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g-Frames in Hilbert Spaces

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Abstract

In this paper, we give recent results on properties and characterizations of generalized frames, called g-frames, in Hilbert spaces. We mention generalized dual frames/dual g-frames and we state some characterization of dual g-frames that fits best for systems having erasures.

1 Introduction

Frames are redundant structures that is necessary in some applications such as image and signal processing. Redundancy of frames let one to have minimal erasure errors in data processing. Additionally, construction of a frame is more flexible than bases. For these reasons, many emphasize have been given to frame theory in recent years (see [3, 4, 5] for introduction to frame theory).

Many different generalizations of frames have been proposed lately such as quasi-projectors [7], frames of subspaces [2, 3], pseudo-frames [11] and oblique frames [6]. The more general definition that includes previous generalizations has been introduced by Sun in [17] and called a g-frame in a complex Hilbert space. In this paper, we talk about g-frames, its properties similar to a frame and its relation between operators. We mention how a frame and g-frame are related and how a frame is constructed from

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a g-frame. Finally, we give recent results on the optimal dual g-frame for erasures in data reconstruction process.

Let U and V two Hilbert spaces with inner product $< \cdot,\cdot >$ and $\{V_i\}_{i\in I}$ be a sequence of closed subspaces of V where I is a subset of $\mathbb{Z}$.

For a sequence $\{V_i\}_{i\in I}$, we define

$$\ell_2\left(\{V_i\}_{i\in I}\right) = \left\{ \{f_i\}_{i\in I} \,|\, f_i \in V_i,\ \|\{f_i\}_{i\in I}\|_2^2 = \sum_{i\in I} \|f_i\|^2 < \infty \right\}.$$

We note that $\ell_2\left(\{V_i\}_{i\in I}\right)$ is a Hilbert space with pointwise operations and inner product

$$< \{f_i\}_{i\in I}, \{g_i\}_{i\in I} >= \sum_{i\in I} < f_i, g_i > .$$

Let $L(U,V_i)$ be the collection of all bounded linear operators from U into V_i. A sequence $\{\Lambda_i \in L(U,V_i) : i \in I\}$ is called a generalized frame, or g-frame, for U with respect to $\{V_i\}_{i\in I}$ if there are two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in U. \tag{1.1}$$

The constants A and B are called the lower and upper g-frame bounds, respectively.

If $A = B$, then the sequence $\{\Lambda_i \in L(U, V_i) : i \in I\}$ is called a tight g-frame, and if $A = B = 1$ it is called a Parseval g-frame. A g-frame is called uniform when $\|\Lambda_i\|$ is a constant for all $i \in I$. The sequences which satisfy only the right side of the inequality in (1.1) are called g-Bessel sequences.

If $\{f \in U : \Lambda_i f = 0, \forall i \in I\} = \{0\}$, then we say that $\{\Lambda_i\}_{i\in I}$ is g-complete. If a g-frame stops to be a g-frame whenever one of its elements is removed, then it is called an exact g-frame.

If $\{\Lambda_i\}_{i\in I}$ is g-complete and there are constants $A,B > 0$ such that for any finite $I' \subset I$ and $g_i \in V_i$, $i \in I'$,

$$A\sum_{i\in I'} \|g_i\|^2 \leq \|\sum_{i\in I'} \Lambda_i^* g_i\|^2 \leq B\sum_{i\in I'} \|g_i\|^2,$$

then $\{\Lambda_i\}_{i\in I}$ is called a g-Riesz basis for U with respect to $\{V_i\}_{i\in I}$.

Now let us look at some examples of g-frames.

Example 1.1. *A frame is a g-frame whenever* $V_i = \mathbb{C}$ *for all* i *and* $\Lambda_i f =< f, f_i >$, $f \in U$, $i \in I$ *(for every functional* $\Lambda_i \in L(U, \mathbb{C})$ *there is an* $f_i \in U$ *such that* $\Lambda_i f =< f, f_i >$ *by the Riesz Representation Theorem.)*

Example 1.2. *Frames of subspaces. For a sequence of subspaces* $\{W_i\}_{i\in I}$ *of* U*, let* P_{W_i} *be the orthogonal projection on* W_i*. Then* $\{W_i\}_{i\in I}$ *is called a frame of subspaces if there exists* A, $B > 0$ *such that*

$$A\|f\|^2 \leq \sum_{i\in I} \|P_{W_i} f\|^2 \leq B\|f\|^2, \quad \forall f \in U.$$

Thus $\{P_{W_i}\}_{i\in I}$ *is a g-frame for* U *with respect to* $\{W_i\}_{i\in I}$*.*

Example 1.3. *Pseudo-frames/oblique frames. Let* U_0 *be a closed subspace of* U*. Let* $\{f_i\}_{i\in I} \subset U$ *be a Bessel sequence in* U_0 *and* $\{\tilde{f}_i\}_{i\in I} \subset U$ *be a Bessel sequence in* U*. Then it was shown in [**?**] that there are constants* $A, B > 0$ *such that* $A\|f\|^2 \leq \sum_{i\in I} \| < f, f_i > \|^2 \leq B\|f\|^2, \quad \forall f \in U_0$*. Letting* Λ_{f_i} *be the functional induced by* f_i*,* $i \in I$*, we get*

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_{f_i} f\|^2 \leq B\|f\|^2, \quad \forall f \in U_0.$$

It is clear that $\{\Lambda_{f_i}\}$ *is a g-frame for* U_0 *with respect to* $\mathbb{C}$*.*

2 G-frame Operators

For a g-Bessel sequence $\{\Lambda_i \in L(U, V_i) : i \in I\}$, the synthesis operator is defined by

$$\ell^2(\{V_i\}_{i\in I}) \to U; \quad T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^*(f_i).$$

The adjoint operator T_Λ^* of T, called as analysis operator of $\{\Lambda_i\}_{i\in I}$, is defined by

$$T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in I}.$$

Then the g-frame operator $S_\Lambda : U \to U$ of $\{\Lambda_i\}_{i\in I}$ is given by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f, \quad \forall f \in U$$

which is a bounded, self-adjoint, positive and invertible operator and

$$AI_U \leq S_\Lambda I_U \leq BI_U,$$

where I is the identity operator on U. Furthermore, we have $B^{-1}I_U \leq S_\Lambda^{-1}I_U \leq A^{-1}I_U$.

Lemma 2.1. *[?] Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for U with respect to $\{V_i\}_{i\in I}$, and T and T^* be synthesis and analysis operators of $\{\Lambda_i\}$, respectively. Then we have*

$$N_T^\perp = T^*(U), \quad N_{T^*} = T(\ell^2(\{V_i\}_{i\in I}))^\perp = \{0\}$$

Theorem 2.1. *[?] A sequence $\{\Lambda_i\}_{i\in I}$ is a g-frame for U with respect to $\{V_i\}_{i\in I}$ if and only if T is a well defined bounded linear operator from $\ell^2(\{V_i\}_{i\in I})$ onto U and the g-frame bounds are $A = \|T^+\|^{-2} = \|S^{-1}\|^{-1}$ and $B = \|T^+\|^2 = \|S\|$ where $T^+ = T^*S^{-1}$ is the pseudo-inverse operator of T and T^* is the adjoint operator of T.*

For $f \in U$ we have

$$f = SS^{-1} = \sum_{i\in I} \Lambda_i^*\Lambda_i S^{-1} f \quad \text{and} \quad f = S^{-1}S = S^{-1}\sum_{i\in I}\Lambda_i^*\Lambda_i = \sum_{i\in I} S^{-1}\Lambda_i^*\Lambda_i.$$

Then, for all $f \in U$, we obtain the following reconstruction formula

$$f = \sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i f = \sum_{i\in I}\tilde{\Lambda}_i^{\,*}\Lambda_i f$$

where $\tilde{\Lambda}_i = \Lambda_i S^{-1}$. It can be shown that $\{\tilde{\Lambda}_i\}_{i\in I}$ is a g-frame for U with respect to $\{V_i\}_{i\in I}$ with frame bounds $1/B$ and $1/A$ ([17]).

Let $\tilde{S}$ be the g-frame operator associated with $\{\tilde{\Lambda}_i\}_{i\in I}$. Then for all $f \in U$,

$$S\tilde{S}f = \sum_{i\in I} S\tilde{\Lambda}_i^*\tilde{\Lambda}_i f = \sum_{i\in I} SS^{-1}\Lambda_i^*\Lambda_i S^{-1} f = \sum_{i\in I}\Lambda_i^*\Lambda_i S^{-1} f = SS^{-1}f = f.$$

Thus, $\{\Lambda_i\}_{i\in I}$ and $\{\tilde{\Lambda}_i\}_{i\in I}$ are (canonical) dual g-frames with respect to each other with $\tilde{S} = S^{-1}$ and $\tilde{\Lambda}_i\tilde{S}^{-1} = \Lambda_i S^{-1}S = \Lambda_i$. We note that one can always get a Parseval g-frame $\{\Upsilon_i\}_{i\in I}$ from a g-frame $\{\Lambda\}_{i\in I}$ with $\Upsilon_i = \Lambda_i S^{-1/2}$.

Let $\{\tilde{f}_i\}_{i\in I}$ be the canonical dual frame of $\{f_i\}_{i\in I}$. Consider the functional Λ_{f_i} induced by f_i such that $\Lambda_{f_i} f = < f, f_i >$ for all $f \in U$. Then for all $f \in U$

$$\Lambda_{f_i} S^{-1} f = < S^{-1} f, f_i > = < f, S^{-1} f_i > = < f, \tilde{f}_i > = \Lambda_{\tilde{f}_i} f.$$

Hence $\Lambda_{f_i} S^{-1} = \Lambda_{\tilde{f}_i}$ which implies that $\{\Lambda_{\tilde{f}_i}\}_{i\in I}$ is the dual g-frame of $\{\Lambda_{f_i}\}_{i\in I}$.

3 Characterization of g-frames

Let $\Lambda_i \in L(U,V_i)$ and $\{e_{i,j} : j \in J_i\}$ be an orthonormal basis for V_i where $J_i \subset \mathbb{Z}$, $i \in I$. Then $f \mapsto < \Lambda_i f, e_{i,j} >$ defines a bounded linear functional on U. Thus, there exists $u_{i,j} \in U$ such that $< f, u_{i,j} > = < \Lambda_i f, e_{i,j} >$ for all $f \in U$. Therefore, for $f \in U$ and $g \in V_i$, we have

$$\Lambda_i f = \sum_{j\in J_i} < f, u_{i,j} > e_{i,j} \quad \text{and}$$

$$< f, \Lambda_i^* g > = < \Lambda_i f, g > = \sum_{j\in J_i} < f, u_{i,j} >< e_{i,j}, g > = \left\langle f, \sum_{j\in J_i} < g, e_{i,j} > u_{i,j} \right\rangle.$$

Thus, $\Lambda_i^* g = \sum_{j\in J_i} < g, e_{i,j} > u_{i,j}$ for all $g \in U_i$. We note that the sequence $\{u_{i,j} : i \in I, j \in J_i\}$ induced by Λ_i with respect to $\{e_{i,j} : i \in I, j \in J_i\}$ is a Bessel sequence for U where $u_{i,j} = \Lambda^* e_{i,j}$.

Let $\Lambda_i \in L(U,V_i)$ and $\{e_{i,j} : j \in J_i\}$ be an orthonormal basis for V_i where $J_i \subset \mathbb{Z}$, $i \in I$. Then $f \mapsto < \Lambda_i f, e_{i,j} >$ defines a bounded linear functional on U. Thus, there exists $u_{i,j} \in U$ such that $< f, u_{i,j} > = < \Lambda_i f, e_{i,j} >$ for all $f \in U$. Therefore, for $f \in U$ and $g \in V_i$, we have

$$\Lambda_i f = \sum_{j\in J_i} < f, u_{i,j} > e_{i,j} \quad \text{and}$$

$$< f, \Lambda_i^* g > = < \Lambda_i f, g > = \sum_{j\in J_i} < f, u_{i,j} >< e_{i,j}, g > = \left\langle f, \sum_{j\in J_i} < g, e_{i,j} > u_{i,j} \right\rangle.$$

Thus, $\Lambda_i^* g = \sum_{j\in J_i} < g, e_{i,j} > u_{i,j}$ for all $g \in U_i$. We note that the sequence $\{u_{i,j} : i \in I, j \in J_i\}$ induced by Λ_i with respect to $\{e_{i,j} : i \in I, j \in J_i\}$ is a Bessel sequence for U where $u_{i,j} = \Lambda^* e_{i,j}$.

Theorem 3.1. *(i)* $\{\Lambda_i\}_{i\in I}$ *is a g-frame for* U *if and only if* $\{u_{i,j} : i \in I, j \in J_i\}$ *is a frame for* U.

(ii) If $\{\Lambda_i\}_{i\in I}$ *is a g-frame for* U *then* $\sum_{i\in I} dimV_i \geq dimU$

(iii) The g-frame operator for $\{\Lambda_i\}_{i\in I}$ *and the frame operator for* $\{u_{i,j} : i \in I, j \in J_i\}$ *are the same.*

(iv) $\{\Lambda_i\}_{i\in I}$ *and* $\{\tilde{\Lambda}_i\}_{i\in I}$ *are a pair of canonical dual g-frames if and only if* $\{u_{i,j} : i \in I, j \in J_i\}$ *and* $\{\tilde{u}_{i,j} : i \in I, j \in J_i\}$ *are a pair of canonical dual frames.*

By the above theorem we see that we need an orthonormal basis $\{e_{i,j}\}_{j\in J_i}$ for V_i to obtain a frame from a g-frame. In practice it is hard to find an orthonormal basis. However, as we will see in the following theorem, we do not need such a basis to construct a frame from a given g-frame.

Theorem 3.2. *[17] Let* $\{\Lambda_i\}_{i\in I}$ *and* $\{\tilde{\Lambda}_i\}_{i\in I}$ *be a pair of dual g-frames for* U *with respect to* $\{V_i\}_{i\in I}$. *Let* $\{g_{i,j}\}_{j\in J_i}$ *and* $\{\tilde{g}_{i,j}\}_{j\in J_i}$ *be a pair of dual frames for* V_i. *Then* $\{\Lambda_i^* g_{i,j}\}_{i\in I, j\in J_i}$ *and* $\{\tilde{\Lambda}_i^* \tilde{g}_{i,j}\}_{i\in I, j\in J_i}$ *are a pair of dual g-frames for* U.

As we have seen so far g-frames and frames share many similar properties. However, there are some properties of frames that g-frames do not have. For instance, an exact g-frame may not be a g-Riesz basis while Riesz bases are exact frames. Moreover, whenever an element of a g-frame is removed, it may remain neither a g-frame nor g-complete (see [17]) while a frame remains a frame or incomplete.

4 Dual g-frames and Erasures

Two g-Bessel sequences $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Theta = \{\Theta_i\}_{i\in I}$ for U with respect to $\{V_i\}_{i\in I}$ are called dual g-frames if

$$f = \sum_{i\in I} \Lambda_i^* \Theta_i f, \quad \forall f \in U.$$

It can be seen that such g-Bessel sequences are g-frames. And it is said that $\{\Theta_i\}_{i\in I}$ is a dual g-frame for $\{\Lambda_i\}_{i\in I}$. If S is the g-frame operator for $\{\Lambda_i\}_{i\in I}$, then $\{\Lambda_i S^{-1}\}_{i\in I}$ is called a canonical dual g-frame, and otherwise a dual is called an alternate dual g-frame.

The connection between the canonical dual g-frame and alternate dual g-frame is given by the following:

Proposition 4.1. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame and S_Λ be a g-frame operator. Then, $\Theta = \{\Theta_i\}_{i\in I}$ is a dual g-frame of Λ if and only if there exists a g-Bessel sequence $\Phi = \{\Phi_i \in L(U,V_i) : i \in I\}$ such that $\Theta_i = \Lambda_i S_\Lambda^{-1} + \Phi_i$ and $T_\Lambda T_\Phi^* = 0$*

In data transmissions naturally erasures occur. For this reason, in frame theory it is important to find frames and dual frames that minimizes data transmission and reconstruction process. To find the optimal dual g-frame that minimizes erasure errors, we look at the following:

$$\min_\Theta \max\{\|T_\Lambda D T_\Theta^*\| : D \in D_m\} = \min_\Phi \max\{\|\sum_{i\in I'} \Lambda_i^* \Lambda_i S^{-1} f + \sum_{i\in I'} \Lambda_i^* \Phi_i f\|\},$$

where T_Λ, T_Θ are associated synthesis operators for Λ and Θ, D_m is the set of all infinite diagonal matrices with m 1's and $I' \subset I$ is the set of indices corresponding to erased data. If a dual g-frame minimizes the maximum m-erasure error in addition to the minimum of the maximum of m-1 erasure error, it is called as m-erasure optimal dual g-frame for $\Lambda = \{\Lambda_i\}_{i\in I}$. In [**?**], the conditions on a canonical dual g-frame to be optimal for one-erasure are given in the following theorem.

Definition 4.1. *We call a sequence $\{\Lambda_i\}_{i\in I}$ an $\ell^2(\{V_i\}_{i\in I})$ linearly independent family if $\sum_{i\in I}\Lambda_i^* g_i = 0$ for $\{g_i\} \in \ell^2(\{V_i\}_{i\in I})$ then $g_i = 0$ for all $i \in I$.*

For a g-frame $\{\Lambda_i\}_{i\in I}$, let $c = \max\left\{\|\Lambda_i\|\|\Lambda_i S_\Lambda^{-1}\| : i \in I\right\}$ and $U_k = \overline{span}\{\Lambda_i^*(V_i)\}_{i\in I_k}$ for $k = 1,2$ where $I_1 = \left\{i : \|\Lambda_i\|\|\Lambda_i S_\Lambda^{-1}\| = c\right\}$ and $I_2 = \{i : i \notin I_1\}$.

Theorem 4.1. *The canonical dual g-frame $\{\Lambda_i S_\Lambda^{-1}\}$ is the unique optimal dual g-frame for 1-erasure if and only if $U_1 \cap U_2 = \{0\}$ and $\{\Lambda_i\}_{i\in I_2}$ is an $\ell^2(\{V_i\}_{i\in I_2})$ linearly independent family.*

There are still many problems that need to be addressed in erasures with g-frames. For instance, what are the properties of g-frames that give minimal erasure errors, how to characterize and construct such frames for any erasures? Additionally, how one can construct optimal dual g-frames for a g-frame and how to characterize them? These are among the questions that need to be investigated in future works.

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