A note on some integral inequalities for (h, m)-convex functions in a generalized framework

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Abstract

In this paper using a general definition of convexity, the (h, m)-convex functions, we present some new integral inequalities, using a generalized integral operator.

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1 Preliminars

Definition 1 *A function $\phi : I \to \mathbb{R}$, $I := [a_1, a_2]$ is said to be convex if $\phi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$ holds for all $x, y \in I$ and $\lambda \in [0, 1]$.*

If the above inequality is reversed, then the function ϕ will be the concave on $[a_1, a_2]$.

One of the most fruitful notions in current Mathematics is that of the convex function, which has spread in various directions (the interested reader can consult [27], where a fairly complete overview of the generalizations and extensions of the convex function concept is presented).

For convex functions, the following inequality is known, undoubtedly one of the most famous in Mathematics, for its multiple connections and applications:

$$\phi\left(\frac{a_1+a_2}{2}\right) \le \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \phi(x)\,dx \le \frac{\phi(a_1)+\phi(a_2)}{2}, \quad (1)$$ {Hadamard}

this is called the Hermite–Hadamard inequality. The interested reader is referred to [1, 6, 7, 8, 11, 12, 13, 17, 18, 21, 23, 24, 28, 34] and references therein for more information and other extensions of the Hermite–Hadamard inequality.

Toader in [35] defined m-convexity in the following way:

{d:(mc}

Definition 2 *The function $\phi : [0, a_2] \to \mathbb{R}$, $a_2 > 0$, is said to be m-convex, where $m \in [0, 1]$, if*

$$\phi\left(tx + (1-t)y\right) \le h(t)\phi(x) + mh(1-t)\phi(y) \quad (2)$$ {e:qc}

holds for all $x, y \in [0, a_2]$ and $t \in [0, 1]$.

If the above inequality holds in reverse, then we say that the function ϕ is m-concave.

In [23] the following definition is introduced that will be used in our work.

{d:(hmc}

Definition 3 *Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function, $h \ne 0$. The nonnegative function $\phi : [0, a_2] \to \mathbb{R}$ with $a_2 > 0$ is said to be (h, m)-convex on $[0, a_2]$ if inequality*

$$\phi\left(tx + (1-t)y\right) \le h(t)\phi(x) + mh(1-t)\phi(y) \quad (3)$$ {e:hmc}

is fulfilled for $m \in [0, 1]$, for all $x, y \in I$ and $t \in [0, 1]$.

If the above inequality is reversed, then f is said to be (h, m)-concave. Note that if $h(t) = t$ then the f above definition reduces to the definition of m-convex function, if in addition, we put $m = 1$ then we obtain the definition of convex function.

In the last 5 decades, we have witnessed the development of new traders, spreads and integrals, which include both fractional and generalized. The latter, in general, are defined as local derivatives and generate integral operators that may or may not be fractional. To date, the study of this area has attracted the attention of many researchers, not only in Pure Mathematics, but in multiple fields of applied science. Between its own theoretical development and the multiplicity of applications, the field has grown rapidly in recent years, in such a way that a single definition of "fractional derivative or integral" does not exist, or at least is not unanimously accepted, in [5] suggests and justifies the idea of a fairly complete classification of the known operators of the Fractional Calculus, on the other hand, in the work [4] some reasons are presented why new operators linked to applications and developments theorists appear every day. These operators had been developed by numerous mathematicians with a barely specific formulation, for instance, the Riemann-Liouville (RL), the Weyl, Erdelyi-Kober, Hadamard integrals, and the Liouville and Katugampola fractional operators and many authors have introduced new fractional operators generated from general classical local derivatives.

In addition, Chapter 1 of [2] presents a history of differential operators, both local and global, from Newton to Caputo and presents a definition of local derivative with new parameter, providing a large number of applications, with a difference qualitative between both types of operators, local and global. Most importantly, Section 1.4 concludes "We can therefore conclude that both the Riemann – Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result [36] that, the local fractional operator is not a fractional derivative" (p.24). As we said before, they are new tools that have demonstrated their usefulness and potential in the modeling of different processes and phenomena (see also [3]).

In Fractional and Generalized Calculus, the Γ (see [31, 33, 37, 38]) and Γ_k (see [9]) functions are used:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t, \quad \Re(z) > 0, \tag{4}$$

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t^k/k}\, \mathrm{d}t, k > 0. \tag{5}$$

Unmistakably if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

One of the first operators that can be called fractional is that of Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}, Re(\alpha) \geq 0$, defined by (see [15]):

{d:RL}

Definition 4 *Let $f \in L^1((a_1,a_2);\mathbb{R}), (a_1,a_2) \in \mathbb{R}^2, a_1 < a_2$. The right and left side Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined by*

$$^{RL}J^{\alpha}_{a_1+} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{t} (t-s)^{\alpha-1} f(s) ds, \quad t > a_1, \tag{6}$$ {e:RL+}

and

$$^{RL}J^{\alpha}_{a_2-} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{a_2} (s-t)^{\alpha-1} f(s) ds, \quad t < a_2. \tag{7}$$ {e:RL-}

And their corresponding differential operators are given by

$$D^{\alpha}_{a_1+} f(t) = \frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a_1+} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{a_1}^{t} \frac{f(t)}{(t-s)^{\alpha}} ds.$$

$$D^{\alpha}_{a_2-} f(t) = -\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a_2-} f(t)\right) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t}^{a_2} \frac{f(t)}{(s-t)^{\alpha}} ds.$$

In [26] a generalized fractional derivative was defined in the following way (see also [14] and [39]).

{d:1}

Definition 5 *Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by*

$$N_F^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon F(t, \alpha)) - f(t)}{\varepsilon} \qquad (8)$$ {e:d}

for all $t > 0$, $\alpha \in (0, 1)$ being $F(t, \alpha)$ is some function.

If f is $\alpha-$differentiable in some $(0, \alpha)$, and $\lim_{t \to 0^+} N_F^{\alpha} f(t)$ exists, then define $N_F^{\alpha} f(0) = \lim_{t \to 0^+} N_F^{\alpha} f(t)$, note that if f is differentiable, then $N_F^{\alpha} f(t) = F(t, \alpha) f'(t)$ where $f'(t)$ is the ordinary derivative.

It is easy to verify that this operator contains, as particular cases, most of the known local differential operators, both conformable and non-conformable.

Now, we give the definition of a general integral. Throughout the work we will consider that the integral operator kernel F defined below is an absolutely continuous function.

{d:01}

Definition 6 *Let I be an interval $I \subseteq \mathbb{R}$, $a_1, t \in I$ and $\alpha \in \mathbb{R}$. The integral operators, right and left, is defined for every locally integrable function f on I as*

$$J_{F, a_1+}^{\alpha}(f)(t) = \int_{a_1}^{t} \frac{f(s)}{F(\frac{t-s}{a_2 - a_1}, \alpha)} ds, \quad t > a_1. \qquad (9)$$ {e:oig+}

$$J_{F, a_2-}^{\alpha}(f)(t) = \int_{t}^{a_2} \frac{f(s)}{F(\frac{s-t}{a_2 - a_1}, \alpha)} ds, \quad a_2 > t. \qquad (10)$$ {e:oig-}

{d:02}

Definition 7 *Let I be an interval $I \subseteq \mathbb{R}$, $a_1, t \in I$ and $\alpha \in \mathbb{R}$. The m-integral operators, right and left, is defined for every locally integrable function f on I as*

$${}^{m}J_{F, a_1+}^{\alpha}(f)(t) = \int_{a_1}^{t} \frac{f(s)}{F(\frac{t-s}{m a_2 - a_1}, \alpha)} ds, \quad t > a_1. \qquad (11)$$ {e:oig+}

$${}^{m}J_{F, a_2-}^{\alpha}(f)(t) = \int_{t}^{a_2} \frac{f(s)}{F(\frac{s-t}{m a_2 - a_1}, \alpha)} ds, \quad a_2 > t. \qquad (12)$$ {e:oig-}

Remark 8 *We will also use the "central" integral operator defined by (see [14] and [39]):*

$$J_{F, a_1}^{\alpha}(f)(a_2) = \int_{a_1}^{a_2} \frac{f(t)}{F(t, \alpha)} dt, a_2 > a_1. \qquad (13)$$ {e:i1}

To cite just two particular cases: if we consider $F \equiv 1$ we obtain the classical Riemann Integral and if we put $F(t, \alpha) = t^{1-\alpha}$ we obtain the Riemann-Liouville integral of the Definition 4. Note that the latter is a fractional integral operator, which can be generated from the conformable differential operator of [20].

Remark 9 *We can define the function space* $L^p_\alpha[a_2,a_1]$ *as the set of functions over* $[a_2,a_1]$ *such that* $(J^\alpha_{F,a_1}[f(t)]^p(a_2)) < +\infty$.

The following statement are analogous to the one known from the Ordinary Calculus (see [16], [14] and [39]).

{t:01}

Theorem 10 *Let* f *be* N*-differentiable function in* (t_0,∞) *with* $\alpha \in (0,1]$. *Then for all* $t > t_0$ *we have*

a) $J^\alpha_{F,t_0}\left(N^\alpha_F f(t)\right) = f(t) - f(t_0)$.

b) $N^\alpha_F\left(J^\alpha_{F,t_0} f(t)\right) = f(t)$.

An important property, and necessary, in our work is that established in the following result.

{t:02}

Theorem 11 *(Integration by parts) Let* u *and* v *be* N*-differentiable function in* (t_0,∞) *with* $\alpha \in (0,1]$. *Then for all* $t > t_0$ *we have*

$$J^\alpha_{F,t_0}\left((uN^\alpha_F v)(t)\right) = [uv(t) - uv(t_0)] - J^\alpha_{F,t_0}\left((vN^\alpha_F u)(t)\right) \tag{14}$$

In this article, using the notion of function (h,m)-convex, we establish new integral inequalities for this functional class, using the generalized integral operators of Definitions 6 and 7.

2 Results

Our first result is the following, a generalization of the inequality (1).

{t:0}

Theorem 12 *Let* $\phi : [0,+\infty) \to \mathbb{R}$ *be a* (h,m)*-convex function with* $m \in (0,1]$. *If* $0 \le a_1 < ma_2 < +\infty$, $\phi \in L^1[a_1, ma_2]$ *and* $h \in L^1[0,1]$, *then we have the following inequality:*

$$\begin{aligned}
&\frac{\mathbb{F}}{h\left(\frac{1}{2}\right)}\phi\left(\frac{a_1+a_2}{2}\right) \le \frac{1}{a_2-a_1}\left(J^\alpha_{F,a_1+}(\phi)(a_2) + J^\alpha_{F,a_2-}(\phi)(a_1)\right) \\
\le\ & (\phi(a_1)+\phi(a_2))\int_0^1 \frac{h(t)dt}{F(t,\alpha)} + m\left(\phi(\frac{a_1}{m}) + \phi(\frac{a_2}{m})\right)\int_0^1 \frac{h(1-t)dt}{F(t,\alpha)},
\end{aligned} \tag{15}$$

{e:001}

with $\mathbb{F} = \int_0^1 \frac{dt}{F(t,\alpha)}$.

Proof. For $x, y \in [0,+\infty)$, $t = \frac{1}{2}$ and $m = 1$, we have

$$\phi\left(\frac{x+y}{2}\right) \le h\left(\frac{1}{2}\right)\phi(x) + h\left(\frac{1}{2}\right)\phi(y),$$

If we choose $x = ta_1 + (1-t)a_2$ and $y = ta_2 + (1-t)a_1$, with $t \in [0,1]$, we get

$$\phi\left(\frac{a_1+a_2}{2}\right) \le h\left(\frac{1}{2}\right)\left(\phi(ta_1+(1-t)a_2)+\phi(ta_2+(1-t)a_1)\right). \tag{16}$$ {e:002}

By integrating this inequality with respect to t, on $[0,1]$, and changing variables brings us to the first inequality of (15). From right member of (16) we obtain

$$\begin{aligned} & h\left(\frac{1}{2}\right)\left(\phi(ta_1+(1-t)a_2)+\phi(ta_2+(1-t)a_1)\right) \\ = & h\left(\frac{1}{2}\right)\left(\phi(ta_1+m(1-t)\frac{a_2}{m})+\phi(ta_2+m(1-t)\frac{a_1}{m})\right) \\ \le & h\left(\frac{1}{2}\right)\left(h(t)\phi(a_1)+mh(1-t)\phi(\frac{a_2}{m}))+h(t)\phi(a_2)+mh(1-t)\phi(\frac{a_1}{m}))\right). \end{aligned}$$

Integrating with respect to t, between 0 and 1, leads us to

$$\begin{aligned} & h\left(\frac{1}{2}\right)\left(J^{\alpha}_{F,a_1+}(\phi)(a_2)+J^{\alpha}_{F,a_2-}(\phi)(a_1)\right) \\ \le & h\left(\frac{1}{2}\right)\left((\phi(a_1)+\phi(a_2))\int_0^1\frac{h(t)dt}{F(t,\alpha)}+m\left(\phi(\frac{a_1}{m})+\phi(\frac{a_2}{m})\right)\int_0^1\frac{h(1-t)dt}{F(t,\alpha)}\right). \end{aligned}$$

What is the second required inequality. ■

Remark 13 *If in the previous Theorem we consider the Riemann integral, or what is the same, $F\equiv 1$ and ϕ is a convex function ($h(t)=t$ and $m=1$), then from* (16) *we obtain the classic Hermite-Hadamard Inequality* (1)*. This result contains as a particular case Theorem 9 of [29], putting $F\equiv 1$.*

As we will see, the following result "complements" the previous one.

{t:1}

Theorem 14 *Let $\phi:[0,+\infty)\to\mathbb{R}$ be a (h,m)-convex function with $m\in(0,1]$. If $0\le a_1<ma_2<+\infty$, $\phi\in L^1[a_1,ma_2]$ and $h\in L^1[0,1]$, then we have the following inequality:*

$$\frac{1}{ma_2-a_1}\left[{}^{m}J^{\alpha}_{F,a_1+}(\phi)(ma_2)\right] \le \phi(a_1)J^{\alpha}_{F,0}(h(s))(1)+\phi(ma_2)J^{\alpha}_{F,0}(h(1-s))(1). \tag{17}$$ {e:01}

Proof. Using the (h,m)-convexity of ϕ we have

$$\phi(ta_1+(1-t)a_2)\le h(t)\phi(a_1)+mh(1-t)\phi(a_2).$$

Integrating the previous inequality, with respect to t between 0 and 1, we obtain

$$\int_0^1 \frac{\phi(ta_1+(1-t)a_2)}{F(t,\alpha)}dt \le \phi(a_1)\int_0^1 \frac{h(t)}{F(t,\alpha)}dt + m\phi(a_2)\int_0^1 \frac{h(1-t)}{F(t,\alpha)}dt.$$

By changing variables in the first integral, the required inequality is obtained.
■

Remark 15 *It is easy to verify that if $F \equiv 1$ y ϕ is a convex function, then from* (17) *we get the right member of the classic Hermite-Hadamard inequality* (1). *In this same direction, working with the classical Riemann integral, that is, taking $F \equiv 1$ and simultaneously taking $\phi(ta_1+(1-t)a_2)$ and with $\phi(ta_2+(1-t)a_1)$, we can easily obtain Theorem 8 from [29]. Analogously, if in the Theorem 14 we put $F \equiv 1$, we obtain Theorem 2.1 of [23], the Remark 2.1 of the aforementioned work is also still valid. If, on the contrary, we consider the kernel $F(t,\alpha) = \frac{t^{1-\alpha}}{\Gamma(\alpha)}$, the following inequality not reported in the literature is obtained, valid for Riemann-Liouville fractional integrals:*

$$\frac{1}{ma_2-a_1}\left[{}^{RL}J^{\alpha}_{a_1+}\phi(t)\right] \le (\phi(a_1)+\phi(ma_2))\left[{}^{RL}J^{\alpha}_{0^+}(h)(1)\right].$$

Of course, if we consider different kernels, we will get new variants of (17).

A more general variation from the previous result, in which two (h,m)-convex functions are involved, is as follows.

{t:2}

Theorem 16 *Let ϕ_1 be a (h_1,m)-convex and ϕ_2 a (h_2,m)-convex functions such that $\phi_1\phi_2 \in L^1[a_1,a_2]$ and $h_1h_2 \in L^1[a_1,a_2]$. So, we have the following inequality*

$$\begin{aligned}
& \frac{1}{ma_2-a_1}\left[{}^{m}J^{\alpha}_{F,a_1+}(\phi)(ma_2) + {}^{m}J^{\alpha}_{F,ma_2-}(\phi)(a_1)\right] \\
\le\ & (\phi_1(a_1)\phi_2(a_1)+\phi_1(a_2)\phi_2(a_2))\, J^{\alpha}_{F,0}(h_1h_2)(1) \\
+\ & m\phi_1(a_1)\phi_2(a_2)\int_0^1 h_1(s)h_2(1-s)d_Fs + m\phi_1(a_2)\phi_2(a_1)\int_0^1 h_1(1-s)h_2(s)d_Fs \\
+\ & m\phi_1(a_2)\phi_2(a_1)\int_0^1 h_1(s)h_2(1-s)d_Fs + m\phi_1(a_1)\phi_2(a_2)\int_0^1 h_1(1-s)h_2(s)d_Fs \\
+\ & m^2(\phi_1(a_1)\phi_2(a_1)+\phi_1(a_2)\phi_2(a_2))\int_0^1 h_1(1-s)h_2(1-s)d_Fs \qquad (18)
\end{aligned}$$

{e:9}

where $d_Fs = \frac{ds}{F(s,\alpha)}$.

Proof. Since ϕ_1 and ϕ_2 are (h_1,m)-convex and (h_2,m)-convex, respectively, we have

$$\begin{aligned}
& \phi_1(ta_1+(1-t)a_2)\phi_2(ta_1+(1-t)a_2) \\
\le\ & (h_1(t)\phi_1(a_1)+mh_1(1-t)\phi_1(a_2))\,(h_2(t)\phi_2(a_1)+mh_2(1-t)\phi_2(a_2)) \qquad (19)
\end{aligned}$$

{e:91}

$$\begin{aligned}
& \phi_1(ta_2+(1-t)a_1)\phi_2(ta_2+(1-t)a_1) \\
\leq\ & (h_1(t)\phi_1(a_2)+mh_1(1-t)\phi_1(a_1))\,(h_2(t)\phi_2(a_2)+mh_2(1-t)\phi_2(a_1)) \qquad (20)
\end{aligned}$$ {e:92}

after multiplying and ordering we get from (19) and (20)

$$\begin{aligned}
& \phi_1(ta_2+(1-t)a_1)\phi_2(ta_2+(1-t)a_1) \\
\leq\ & h_1(t)h_2(t)\left(\phi_1(a_1)\phi_2(a_1)+\phi_1(a_2)\phi_2(a_2)\right) \\
+\ & mh_1(t)h_2(1-t)\phi_1(a_1)\phi_2(a_2)+mh_1(1-t)h_2(t)\phi_1(a_2)\phi_2(a_1) \\
+\ & mh_1(t)h_2(1-t)\phi_1(a_2)\phi_2(a_1)+mh_1(1-t)h_2(t)\phi_1(a_1)\phi_2(a_2) \\
+\ & m^2h_1(1-t)h_2(1-t)\left(\phi_1(a_1)\phi_2(a_1)+\phi_1(a_2)\phi_2(a_2)\right).
\end{aligned}$$

Integrating in the previous inequality, with respect to t between 0 and 1, we obtain the inequality sought. ■

Remark 17 *If in the Theorem 16, we put $F \equiv 1$, and we consider only* (19) *we obtain Theorem 2.2 of [23]. If we consider the kernel $F(t,\alpha) = t^{1-\alpha}$, we obtain new inequalities under the Riemann-Liouville fractional integrals. If we use another kernel F, we will obtain inequalities not reported in the literature.*

A more general conclusion to Theorem 14, is given in the following result.

{t:3}

Theorem 18 *Under the conditions on ϕ_1 and ϕ_2 of the previous Theorem, the following inequality is satisfied:*

$$\frac{1}{ma_2-a_1}\left[{}^mJ^{\alpha}_{F,a_1+}\phi_1\phi_2(ma_2)+{}^mJ^{\alpha}_{F,a_2-}\phi_1\phi_2(a_1)\right] \leq MinA \qquad (21)$$

with

$$\begin{aligned}
& A=\left(\phi_1(a_1)\phi_2(a_1)+\phi_1(a_2)\phi_2(a_2)\right)J^{\alpha}_{F,0}(h_1h_2)(1) \\
+\ & m_1m_2\left[\phi_1\left(\frac{a_2}{m_1}\right)\phi_2\left(\frac{a_2}{m_2}\right)\right]J^{\alpha}_{F,0}(h_1h_2)(1) \\
+\ & \left[\phi_1\left(\frac{a_2}{m_1}\right)\phi_2(a_1)+m_1\phi_1\left(\frac{a_1}{m_1}\right)\phi_2(a_2)\right]\int_0^1\left(h_1(1-s)h_2(s)\right)d_Fs \\
+\ & m_2\left[\phi_1(a_1)\phi_2\left(\frac{a_2}{m_2}\right)+\phi_1(a_2)\phi_2\left(\frac{a_1}{m_2}\right)\right]\int_0^1\left(h_1(s)h_2(1-s)\right)d_Fs.
\end{aligned}$$

Proof. The equations (19) and (20) can be rewritten as follows:

$$\begin{aligned}
& \phi_1(ta_1+(1-t)a_2)\phi_2(ta_1+(1-t)a_2) \\
=\ & \phi_1(ta_1+m_1(1-t)\frac{a_2}{m_1})\phi_2(ta_1+m_2(1-t)\frac{a_2}{m_2}) \\
\leq\ & \left(h_1(t)\phi_1(a_1)+m_1h_1(1-t)\phi_1(\frac{a_2}{m_1})\right)\left(h_2(t)\phi_2(a_1)+m_2h_2(1-t)\phi_2(\frac{a_2}{m_1})\right) \qquad (22)
\end{aligned}$$ {e:101}

$$
\begin{aligned}
& \phi_1(ta_2+(1-t)a_1)\phi_2(ta_2+(1-t)a_1) \\
= \; & \phi_1(ta_2+m_1(1-t)\frac{a_1}{m_1})\phi_2(ta_2+m_2(1-t)\frac{a_1}{m_2}) \\
\leq \; & \left(h_1(t)\phi_1(a_2)+m_1h_1(1-t)\phi_1(\frac{a_1}{m_1})\right)\left(h_2(t)\phi_2(a_2)+m_2h_2(1-t)\phi_2(\frac{a_1}{m_2})\right) \quad (23) \quad \{e{:}102\}
\end{aligned}
$$

Adding member to member we obtain

$$
\begin{aligned}
& \phi_1(ta_1+(1-t)a_2)\phi_2(ta_1+(1-t)a_2)\phi_1(ta_2+(1-t)a_1)\phi_2(ta_2+(1-t)a_1) \\
\leq \; & \left(h_1(t)\phi_1(a_1)+m_1h_1(1-t)\phi_1(\frac{a_2}{m_1})\right)\left(h_2(t)\phi_2(a_1)+m_2h_2(1-t)\phi_2(\frac{a_2}{m_1})\right) \\
+ \; & \left(h_1(t)\phi_1(a_2)+m_1h_1(1-t)\phi_1(\frac{a_1}{m_1})\right)\left(h_2(t)\phi_2(a_2)+m_2h_2(1-t)\phi_2(\frac{a_1}{m_2})\right).
\end{aligned}
$$

Proceeding as in the previous Theorem, we obtain the required inequality.
■

Remark 19 *If we consider the kernel* $F \equiv 1$ *and use only* (22)*, we get Theorem 2.3 of [23]. Using different kernels, we obtain new integral inequalities, in particular, for the Riemann-Liouville Fractional Integrals.*

The following result, will be basic from now on.

Lemma 20 *Let* $\phi : I \to \mathbb{R}$*,* $I \subset \mathbb{R}$*, be a differentiable mapping on* I^o*, and* $a_1, a_2 \in I$*,* $m \in [0,1]$ *and* $a_1 < ma_2$*. If* $\phi' \in L^1[a_1, ma_2]$*, then*

$$
\begin{aligned}
& \frac{\phi(a_1)+\phi(a_2)}{2} - \frac{1}{ma_2-a_1} J^{\alpha}_{F,ma_2-}(\phi)(a_1) \\
= \; & \frac{ma_2-a_1}{2} J^{\alpha}_{F,0}\left[(1-2t)N^{\alpha}_{F}\phi(ta_1+m(1-t)a_2))\right](1). \qquad (24) \quad \{e{:}11\}
\end{aligned}
$$

Proof. It is enough to apply the Theorem 11 to the integral of the right member of (24), to obtain the left member. ■

Remark 21 *Putting* $F \equiv 1$ *we obtain the Lemma 2.1 of [23]. If we additionally consider* $m = 1$ *we have Lemma 2.1 of [10]. On the other hand, it is clear that we can "work" with the* ϕ *function in* $ta_1 + (1-t)a_2$ *and* $ta_2 + (1-t)a_1$*, a result already known for several integral operators.*

Hereafter, we will provide various extensions of the hermite-Hadamard Inequality, using the Lemma above.

Theorem 22 *Under the considerations of Lemma 2.1, if* $|N_F^\alpha \phi|$ *is a* (h,m)*-convex function, then we have*

$$\left|\frac{\phi(a_1)+\phi(a_2)}{2} - \frac{1}{ma_2-a_1} J^\alpha_{F,ma_2-}(\phi)(a_1)\right| \leq \frac{ma_2-a_1}{2} J^\alpha_{F,0}\left[\left|N_F^\alpha \phi(a_1)\right| h(t) + m\left|N_F^\alpha \phi(a_2)\right| h(1-t)\right](1). \quad (25)$$

{e:12}

Proof. Using Lemma 2.1 and (h,m)-convexity of $|N_F^\alpha \phi|$ we have

$$\begin{aligned}
&\left|\frac{\phi(a_1)+\phi(a_2)}{2} - \frac{1}{ma_2-a_1} J^\alpha_{F,ma_2-}(\phi)(a_1)\right| \\
\leq\ & \frac{ma_2-a_1}{2} J^\alpha_{F,0}\left[|1-2t|\left|N_F^\alpha \phi(ta_1+m(1-t)a_2))\right|\right](1) \\
\leq\ & \frac{ma_2-a_1}{2} J^\alpha_{F,0}\left[\left|N_F^\alpha \phi(a_1)\right| h(t) + m\left|N_F^\alpha \phi(a_2)\right| h(1-t)\right](1).
\end{aligned}$$

which is the required inequality. ■

Remark 23 *In the case of considering that the kernel* F *is symmetric in* t *with respect to* $\frac{1}{2}$ *then we obtain Theorem 2.6 of [23].*

Theorem 24 *Under the considerations of Lemma 2.1, if* $|N_F^\alpha \phi|^q$ *is a* (h,m)*-convex function, with* $p,q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$*, then we have*

$$\begin{aligned}
&\left|\frac{\phi(a_1)+\phi(a_2)}{2} - \frac{1}{ma_2-a_1} J^\alpha_{F,ma_2-}(\phi)(a_1)\right| \\
\leq\ & \frac{ma_2-a_1}{2}\left(J^\alpha_{F,0}|1-2t|^p\big(1)\right)^{\frac{1}{p}}\left(J^\alpha_{F,0}\left[\left|N_F^\alpha \phi(a_1)\right|^q h(t) + m\left|N_F^\alpha \phi(a_2)\right|^q h(1-t)\right]\big(1)\right)^{\frac{1}{q}}
\end{aligned} \quad (26)$$

Proof. Using Lemma 2.1, the (h,m)-convexity of $|N_F^\alpha \phi|^q$ and using well known Hölder's integral inequality, we get

$$\begin{aligned}
&\left|\frac{\phi(a_1)+\phi(a_2)}{2} - \frac{1}{ma_2-a_1} J^\alpha_{F,ma_2-}(\phi)(a_1)\right| \\
\leq\ & \frac{ma_2-a_1}{2} J^\alpha_{F,0}\left[\left|N_F^\alpha \phi(a_1)\right| h(t) + m\left|N_F^\alpha \phi(a_2)\right| h(1-t)\right](1) \\
\leq\ & \frac{ma_2-a_1}{2}\left(J^\alpha_{F,0}|1-2t|^p\big(1)\right)^{\frac{1}{p}}\left(J^\alpha_{F,0}\left|N_F^\alpha \phi(ta_1+m(1-t)a_2))\right|^q\big(1)\right)^{\frac{1}{q}} \\
\leq\ & \leq \frac{ma_2-a_1}{2}\left(J^\alpha_{F,0}|1-2t|^p\big(1)\right)^{\frac{1}{p}}\left(J^\alpha_{F,0}\left[\left|N_F^\alpha \phi(a_1)\right|^q h(t) + m\left|N_F^\alpha \phi(a_2)\right|^q h(1-t)\right]\big(1)\right)^{\frac{1}{q}}.
\end{aligned}$$

Which is the desired result. ■

Theorem 25 *Under the considerations of Lemma 2.1, if $|N_F^\alpha \phi|^q$ is a (h,m)-convex function, with $q \geq 1$, then*

$$\begin{aligned}
&\left|\frac{\phi(a_1)+\phi(a_2)}{2} - \frac{1}{ma_2-a_1} J_{F,ma_2-}^\alpha(\phi)(a_1)\right| \\
\leq\ & \frac{ma_2-a_1}{2}\left(J_{F,0}^\alpha\left|1-2t\right|(1)\right)^{1-\frac{1}{q}}\left(J_{F,0}^\alpha\left[\left|N_F^\alpha\phi(a_1)\right|^q h(t) + m\left|N_F^\alpha\phi(a_2)\right|^q h(1-t)\right]\left(1\right)\right)^{\frac{1}{q}}. \qquad (27)
\end{aligned}$$

Proof. Using Lemma 2.1, the (h,m)-convexity of $|N_F^\alpha \phi|^q$ and taking into account the well known power mean inequality, have

$$\begin{aligned}
&\left|\frac{\phi(a_1)+\phi(a_2)}{2} - \frac{1}{ma_2-a_1} J_{F,ma_2-}^\alpha(\phi)(a_1)\right| \\
\leq\ & \frac{ma_2-a_1}{2} J_{F,0}^\alpha\left[\left|N_F^\alpha\phi(a_1)\right| h(t) + m\left|N_F^\alpha\phi(a_2)\right| h(1-t)\right](1) \\
\leq\ & \frac{ma_2-a_1}{2}\left(J_{F,0}^\alpha\left|1-2t\right|(1)\right)^{1-\frac{1}{q}}\left(J_{F,0}^\alpha\left|N_F^\alpha\phi(ta_1+m(1-t)a_2))\right|^q\left(1\right)\right)^{\frac{1}{q}} \\
\leq\ & \frac{ma_2-a_1}{2}\left(J_{F,0}^\alpha\left|1-2t\right|(1)\right)^{1-\frac{1}{q}}\left(J_{F,0}^\alpha\left[\left|N_F^\alpha\phi(a_1)\right|^q h(t) + m\left|N_F^\alpha\phi(a_2)\right|^q h(1-t)\right]\left(1\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

Wanted inequality. ■

Remark 26 *If we consider the kernel $F \equiv 1$ and work with convex functions, that is, h is the identity function and $m = 1$, this result becomes Theorem 1 of [30].*

3 Conclusions

In this paper, using a generalized integral operator, we have obtained various integral inequalities, which generalize several interesting results reported in the literature and which open up new work possibilities, depending on the kernel used, for example, if we use that of the generalized fractional integral of Hilfer (see [25]), we can derive new inequalities not yet published.

Finally, we want to emphasize that the results presented contain generalized inequalities valid for various functional classes such as convex, h-convex functions, m-convex functions, Godunova-Levin functions, p and s-convex functions in the second sense, defined on a closed interval of non-negative real numbers.

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