*)\langle \pi(x,x^*), \nabla^2 f(x^*)z\rangle$$
$$-\frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^m,$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and $\langle a,b\rangle$ is the inner product of the vectors a and b.

The function f is said to be (strictly) $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-sounivex *on X if it is* (strictly) $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-sounivex *at each $x^* \in X$.*

Definition 1.2. *The function f is said to be* (strictly) ($\mathscr{F}$, β, ϕ, π, ω, ρ, θ, m)-pseudosounivex *at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\pi, \omega, \theta : X \times X \to \mathbb{R}^n$, a sublinear function $\mathscr{F}(x,x^*;\cdot) : \mathbb{R}^n \to \mathbb{R}$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,*

$$\mathscr{F}\big(x,x^*;\beta(x,x^*)\nabla f(x^*)\big) + \beta(x,x^*)\langle \pi(x,x^*), \nabla^2 f(x^*)z\rangle - \frac{1}{2}\langle \omega(x,x^*), \nabla^2 f(x^*)z\rangle$$
$$\geq -\rho(x,x^*)\|\theta(x,x^*)\|^m \;\Rightarrow\; \phi\big(f(x) - f(x^*)\big)(>) \geq 0.$$

The function f is said to be (strictly) $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-pseudosounivex *on X if it is* (strictly) $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-pseudosounivex *at each $x^* \in X$.*

Definition 1.3. *The function f is said to be* (prestrictly) ($\mathscr{F}$, β, ϕ, π, ω, ρ, θ, m)-quasisounivex *at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\pi, \omega, \theta : X \times X \to \mathbb{R}^n$, a sublinear* *function* $\mathscr{F}(x,x^*;\cdot):\mathbb{R}^n\to\mathbb{R}$, *and a positive integer* m *such that for each* $x\in X$ *and* $z\in\mathbb{R}^n$,

$$\phi\big(f(x)-f(x^*)\big)(<)\leq 0 \;\Rightarrow\; \mathscr{F}\big(x,x^*;\beta(x,x^*)\nabla f(x^*)\big)+\beta(x,x^*)\langle\pi(x,x^*),\nabla^2 f(x^*)z\rangle - \frac{1}{2}\langle\omega(x,x^*),\nabla^2 f(x^*)z\rangle \leq -\rho(x,x^*)\|\theta(x,x^*)\|^m.$$

The function f *is said to be* (prestrictly) $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-quasisounivex *on* X *if it is* (prestrictly) $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-quasisounivex *at each* $x^*\in X$.,

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, ($\mathscr{F}$, β, ϕ, π, ω, ρ, θ, m)-quasisounivexity can be defined in the following equivalent way:

The function f is said to be $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-quasisounivex at x^* if there exist functions $\beta: X\times X\to\mathbb{R}_+$, $\phi:\mathbb{R}\to\mathbb{R}$, $\rho: X\times X\to\mathbb{R}$, $\pi,\omega,\theta: X\times X\to\mathbb{R}^n$, a sublinear function $\mathscr{F}(x,x^*;\cdot):\mathbb{R}^n\to\mathbb{R}$, and a positive integer m such that for each $x\in X$ and $z\in\mathbb{R}^n$,

$$\mathscr{F}\big(x,x^*;\beta(x,x^*)\nabla f(x^*)\big)+\beta(x,x^*)\langle\pi(x,x^*),\nabla^2 f(x^*)z\rangle - \frac{1}{2}\langle\omega(x,x^*),\nabla^2 f(x^*)z\rangle > -\rho(x,x^*)\|\theta(x,x^*)\|^m \;\Rightarrow\; \phi\big(f(x)-f(x^*)\big)>0.$$

We observe that the new classes of generalized convex functions specified in Definitions 1.1 - 1.3 contain a variety of special cases that can easily be identified by appropriate choices of $\mathscr{F}(x,x^*)$, ϕ, ρ, θ, and m. For example, if let $\mathscr{F}\big(x,x^*;\nabla f(x^*)\big)=\beta(x,x^*)\langle\nabla f(x^*),\pi(x,x^*)\rangle$, where π is a function from $X\times X$ to $\mathbb{R}^n$, then we obtain the definitions of (strictly) (ϕ,π,ρ,θ,m)-sonvex, (strictly) (ϕ,π,ρ,θ,m)-pseudosonvex, and (prestrictly) (ϕ,π,ρ,θ,m)-quasisonvex functions introduced recently in [5].

We conclude this section by recalling a set of second-order parameter-free necessary optimality conditions for (P). This result is obtained from Theorem 3.1 of [5] by eliminating the parameter λ^* and redefining the Lagrange multipliers. We shall use the form and features of this result to formulate our generalized second-order parameter-free sufficient optimality conditions for (P).

Theorem 1.1. [4] *Let x^* be a normal optimal solution of (P), let $\lambda^* = \varphi(x^*) \equiv \max_{1\leq i\leq p} f_i(x^*)/g_i(x^*)$, and assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are twice continuously differentiable at x^*. Then for each $z^* \in C(x^*)$, there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0, \sum_{i=1}^p u_i = 1\}$, $v^* \in \mathbb{R}^q_+ \equiv \{v \in \mathbb{R}^q : v \geq 0\}$, and $w^* \in \mathbb{R}^r$ such that*

$$\sum_{i=1}^p u_i^*[D(x^*,u^*)\nabla f_i(x^*) - N(x^*,u^*)\nabla g_i(x^*)] + \sum_{j=1}^q v_j^*\nabla G_j(x^*) + \sum_{k=1}^r w_k^*\nabla H_k(x^*) = 0,$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^p u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*) - N(x^*,u^*)\nabla^2 g_i(x^*)] + \sum_{j=1}^q v_j^*\nabla^2 G_j(x^*) + \sum_{k=1}^r w_k^*\nabla^2 H_k(x^*)\Big\} z^* \Big\rangle \geq 0,$$

$$u_i^*[D(x^*,u^*)f_i(x^*) - N(x^*,u^*)g_i(x^*)] = 0, \ \ i \in \underline{p},$$

$$\max_{1\leq i\leq p} \frac{f_i(x^*)}{g_i(x^*)} = \frac{N(x^*,u^*)}{D(x^*,u^*)},$$

$$v_j^* G_j(x^*) = 0, \ \ j \in \underline{q},$$

where $C(x^*)$ is the set of all critical directions of (P) at x^*, that is,

$$C(x^*) = \{z \in \mathbb{R}^n : \langle \nabla f_i(x^*) - \lambda \nabla g_i(x^*), z\rangle = 0, \ \ i \in A(x^*),$$

$$\langle \nabla G_j(x^*), z\rangle \leq 0, \ \ j \in B(x^*),$$

$$\langle \nabla H_k(x^*), z\rangle = 0, \ \ k \in \underline{r}\},$$

$A(x^*) = \{j \in \underline{p} : f_j(x^*)/g_j(x^*) = \max_{1\leq i\leq p} f_i(x^*)/g_i(x^*)\}$, $B(x^*) = \{j \in \underline{q} : G_j(x^*) = 0\}$, $N(x^*,u^*) = \sum_{i=1}^p u_i^* f_i(x^*)$, and $D(x^*,u^*) = \sum_{i=1}^p u_i^* g_i(x^*)$.

In the above theorem, a *normal* optimal solution refers to an optimal solution at which an appropriate second-order constraint qualification is satisfied.

2 Second-Order Sufficient Optimality Conditions

In this section, we discuss several families of second-order parameter-free sufficient optimality results under various generalized $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-sounivexity hypotheses imposed on certain combinations of the problem functions. This is accomplished by employing a certain partitioning scheme which was originally proposed in [3] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let $\{J_0,J_1,\ldots,J_M\}$ and $\{K_0,K_1,\ldots,K_M\}$ be partitions of the index sets $\underline{q}$ and $\underline{r}$, respectively; thus, $J_\mu \subseteq \underline{q}$ for each $\mu \in \underline{M}\cup\{0\}$, $J_\mu \cap J_\nu = \emptyset$ for each $\mu,\nu \in \underline{M}\cup\{0\}$ with $\mu \neq \nu$, and $\cup_{\mu=0}^{M} J_\mu = \underline{q}$. Obviously, similar properties hold for $\{K_0,K_1,\ldots,K_M\}$. Moreover, if M_1 and M_2 are the numbers of the partitioning sets of $\underline{q}$ and $\underline{r}$, respectively, then $M=\max\{M_1,M_2\}$ and $J_\mu=\emptyset$ or $K_\mu=\emptyset$ for $\mu>\min\{M_1,M_2\}$

In addition, we use the real-valued functions $\xi \to \Phi_i(\xi,x,v,w),\ i\in\underline{p},\ \xi \to$ $\Phi(\xi,x,u,v,w)$, and $\xi \to \Lambda_t(\xi,v,w)$ defined, for fixed x,u,v, and w, on X as follows:

$$\Phi_i(\xi,x,u,v,w)=D(x,u)f_i(\xi)-N(x,u)g_i(\xi)+\sum_{j\in J_0} v_jG_j(\xi)+\sum_{k\in K_0} w_kH_k(\xi),\ \ i\in\underline{p},$$

$$\Phi(\xi,x,u,v,w)=\sum_{i=1}^{p} u_i[D(x,y)f_i(\xi)-N(x,u)g_i(\xi)]+\sum_{j\in J_0} v_jG_j(\xi)+\sum_{k\in K_0} w_kH_k(\xi),$$

$$\Lambda_t(\xi,v,w)=\sum_{j\in J_t} v_jG_j(\xi)+\sum_{k\in K_t} w_kH_k(\xi),\quad t\in\underline{M}.$$

In the proofs of our sufficiency theorems, we shall make frequent use of the following auxiliary result which provides an alternative expression for the objective function of (P).

Lemma 2.1. [8] *For each $x\in X$,*

$$\varphi(x)\equiv \max_{1\le i\le p}\frac{f_i(x)}{g_i(x)}=\max_{u\in U}\frac{\sum_{i=1}^{p}u_if_i(x)}{\sum_{i=1}^{p}u_ig_i(x)}.$$

Making use of the sets and functions defined above, we can now formulate our first collection of generalized second-order parameter-free sufficient optimality results for (P) as follows.

Theorem 2.1. *Let $x^* \in \mathbb{F}$ and assume that the functions $f_i,\ g_i,\ i \in \underline{p},\ G_j,\ j \in \underline{q}$, and $H_k,\ k \in \underline{r}$, are twice differentiable at x^*, and that for each $z^* \in C(x^*)$, there exist $u^* \in U,\ v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that*

$$\sum_{i=1}^{p} u_i^* [D(x^*,u^*)\nabla f_i(x^*) - N(x^*,u^*)\nabla g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \tag{2.1}$$

$$\begin{aligned}\beta(x,x^*)\Big\langle \pi(x,x^*), &\sum_{i=1}^{p} u_i^* [D(x^*,u^*)\nabla^2 f_i(x^*) - N(x^*,u^*)\nabla^2 g_i(x^*)]z^* \\ &+ \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*)z^* + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*)z^* \Big\rangle \\ &- \frac{1}{2}\Big\langle \omega(x,x^*), \sum_{i=1}^{p} u_i^* [D(x^*,u^*)\nabla^2 f_i(x^*) - N(x^*,u^*)\nabla^2 g_i(x^*)]z^* \\ &+ \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*)z^* + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*)z^* \Big\rangle \geq 0\ \forall\, x \in \mathbb{F},\end{aligned} \tag{2.2}$$

$$\max_{1 \leq i \leq p} \frac{f_i(x^*)}{g_i(x^*)} = \frac{N(x^*,u^*)}{D(x^*,u^*)}, \tag{2.3}$$

$$v_j^* G_j(x^*) = 0,\ \ j \in \underline{q}. \tag{2.4}$$

Assume, furthermore, that any one of the following four sets of hypotheses is satisfied:

(a) *(i)* $\xi \to \Phi(\xi,x^*,u^*,v^*,w^*)$ *is prestrictly* $(\mathscr{F},\beta,\bar{\phi},\pi,\omega,\bar{\rho},\theta,m)$*-quasisounivex at* x^* *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi,v^*,w^*)$ *is strictly* $(\mathscr{F},\beta,\tilde{\phi}_t,\pi,\omega,\tilde{\rho}_t,\theta,m)$*-pseudosounivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$*;*

(iii) $\bar{\rho}(x,x^*) + \sum_{t=1}^{M} \tilde{\rho}_t(x,x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(b) *(i)* $\xi \to \Phi(\xi,x^*,u^*,v^*,w^*)$ *is* $(\mathscr{F},\beta,\bar{\phi},\pi,\omega,\bar{\rho},\theta,m)$*-pseudosounivex at* x^* *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi,v^*,w^*)$ *is* $(\mathscr{F},\beta,\tilde{\phi}_t,\pi,\omega,\tilde{\rho}_t,\theta,m)$*-quasisounivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$*;*

(iii) $\bar{\rho}(x,x^*)+\sum_{t=1}^{M}\tilde{\rho}_t(x,x^*)\geq 0$ *for all* $x\in\mathbb{F}$*;*

(c) (i) $\xi\to\Phi(\xi,x^*,u^*,v^*,w^*)$ *is prestrictly* $(\mathscr{F},\beta,\bar{\phi},\pi,\omega,\bar{\rho},\theta,m)$*-quasisounivex at* x^* *and* $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$*;*

(ii) *for each* $t\in\underline{M}$, $\xi\to\Lambda_t(\xi,v^*,w^*)$ *is* $(\mathscr{F},\beta,\tilde{\phi}_t,\pi,\omega,\tilde{\rho}_t,\theta,m)$*-quasisounivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0)=0$*;*

(iii) $\bar{\rho}(x,x^*)+\sum_{t=1}^{M}\tilde{\rho}_t(x,x^*)>0$ *for all* $x\in\mathbb{F}$*;*

(d) (i) $\xi\to\Phi(\xi,x^*,u^*,v^*,w^*)$ *is prestrictly* $(\mathscr{F},\beta,\bar{\phi},\pi,\omega,\bar{\rho},\theta,m)$*-quasisounivex at* x^* *and* $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$*;*

(ii) *for each* $t\in\underline{M_1}$, $\xi\to\Lambda_t(\xi,v^*,w^*)$ *is* $(\mathscr{F},\beta,\tilde{\phi}_t,\pi,\omega,\tilde{\rho}_t,\theta,m)$*-quasisounivex at* x^**, for each* $t\in\underline{M_2}\neq\emptyset$, $\xi\to\Lambda_t(\xi,v^*,w^*)$ *is strictly* $(\mathscr{F},\beta,\tilde{\phi}_t,\pi,\omega,\tilde{\rho}_t,\theta,m)$*-pseudosounivex at* x^**, and for each* $t\in\underline{M}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0)=0$*, where* $\{\underline{M_1},\underline{M_2}\}$ *is a partition of* $\underline{M}$*;*

(iii) $\bar{\rho}(x,x^*)+\sum_{t=1}^{M}\tilde{\rho}_t(x,x^*)\geq 0$ *for all* $x\in\mathbb{F}$.

Then x^* *is an optimal solution of (P).*

Proof. Let x be an arbitrary feasible solution of (P).

(a) : It is clear that (2.1) and (2.2) can be expressed as follows:

$$\sum_{i=1}^{p}u_i^*[D(x^*,u^*)\nabla f_i(x^*)-N(x^*,u^*)\nabla g_i(x^*)]+\sum_{j\in J_0}v_j^*\nabla G_j(x^*)+\sum_{k\in K_0}w_k^*\nabla H_k(x^*)+\sum_{t=1}^{M}\Big[\sum_{j\in J_t}v_j^*\nabla G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\Big]=0, \quad (2.5)$$

$$\beta(x,x^*)\Big\langle \pi(x,x^*),\Big\{\sum_{i=1}^{p}u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*)-N(x^*,u^*)\nabla^2 g_i(x^*)]$$
$$+\sum_{j\in J_0}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_0}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle$$
$$+\beta(x,x^*)\Big\langle \pi(x,x^*),\sum_{t=1}^{M}\Big[\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$
$$-\frac{1}{2}[\Big\langle \omega(x,x^*),\Big\{\sum_{i=1}^{p}u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*)-N(x^*,u^*)\nabla^2 g_i(x^*)]+\sum_{j\in J_0}v_j^*\nabla^2 G_j(x^*)+$$
$$\sum_{k\in K_0}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle$$
$$+\Big\langle \omega(x,x^*),\sum_{t=1}^{M}\Big[\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle]\geq 0. \quad (2.6)$$

Since for each $t\in \underline{M}$,

$$\begin{aligned}\Lambda_t(x,v^*,w^*) &= \sum_{j\in J_t}v_j^*G_j(x)+\sum_{k\in K_t}w_k^*H_k(x)\\ &\leq 0 \ \text{(by the feasibility of } x)\\ &= \sum_{j\in J_t}v_j^*G_j(x^*)+\sum_{k\in K_t}w_k^*H_k(x^*) \ \text{(by (2.4) and the feasibility of } x^*)\\ &= \Lambda_t(x^*,v^*,w^*),\end{aligned}$$

and hence $\tilde{\phi}_t\big(\Lambda_t(x,v^*,w^*)-\Lambda_t(x^*,v^*,w^*)\big)\leq 0$, it follows from (ii) that

$$\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big[\sum_{j\in J_t}v_j^*\nabla G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\Big]\Big)$$
$$+\beta(x,x^*)\Big\langle \pi(x,x^*),\Big[\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$
$$-\frac{1}{2}\Big\langle \omega(x,x^*),\Big[\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$
$$<-\tilde{\rho}_t(x,x^*)\|\theta(x,x^*)\|^m.$$

Summing over $t \in \underline{M}$ and using the sublinearity of $\mathscr{F}(x,x^*;\cdot)$, we obtain

$$\mathscr{F}\Big(x,x^*;\beta(x,x^*)\sum_{t=1}^{M}\Big[\sum_{j\in J_t} v_j^*\nabla G_j(x^*)+\sum_{k\in K_t} w_k^*\nabla H_k(x^*)\Big]\Big)$$

$$+\beta(x,x^*)\Big\langle \pi(x,x^*),\sum_{t=1}^{M}\Big[\sum_{j\in J_t} v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t} w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$

$$-\frac{1}{2}\Big\langle \omega(x,x^*),\sum_{t=1}^{M}\Big[\sum_{j\in J_t} v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t} w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$

$$< -\sum_{t=1}^{M}\tilde{\rho}_t(x,x^*)\|\theta(x,x^*)\|^m. \quad (2.7)$$

Combining (2.5) - (2.7), and using (iii) we get

$$\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big\{\sum_{i=1}^{p} u_i^*[D(x^*,u^*)\nabla f_i(x^*)-N(x^*,u^*)\nabla g_i(x^*)]+\sum_{j\in J_0} v_j^*\nabla G_j(x^*)$$
$$+\sum_{k\in K_0} w_k^*\nabla H_k(x^*)\Big\}\Big)$$

$$+\beta(x,x^*)\Big\langle \pi(x,x^*),\Big\{\sum_{i=1}^{p} u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*)-N(x^*,u^*)\nabla^2 g_i(x^*)]$$

$$+\sum_{j\in J_0} v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_0} w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle$$

$$-\frac{1}{2}[\Big\langle \omega(x,x^*),\Big\{\sum_{i=1}^{p} u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*)-N(x^*,u^*)\nabla^2 g_i(x^*)]+\sum_{j\in J_0} v_j^*\nabla^2 G_j(x^*)$$
$$+\sum_{k\in K_0} w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle]$$

$$> \sum_{t=1}^{M}\tilde{\rho}_t(x,x^*)\|\theta(x,x^*)\|^m \geq -\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^m,$$

which by virtue of (i) implies that

$$\bar{\phi}\big(\Phi(x,x^*,u^*,v^*,w^*)-\Phi(x^*,x^*,u^*,v^*,w^*)\big)\geq 0.$$

But $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$, and hence we get

$$\Phi(x,x^*,u^*,v^*,w^*)\geq\Phi(x^*,x^*,u^*,v^*,w^*)=0,$$

where the equality follows from the feasibility of x^* and definitions of $D(x^*,u^*)$ and $D(x^*,u^*)$. Since $x\in\mathbb{F}$, the above inequality reduces to

$$\sum_{i=1}^{p}u_i^*[D(x^*,u^*)f_i(x)-N(x^*,u^*)g_i(x)]\geq 0. \tag{2.8}$$

Now using (2.3), (2.8), and Lemma 2.1, we see that

$$\varphi(x^*)=\frac{N(x^*,u^*)}{D(x^*,u^*)}\leq\frac{\sum_{i=1}^{p}u_i^*f_i(x)}{\sum_{i=1}^{p}u_i^*g_i(x)}\leq\max_{u\in U}\ \frac{\sum_{i=1}^{p}u_if_i(x)}{\sum_{i=1}^{p}u_ig_i(x)}=\varphi(x).$$

Since $x\in\mathbb{F}$ was arbitrary, we conclude from this inequality that x^* is an optimal solution of (P).

(b) : Proceeding as in the proof of part (a), we see that (ii) leads to the following inequality:

$$\begin{aligned}\mathscr{F}\Big(x,x^*;\beta(x,x^*)\sum_{t=1}^{M}\Big[\sum_{j\in J_t}v_j^*\nabla G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\Big)\\
+\beta(x,x^*)\Big\langle\pi(x,x^*),\sum_{t=1}^{M}\Big[\sum_{j\in J_t}v_j^*\nabla^2G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2H_k(x^*)\Big]z^*\Big\rangle\\
-\frac{1}{2}\Big\langle z^*,\sum_{t=1}^{M}\Big[\sum_{j\in J_t}v_j^*\nabla^2G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2H_k(x^*)\Big]z^*\Big\rangle\leq-\sum_{t=1}^{M}\tilde{\rho}_t(x,x^*)\|\theta(x,x^*)\|^m.\end{aligned}$$

Combining this inequality with (2.5) and (2.6), and using (iii) we get

$$\begin{aligned}
\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big\{\sum_{i=1}^{p}u_i^*[D(x^*,u^*)\nabla f_i(x^*)-N(x^*,u^*)\nabla g_i(x^*)]+\sum_{j\in J_0}v_j^*\nabla G_j(x^*)\\
+\sum_{k\in K_0}w_k^*\nabla H_k(x^*)\Big\}\Big)\\
+\beta(x,x^*)\Big\langle\pi(x,x^*),\Big\{\sum_{i=1}^{p}u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*)-N(x^*,u^*)\nabla^2 g_i(x^*)]\\
+\sum_{j\in J_0}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_0}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle\\
-\frac{1}{2}\Big\langle\omega(x,x^*),\Big\{\sum_{i=1}^{p}u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*)-N(x^*,u^*)\nabla^2 g_i(x^*)]\\
+\sum_{j\in J_0}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_0}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle\\
\geq\sum_{t=1}^{M}\tilde{\rho}_t(x,x^*)\|\theta(x,x^*)\|^m\geq-\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^m,
\end{aligned}$$

which by virtue of (i) implies that

$$\bar{\phi}\big(\Phi(x,x^*,u^*,v^*,w^*)-\Phi(x^*,x^*,u^*,v^*,w^*)\big)\geq 0.$$

The rest of the proof is identical to that of part (a).

(c) and (d) : The proofs are similar to those of parts (a) and (b). □

Theorem 2.2. *Let $x^*\in\mathbb{F}$ and assume that the functions $f_i,\ g_i,\ i\in\underline{p},\ G_j,\ j\in\underline{q}$, and $H_k,\ k\in\underline{r}$, are twice differentiable at x^*, and that for each $z^*\in C(x^*)$, there exist $u^*\in U,\ v^*\in\mathbb{R}_+^q$, and $w^*\in\mathbb{R}^r$ such that (2.1) - (2.4) and the following relations hold:*

$$u_i^*[D(x^*,u^*)f_i(x^*)-N(x^*,u^*)g_i(x^*)]=0,\ \ i\in\underline{p}. \tag{2.9}$$

Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:

(a) (i) for each $i\in I_+\equiv\{i\in\underline{p}\}$, $\xi\to\Phi_i(\xi,x^*,u^*,v^*,w^*)$ is $(\mathscr{F},\beta,\bar{\phi}_i,\pi,\omega,\bar{\rho}_i,\theta,m)$-pseudosounivex at x^*, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0)=0$;

(ii) *for each* $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ *is* $(\mathscr{F}, \beta, \tilde{\phi}_t, \pi, \omega, \tilde{\rho}_t, \theta, m)$*-quasisounivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*) + \sum_{t=1}^{M} \tilde{\rho}_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(b) *(i)* *for each* $i \in I_+$, $\xi \to \Phi_i(\xi, x^*, u^*, v^*, w^*)$ *is prestrictly* $(\mathscr{F}, \beta, \bar{\phi}_i, \pi, \omega, \bar{\rho}_i, \theta, m)$*-quasisounivex at* x^*, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) *for each* $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ *is strictly* $(\mathscr{F}, \beta, \tilde{\phi}_t, \pi, \omega, \tilde{\rho}_t, \theta, m)$*-pseudosounivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*) + \sum_{t=1}^{M} \tilde{\rho}_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(c) *(i)* *for each* $i \in I_+$, $\xi \to \Phi_i(\xi, x^*, u^*, v^*, w^*)$ *is prestrictly* $(\mathscr{F}, \beta, \bar{\phi}_i, \pi, \omega, \bar{\rho}_i, \theta, m)$*-quasisounivex at* x^*, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) *for each* $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ *is* $(\mathscr{F}, \beta, \tilde{\phi}_t, \pi, \omega, \tilde{\rho}_t, \theta, m)$*-quasisounivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*) + \sum_{t=1}^{M} \tilde{\rho}_t(x, x^*) > 0$ *for all* $x \in \mathbb{F}$*;*

(d) *(i)* *for each* $i \in I_{1+}$, $\xi \to \Phi_i(\xi, x^*, u^*, v^*, w^*)$ *is* $(\mathscr{F}, \beta, \bar{\phi}_i, \pi, \omega, \bar{\rho}_i, \theta, m)$*-pseudosounivex at* x^*, *for each* $i \in I_{2+}$, $\xi \to \Phi_i(\xi, x^*, u^*, v^*, w^*)$ *is prestrictly* $(\mathscr{F}, \beta, \bar{\phi}_i, \pi, \omega, \bar{\rho}_i, \theta, m)$*-quasisounivex at* x^**, and for each* $i \in I_+$, $\bar{\phi}_i$ *is strictly increasing and* $\bar{\phi}_i(0) = 0$*, where* $\{I_{1+}, I_{2+}\}$ *is a partition of* I_+*;*

(ii) *for each* $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ *is strictly* $(\mathscr{F}, \beta, \tilde{\phi}_t, \pi, \omega, \tilde{\rho}_t, \theta, m)$*-pseudosounivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*) + \sum_{t=1}^{M} \tilde{\rho}_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(e) *(i)* for each $i \in I_{1+} \neq \emptyset$, $\xi \to \Phi_i(\xi, x^*, u^*, v^*, w^*)$ is $(\mathscr{F}, \beta, \bar{\phi}_i, \pi, \omega, \bar{\rho}_i, \theta, m)$-pseudosounivex at x^*, for each $i \in I_{2+}$, $\xi \to \Phi_i(\xi, x^*, u^*, v^*, w^*)$ is prestrictly $(\mathscr{F}, \beta, \bar{\phi}_i, \pi, \omega, \bar{\rho}_i, \theta, m)$-quasisounivex at x^*, and for each $i \in I_+$, $\bar{\phi}_i$ is strictly increasing and $\bar{\phi}_i(0) = 0$, where $\{I_{1+}, I_{2+}\}$ is a partition of I_+;

(ii) for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ is $(\mathscr{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \pi, \omega, \theta, m)$-quasisounivex at x^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*) + \sum_{t=1}^{M} \tilde{\rho}_t(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(f) *(i)* for each $i \in I_+$, $\xi \to \Phi_i(\xi, x^*, u^*, v^*, w^*)$ is prestrictly $(\mathscr{F}, \beta, \bar{\phi}_i, \pi, \omega, \bar{\rho}_i, \theta, m)$-quasisounivex at x^*, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0) = 0$;

(ii) for each $t \in \underline{M}_1 \neq \emptyset$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ is strictly $(\mathscr{F}, \beta, \tilde{\phi}_t, \pi, \omega, \tilde{\rho}_t, \theta, m)$-pseudosounivex at x^*, for each $t \in \underline{M}_2$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ is $(\mathscr{F}, \beta, \tilde{\phi}_t, \pi, \omega, \tilde{\rho}_t, \theta, m)$-quasisounivex at x^*, and for each $t \in \underline{M}$, $\tilde{\phi}_t$ is increasing and $\tilde{\phi}_t(0) = 0$, where $\{\underline{M}_1, \underline{M}_2\}$ is a partition of $\underline{M}$;

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*) + \sum_{t=1}^{M} \tilde{\rho}_t(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(g) *(i)* for each $i \in I_{1+}$, $\xi \to \Phi_i(\xi, x^*, u^*, v^*, w^*)$ is $(\mathscr{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \pi, \omega, \theta, m)$-pseudosounivex at x^*, for each $i \in I_{2+}$, $\xi \to \Phi_i(\xi, x^*, u^*, v^*, w^*)$ is prestrictly $(\mathscr{F}, \beta, \bar{\phi}_i, \pi, \omega, \bar{\rho}_i, \theta, m)$-quasisounivex at x^*, and for each $i \in I_+$, $\bar{\phi}_i$ is strictly increasing and $\bar{\phi}_i(0) = 0$, where $\{I_{1+}, I_{2+}\}$ is a partition of I_+;

(ii) for each $t \in \underline{M}_1$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ is strictly $(\mathscr{F}, \beta, \tilde{\phi}_t, \pi, \omega, \tilde{\rho}_t, \theta, m)$-pseudosounivex at x^*, for each $t \in \underline{M}_2$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ is $(\mathscr{F}, \beta, \tilde{\phi}_t, \pi, \omega, \tilde{\rho}_t, \theta, m)$-quasisounivex at x^*, and for each $t \in \underline{M}$, $\tilde{\phi}_t$ is increasing and $\tilde{\phi}_t(0) = 0$, where $\{\underline{M}_1, \underline{M}_2\}$ is a partition of $\underline{M}$;

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*) + \sum_{t=1}^{M} \tilde{\rho}_t(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(iv) $I_{1+} \neq \emptyset$, $\underline{M_1} \neq \emptyset$, or $\sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*) + \sum_{t=1}^{M} \tilde{\rho}_t(x, x^*) > 0$.

Then x^* is an optimal solution of (P).

Proof. (a) : Suppose to the contrary that x^* is not an optimal solution of (P). Then there is a feasible solution $\bar{x}$ of (P) such that $\varphi(\bar{x}) < \varphi(x^*)$ and hence it follows that

$$D(\bar{x},u^*)f_i(\bar{x}) - N(\bar{x},u^*)g_i(\bar{x}) < 0 \text{ for each } i \in \underline{p}.$$

Keeping in mind that $v^* \geq 0$ and using this strict inequality, we see that

$$\begin{aligned}
\Phi_i(\bar{x},x^*,v^*,w^*) &= D(\bar{x},u^*)f_i(\bar{x}) - N(\bar{x},u^*)g_i(\bar{x}) + \sum_{j\in J_0} v_j^* G_j(\bar{x}) + \sum_{k\in K_0} w_k^* H_k(\bar{x}) \\
&\leq D(\bar{x},u^*)f_i(\bar{x}) - N(\bar{x},u^*)g_i(\bar{x}) \text{ (by the feasibility of } \bar{x}) \\
&< 0 \\
&= D(\bar{x},u^*)f_i(x^*) - N(\bar{x},u^*)g_i(x^*) + \sum_{j\in J_0} v_j^* G_j(x^*) + \sum_{k\in K_0} w_k^* H_k(x^*) \\
&\quad \text{(by (2.4), (2.9), and the feasibility of } x^*) \\
&= \Phi_i(x^*,x^*,v^*,w^*),
\end{aligned}$$

and so using the properties of the function $\bar{\phi}_i$, we get

$$\bar{\phi}_i\big(\Phi_i(\bar{x},x^*,v^*,w^*) - \Phi_i(x^*,x^*,v^*,w^*)\big) < 0,$$

which in view of (i) implies that for each $i \in I_+$,

$$\begin{aligned}
&\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big\{D(\bar{x},u^*)\nabla f_i(x^*) - N(\bar{x},u^*)\nabla g_i(x^*) \\
&\qquad + \sum_{j\in J_0} v_j^* \nabla G_j(x^*) + \sum_{k\in K_0} w_k^* \nabla H_k(x^*)\Big\}\Big) \\
&\quad + \beta(x,x^*)\Big\langle \pi(\bar{x},x^*), \Big[D(\bar{x},u^*)\nabla^2 f_i(x^*) - N(\bar{x},u^*)\nabla^2 g_i(x^*) \\
&\qquad + \sum_{j\in J_0} v_j^* \nabla^2 G_j(x^*) + \sum_{k\in K_0} w_k^* \nabla^2 H_k(x^*)\Big] z\Big\rangle \\
&\quad - \frac{1}{2}\Big\langle \omega(x,x^*), \Big[D(\bar{x},u^*)\nabla^2 f_i(x^*) - N(\bar{x},u^*)\nabla^2 g_i(x^*) + \sum_{j\in J_0} v_j^* \nabla^2 G_j(x^*) \\
&\qquad + \sum_{k\in K_0} w_k^* \nabla^2 H_k(x^*)\Big] z\Big\rangle < -\bar{\rho}_i(x,x^*)\|\theta(\bar{x},x^*)\|^m.
\end{aligned}$$

Since $u^* \geq 0,\ u_i^* = 0$ for each $i \in \underline{p} \backslash I_+,\ \ \sum_{i=1}^p u_i^* = 1$, and $\mathscr{F}(x,x^*;\cdot)$ is sublinear, the above inequalities yield

$$\begin{aligned}
&\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big\{\sum_{i=1}^p u_i^*[D(\bar{x},u^*)\nabla f_i(x^*) - N(\bar{x},u^*)\nabla g_i(x^*)] \\
&\qquad + \sum_{j\in J_0} v_j^* \nabla G_j(x^*) + \sum_{k\in K_0} w_k^* \nabla H_k(x^*)\Big\}\Big) \\
&\quad + \beta(x,x^*)\Big\langle \pi(\bar{x},x^*), \Big\{\sum_{i=1}^p u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*) - N(\bar{x},u^*)\nabla^2 g_i(x^*)] \\
&\qquad + \sum_{j\in J_0} v_j^* \nabla^2 G_j(x^*) + \sum_{k\in K_0} w_k^* \nabla^2 H_k(x^*)\Big\} z^*\Big\rangle \\
&\quad - \frac{1}{2}\Big\langle \omega(x,x^*), \Big\{\sum_{i=1}^p u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*) - N(\bar{x},u^*)\nabla^2 g_i(x^*)] \\
&\quad + \sum_{j\in J_0} v_j^* \nabla^2 G_j(x^*) + \sum_{k\in K_0} w_k^* \nabla^2 H_k(x^*)\Big\} z^*\Big\rangle < -\sum_{i\in I_+} u_i^* \bar{\rho}_i(x,x^*)\|\theta(\bar{x},x^*)\|^m. \quad (2.10)
\end{aligned}$$

As seen in the proof of Theorem 2.1, our assumptions in (ii) lead to

$$\begin{aligned}
&\mathscr{F}\Big(x,x^*;\beta(x,x^*)\sum_{t=1}^M \Big[\sum_{j\in J_t} v_j^* \nabla G_j(x^*) + \sum_{k\in K_t} w_k^* \nabla H_k(x^*)\Big]\Big) \\
&\quad + \beta(x,x^*)\Big\langle \pi(\bar{x},x^*), \sum_{t=1}^M \Big[\sum_{j\in J_t} v_j^* \nabla^2 G_j(x^*) + \sum_{k\in K_t} w_k^* \nabla^2 H_k(x^*)\Big] z^*\Big\rangle \\
&\quad - \frac{1}{2}\Big\langle \omega(x,x^*), \sum_{t=1}^M \Big[\sum_{j\in J_t} v_j^* \nabla^2 G_j(x^*) + \sum_{k\in K_t} w_k^* \nabla^2 H_k(x^*)\Big] z^*\Big\rangle \\
&\qquad\qquad \leq -\sum_{t=1}^M \tilde{\rho}_t(x,x^*)\|\theta(\bar{x},x^*)\|^m,
\end{aligned}$$

which when combined with (2.5) and (2.6) results in

$$\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big\{\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla f_i(x^*)-N(\bar{x},u^*)\nabla g_i(x^*)] + \sum_{j\in J_0}v_j^*\nabla G_j(x^*)+\sum_{k\in K_0}w_k^*\nabla H_k(x^*)\Big\}\Big)$$
$$+\beta(x,x^*)\Big\langle\pi(\bar{x},x^*),\Big[\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)] + \sum_{j\in J_0}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_0}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$
$$-\frac{1}{2}\Big\langle\omega(x,x^*),\Big[\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)] + \sum_{j\in J_0}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_0}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle \geq \sum_{t=1}^{M}\tilde{\rho}_t(x,x^*)\|\theta(\bar{x},x^*)\|^m.$$

In view of (iii), this inequality contradicts (2.10). Hence, x^* is an optimal solution of (P).

(b) - (g) : The proofs are similar to that of part (a). □

In the next theorem, we present another collection of sufficient optimality results which are somewhat different from those stated in Theorems 2.1 and 2.2. These results are formulated by utilizing a partition of $\underline{p}$ in addition to those of $\underline{q}$ and $\underline{r}$, and by placing appropriate generalized $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$-sounivexity requirements on certain combinations of the problem functions.

Let $\{I_0,I_1,\ldots,I_\ell\}$ be a partition of $\underline{p}$ such that $\mathscr{L}=\{0,1,2,\ldots,\ell\}\subset\mathscr{M}=\{0,1,\ldots,M\}$, and let the real-valued function $\xi\to\Pi_t(\xi,x,u,v,w)$ be defined, for fixed u,v,w by

$$\Pi_t(\xi,x,u,v,w)=\sum_{i\in I_t}u_i[D(x,u)f_i(\xi)-N(x,u)g_i(\xi)]+\sum_{j\in J_t}v_jG_j(\xi) + \sum_{k\in K_t}w_kH_k(\xi),\ \ t\in\underline{M}.$$

Theorem 2.3. *Let $x^*\in\mathbb{F}$ and assume that the functions $f_i,\ g_i,\ i\in\underline{p},\ G_j,\ j\in\underline{q}$, and $H_k,\ k\in\underline{r}$, are twice differentiable at x^*, and that for each $z^*\in C(x^*)$, there exist $u^*\in U,\ v^*\in\mathbb{R}^q_+$, and $w^*\in\mathbb{R}^r$ such that (2.1) - (2.4) and (2.9) hold. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:*

(a) (i) *for each* $t \in \mathscr{L}$, $\xi \to \Pi_t(\xi, x^*, u^*, v^*, w^*)$ *is strictly* $(\mathscr{F}, \beta, \bar{\phi}_t, \pi, \omega, \rho_t, \theta, m)$*- pseudosounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$;

(ii) *for each* $t \in \mathscr{M} \setminus \mathscr{L}$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ *is* $(\mathscr{F}, \beta, \phi_t, \pi, \omega, \rho_t, \theta, m)$*-quasisounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$;

(iii) $\sum_{t \in \mathscr{M}} \rho_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$;

(b) (i) *for each* $t \in \mathscr{L}$, $\xi \to \Pi_t(\xi, x^*, u^*, v^*, w^*)$ *is prestrictly* $(\mathscr{F}, \beta, \phi_t, \pi, \omega, \rho_t, \theta, m)$*-quasisounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$;

(ii) *for each* $t \in \mathscr{M} \setminus \mathscr{L}$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ *is strictly* $(\mathscr{F}, \beta, \phi_t, \pi, \omega, \rho_t, \theta, m)$*-pseudosounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$;

(iii) $\sum_{t \in \mathscr{M}} \rho_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$;

(c) (i) *for each* $t \in \mathscr{L}$, $\xi \to \Pi_t(\xi, x^*, u^*, v^*, w^*)$ *is prestrictly* $(\mathscr{F}, \beta, \phi_t, \pi, \omega, \rho_t, \theta, m)$*-quasisounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$;

(ii) *for each* $t \in \mathscr{M} \setminus \mathscr{L}$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ *is* $(\mathscr{F}, \beta, \phi_t, \pi, \omega, \rho_t, \theta, m)$*-quasisounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$;

(iii) $\sum_{t \in \mathscr{M}} \rho_t(x, x^*) > 0$ *for all* $x \in \mathbb{F}$;

(d) (i) *for each* $t \in \mathscr{L}_1$, $\xi \to \Pi_t(\xi, x^*, u^*, v^*, w^*)$ *is strictly* $(\mathscr{F}, \beta, \phi_t, \pi, \omega, \rho_t, \theta, m)$*-pseudosounivex at* x^*, *for each* $t \in \mathscr{L}_2$, $\xi \to \Pi_t(\xi, x^*, u^*, v^*, w^*)$ *is prestrictly* $(\mathscr{F}, \beta, \phi_t, \pi, \omega, \rho_t, \theta, m)$*-quasisounivex at* x^*, *and for each* $t \in \mathscr{L}$, ϕ_t *is increasing and* $\phi_t(0) = 0$, *where* $\{\mathscr{L}_1, \mathscr{L}_2\}$ *is a partition of* $\mathscr{L}$;

(ii) *for each* $t \in \mathscr{M} \setminus \mathscr{L}$, $\xi \to \Lambda_t(\xi, v^*, w^*)$ *is strictly* $(\mathscr{F}, \beta, \phi_t, \pi, \omega, \rho_t, \theta)$*-pseudosounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$;

(iii) $\sum_{t\in\mathscr{M}}\rho_t(x,x^*)\geq 0$ *for all* $x\in\mathbb{F}$*;*

(e) *(i) for each* $t\in\mathscr{L}_1\neq\emptyset$*,* $\xi\to\Pi_t(\xi,x^*,u^*,v^*,w^*)$ *is strictly* $(\mathscr{F},\beta,\phi_t,\pi,\omega,\rho_t,\theta,m)$*-pseudosounivex at* x^**, for each* $t\in\mathscr{L}_2$*,* $\xi\to\Pi_t(\xi,u^*,v^*,w^*,\lambda^*)$ *is prestrictly* $(\mathscr{F},\beta,\phi_t,\pi,\omega,\rho_t,\theta,m)$*-quasisounivex at* x^**, and for each* $t\in\mathscr{L}$*,* ϕ_t *is increasing and* $\phi_t(0)=0$*, where* $\{\mathscr{L}_1,\mathscr{L}_2\}$ *is a partition of* $\mathscr{L}$*;*

(ii) for each $t\in\mathscr{M}\setminus\mathscr{L}$*,* $\xi\to\Lambda_t(\xi,v^*,w^*)$ *is* $(\mathscr{F},\beta,\phi_t,\rho_t,\theta,m)$*-quasisounivex at* x^**,* ϕ_t *is increasing, and* $\phi_t(0)=0$*;*

(iii) $\sum_{t\in\mathscr{M}}\rho_t(x,x^*)\geq 0$ *for all* $x\in\mathbb{F}$*;*

(f) *(i) for each* $t\in\mathscr{L}$*,* $\xi\to\Pi_t(\xi,x^*,u^*,v^*,w^*)$ *is prestrictly* $(\mathscr{F},\beta,\phi_t,\pi,\omega,\rho_t,\theta,m)$ *-quasisounivex at* x^**,* ϕ_t *is increasing, and* $\phi_t(0)=0$*;*

(ii) for each $t\in(\mathscr{M}\setminus\mathscr{L})_1\neq\emptyset$*,* $\xi\to\Lambda_t(\xi,v^*,w^*)$ *is strictly* $(\mathscr{F},\beta,\phi_t,\pi,\omega,\rho_t,\theta,m)$ *-pseudosounivex at* x^**, for each* $t\in(\mathscr{M}\setminus\mathscr{L})_2$*,* $\xi\to\Lambda_t(\xi,v^*,w^*)$ *is* $(\mathscr{F},\beta,\phi_t,\pi,\omega,\rho_t,\theta,m)$*-quasisounivex at* x^**, and for each* $t\in\mathscr{L}$*,* ϕ_t *is increasing and* $\phi_t(0)=0$*, where* $\{(\mathscr{M}\setminus\mathscr{L})_1,(\mathscr{M}\setminus\mathscr{L})_2\}$ *is a partition of* $\mathscr{M}\setminus\mathscr{L}$*;*

(iii) $\sum_{t\in\mathscr{M}}\rho_t(x,x^*)\geq 0$ *for all* $x\in\mathbb{F}$*;*

(g) *(i) for each* $t\in\mathscr{L}_1$*,* $\xi\to\Pi_t(\xi,x^*,u^*,v^*,w^*)$ *is* $(\mathscr{F},\beta,\phi_t,\pi,\omega,\rho_t,\theta,m)$*-pseudosounivex at* x^**, for each* $t\in\mathscr{L}_2$*,* $\xi\to\Pi_t(\xi,x^*,u^*,v^*,w^*)$ *is prestrictly* $(\mathscr{F},\beta,\phi_t,\pi,\omega,\rho_t,\theta,m)$*-quasisounivex at* x^**, and for each* $t\in\mathscr{L}$*,* ϕ_t *is increasing and* $\phi_t(0)=0$*, where* $\{\mathscr{L}_1,\mathscr{L}_2\}$ *is a partition of* $\mathscr{L}$*;*

(ii) for each $t\in(\mathscr{M}\setminus\mathscr{L})_1$*,* $\xi\to\Lambda_t(\xi,v^*,w^*)$ *is strictly* $(\mathscr{F},\beta,\phi_t,\pi,\omega,\rho_t,\theta,m)$*-pseudosounivex at* x^**, for each* $t\in(\mathscr{M}\setminus\mathscr{L})_2$*,* $\xi\to\Lambda_t(\xi,v^*,w^*)$ *is* $(\mathscr{F},\beta,\phi_t,\pi,\omega,\rho_t,\theta,m)$*-quasisounivex at*

x^*, and for each $t \in \mathscr{M} \setminus \mathscr{L}$, ϕ_t is increasing and $\phi_t(0) = 0$, where $\{(\mathscr{M} \setminus \mathscr{L})_1, (\mathscr{M} \setminus \mathscr{L})_2\}$ is a partition of $\mathscr{M} \setminus \mathscr{L}$;

(iii) $\sum_{t\in\mathscr{M}} \rho_t(x,x^*) \geqq 0$ *for all* $x \in \mathbb{F}$;

(iv) $\mathscr{L}_1 \neq \emptyset$, $(\mathscr{M} \setminus \mathscr{L})_1 \neq \emptyset$, *or* $\sum_{t\in\mathscr{M}} \rho_t(x,x^*) > 0$.

Then x^* is an optimal solution of (P).

Proof. (a): Suppose to the contrary that x^* is not an optimal solution of (P). As seen in the proof of Theorem 2.2, this supposition leads to the inequalities

$$D(\bar{x},u^*)f_i(\bar{x}) - N(\bar{x},u^*)g_i(\bar{x}) < 0, \quad i \in \underline{p},$$

for some $\bar{x} \in \mathbb{F}$. Since $u^* \geq 0$, we see that for each $t \in \mathscr{L}$,

$$\sum_{i\in I_t} u_i^*[D(\bar{x},u^*)f_i(\bar{x}) - N(\bar{x},u^*)g_i(\bar{x})] \leq 0. \tag{2.11}$$

Now using this inequality, we see that

$$\begin{aligned}
\Pi_t(\bar{x},x^*,u^*,v^*,w^*) &= \sum_{i\in I_t} u_i^*[D(\bar{x},u^*)f_i(\bar{x}) - N(\bar{x},u^*)g_i(\bar{x})] \\
&\quad + \sum_{j\in J_t} v_j^* G_j(\bar{x}) + \sum_{k\in K_t} w_k^* H_k(\bar{x}) \\
&\leq \sum_{i\in I_t} u_i^*[D(\bar{x},u^*)f_i(\bar{x}) - N(\bar{x},u^*)g_i(\bar{x})] \ \text{(by the feasibility of } \bar{x}) \\
&\leq 0 \quad \text{(by (2.11))} \\
&= \sum_{i\in I_t} u_i^*[D(\bar{x},u^*)f_i(x^*) - N(\bar{x},u^*)g_i(x^*)] + \sum_{j\in J_t} v_j^* G_j(x^*) \\
&\quad + \sum_{k\in K_t} w_k^* H_k(x^*) \ \text{(by (2.4), (2.9), and the feasibility of } x^*) \\
&= \Pi_t(x^*,x^*,u^*,v^*,w^*),
\end{aligned}$$

and hence

$$\phi_t\big(\Pi_t(\bar{x},x^*,u^*,v^*,w^*) - \Pi_t(x^*,x^*,u^*,v^*,w^*)\big) \leq 0,$$

which in view of (i) implies that

$$\begin{aligned}
\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big\{&\sum_{i\in I_t}u_i^*[D(\bar{x},u^*)\nabla f_i(x^*)-N(\bar{x},u^*)\nabla g_i(x^*)] \\
&+\sum_{j\in J_t}v_j^*\nabla G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\Big\}\Big) \\
&+\beta(x,x^*)\Big\langle\pi(\bar{x},x^*),\Big[\sum_{i\in I_t}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)] \\
&+\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle \\
&-\frac{1}{2}\Big\langle\omega(x,x^*),\Big[\sum_{i\in I_t}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)] \\
&+\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle \\
&<-\rho_t(\bar{x},x^*)\|\theta(\bar{x},x^*)\|^m.
\end{aligned}$$

Summing over $t\in\mathscr{L}$ and using the sublinearity of $\mathscr{F}(x,x^*;\cdot)$, we obtain

$$\begin{aligned}
\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big\{&\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla f_i(x^*)-N(\bar{x},u^*)\nabla g_i(x^*)] \\
&+\sum_{t\in\mathscr{L}}\Big[\sum_{j\in J_t}v_j^*\nabla G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\Big]\Big\}\Big) \\
&+\beta(x,x^*)\Big\langle\pi(\bar{x},x^*),\sum_{t\in\mathscr{L}}\Big\{\sum_{i\in I_t}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)] \\
&+\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle \\
&-\frac{1}{2}\Big\langle\omega(x,x^*),\sum_{t\in\mathscr{L}}\Big\{\sum_{i\in I_t}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)] \\
&+\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle \\
&\leq-\sum_{t\in\mathscr{L}}\rho_t(\bar{x},x^*)\|\theta(\bar{x},x^*)\|^m. \qquad (2.12)
\end{aligned}$$

Proceeding as in the proof of Theorem 2.1, we obtain for each $t\in\mathscr{M}\backslash\mathscr{L}$,

$$\phi_t\big(\Lambda_t(\bar{x},v^*,w^*)-\Lambda_t(x^*,v^*,w^*)\big)\leq 0,$$

which in view of (ii) implies that

$$\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big[\sum_{j\in J_t}v_j^*\nabla G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\Big]\Big)$$
$$+\beta(x,x^*)\Big\langle\pi(\bar{x},x^*),\Big[\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$
$$-\frac{1}{2}\Big\langle\omega(x,x^*),\Big[\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$
$$\leq-\rho_t(\bar{x},x^*)\|\theta(\bar{x},x^*)\|^m.$$

Summing over $t\in\mathscr{M}\backslash\mathscr{L}$ and using the sublinearity of $\mathscr{F}(x,x^*;\cdot)$, we get

$$\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big[\sum_{t\in\mathscr{M}\backslash\mathscr{L}}\Big[\sum_{j\in J_t}v_j^*\nabla G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\Big]\Big)$$
$$+\beta(x,x^*)\Big\langle\pi(\bar{x},x^*),\sum_{t\in\mathscr{M}\backslash\mathscr{L}}\Big[\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$
$$-\frac{1}{2}\Big\langle\omega(x,x^*),\sum_{t\in\mathscr{M}\backslash\mathscr{L}}\Big[\sum_{j\in J_t}v_j^*\nabla^2 G_j(x^*)+\sum_{k\in K_t}w_k^*\nabla^2 H_k(x^*)\Big]z^*\Big\rangle$$
$$<-\sum_{t\in\mathscr{M}\backslash\mathscr{L}}\rho_t(\bar{x},x^*)\|\theta(\bar{x},x^*)\|^m. \quad (2.13)$$

Now combining (2.12) and (2.13) and using (iii), we obtain

$$\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big\{\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla f_i(x^*)-N(\bar{x},u^*)\nabla g_i(x^*)]$$
$$+\sum_{j=1}^{q}v_j^*\nabla G_j(x^*)+\sum_{k=1}^{r}w_k^*\nabla H_k(x^*)\Big\}\Big)$$
$$+\beta(x,x^*)\Big\langle\pi(\bar{x},x^*),\Big\{\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)]$$
$$+\sum_{j=1}^{q}v_j^*\nabla^2 G_j(x^*)+\sum_{k=1}^{r}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle$$
$$-\frac{1}{2}\Big\langle\omega(x,x^*),\Big\{\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)]$$
$$+\sum_{j=1}^{q}v_j^*\nabla^2 G_j(x^*)+\sum_{k=1}^{r}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle<-\sum_{t\in\mathscr{M}}\rho_t(\bar{x},x^*)\|\theta(\bar{x},x^*)\|^m\leq 0. \quad (2.14)$$

Now multiplying (2.1) by $\beta(x,x^*)$, applying the sublinear function $\mathscr{F}(x,x^*;\cdot)$ to both sides of the resulting equation, and then adding the equation to (2.2), we get

$$\mathscr{F}\Big(x,x^*;\beta(x,x^*)\Big\{\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla f_i(x^*)-N(\bar{x},u^*)\nabla g_i(x^*)] + \sum_{j=1}^{q}v_j^*\nabla G_j(x^*)+\sum_{k=1}^{r}w_k^*\nabla H_k(x^*)\Big\}\Big)$$
$$+\beta(x,x^*)\Big\langle\pi(\bar{x},x^*),\Big\{\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)] + \sum_{j=1}^{q}v_j^*\nabla^2 G_j(x^*)+\sum_{k=1}^{r}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle$$
$$-\frac{1}{2}\Big\langle\omega(x,x^*),\Big\{\sum_{i=1}^{p}u_i^*[D(\bar{x},u^*)\nabla^2 f_i(x^*)-N(\bar{x},u^*)\nabla^2 g_i(x^*)] + \sum_{j=1}^{q}v_j^*\nabla^2 G_j(x^*)+\sum_{k=1}^{r}w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle\geq 0,$$

which contradicts (2.14). Therefore, we conclude that x^* is an optimal solution of (P).

(b) - (g) : The proofs are similar to that of part (a). □

3 Concluding Remarks

Remark 3.1. Based on a Dinkelbach-type [1] parametric approach, we have in this paper established numerous sets of generalized second-order sufficient optimality criteria for a discrete minmax fractional programming problem using a variety of generalized $(\mathscr{F},\beta,\phi,\pi,\omega,\rho,\theta,m)$- sounivexity assumptions. These optimality results can be used for constructing various duality models as well as for developing new algorithms for the numerical solution of minmax fractional programming problems. Furthermore, main results can be used, for example, employing similar techniques, one can investigate the second-order sufficient optimality aspects of the following semiinfinite minmax fractional programming problem:

$$\text{Minimize } \max_{1\leq i\leq p}\frac{f_i(x)}{g_i(x)}$$

subject to

$$G_j(x,t) \leq 0 \text{ for all } t \in T_j, \ \ j \in \underline{q}; \ \ H_k(x,s) = 0 \text{ for all } s \in S_k, \ \ k \in \underline{r}; \ \ x \in X,$$

where X, f_i, and g_i, $i \in \underline{p}$, are as defined in the description of (P), for each $j \in \underline{q}$ and $k \in \underline{r}$, T_j and S_k are compact subsets of complete metric spaces, for each $j \in \underline{q}$, $\xi \to G_j(\xi,t)$ is a real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r}$, $\xi \to H_k(\xi,s)$ is a real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $t \to G_j(x,t)$ and $s \to H_k(x,s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$.

References

1. W. Dinkelbach, On nonlinear fractional programming, *Management Sci.* **13** (1967), 492 - 498.

2. M. A. Hanson, Second order invexity and duality in mathematical programming, *Opsearch* **30** (1993), 313 - 320.

3. B. Mond, and T. Weir, Generalized concavity and duality, *Generalized Concavity in Optimization and Economics* (S. Schaible and W. T. Ziemba, eds.), Academic Press, New York, 1981, pp. 263 - 279.

4. R. U. Verma, A generalization to Zalmai type univexities and applications to parametric duality models in discrete minimax fractional programming, *Advances in Nonlinear Variational Inequalities* **15** (2) (2012), 113–123.

5. Ram U. Verma and G. J. Zalmai, Second-order parametric optimality conditions in discrete minmax fractional programming, *Communications on Applied Nonlinear Analysis* **23 (3)** (2016), 1 - 32.

6. R. U. Verma, Hybrid parametric duality models for discrete minmax fractional programming problems on second-order optimality conditions, *Transactions on Mathematical programming and Applications* **5(1)** (2017), 89 - 120.

7. Ram U. Verma and G. J. Zalmai, Generalized second-order parametric optimality conditions in discrete minmax fractional programming, *Transactions on Mathematical programming and Applications* **2 (12)** (2014), 1 - 20.

8. X. Yang, Generalized convex duality for multiobjective fractional programs, *Opsearch* **31** (1994), 155 - 163.

9. G. J. Zalmai, Optimality conditions and duality for constrained measurable subset selection problems with minmax objective functions, *Optimization* **20** (1989), 377 - 395.